① $y' = x e^{-4x}$, $y(0)=2$. Mennyi $y(1)$?

$$y_{ált}(x) = \int x e^{-4x} dx = \left| \begin{matrix} f' = e^{-4x} & g = x \\ F = \frac{e^{-4x}}{-4} & g' = 1 \end{matrix} \right| = -\frac{e^{-4x}}{4} x - \int \frac{e^{-4x}}{-4} \cdot 1 \, dx$$

$$= -\frac{e^{-4x}}{4} x - \frac{e^{-4x}}{16} + C$$

$$y(0)=2 \iff -\frac{e^{-4\cdot 0}}{4} \cdot 0 - \frac{e^{-4\cdot 0}}{16} + C = 2 \longrightarrow C = 2 + \frac{1}{16e^4}$$

$$y_{part}(x) = -\frac{e^{-4x}}{4} x - \frac{e^{-4x}}{16} + \left(2 + \frac{1}{16e^4}\right) \longrightarrow y(1) = -\frac{e^{-4\cdot 1}}{4} \cdot 1 - \frac{e^{-4\cdot 1}}{16} + \left(2 + \frac{1}{16e^4}\right)$$

② $y' = 2y - 7$, $y(0)=1$, Mennyi $y(1)$?

$$\frac{dy}{dx} = 2y-7 \longrightarrow \frac{dx}{dy} = \frac{1}{2y-7} \longrightarrow x = \int \frac{1}{2y-7} dy = \frac{\ln|2y-7|}{2} + C \longrightarrow$$

$$\longrightarrow e^{2x-C} = |2y-7| \longrightarrow y = \left(\pm e^{2x+C} + 7\right) \cdot \frac{1}{2} = K e^{2x} + \frac{7}{2}$$

$$y(0)=1 \iff K e^{2\cdot 0} + \frac{7}{2} = 1 \longrightarrow K = -\frac{5}{2} \longrightarrow y(1) = -\frac{5}{2} e^{2\cdot 1} + \frac{7}{2}.$$

Az általános megoldás struktúrája:

$$y = \frac{1}{2}\left(\pm e^{2(x+\tilde{c})} + 7\right) = K \cdot e^{2x} + \frac{7}{2}$$

$\frac{7}{2}$: az inhomogén egyenlet egyik partikuláris megoldása $\left(\left(\frac{7}{2}\right)' = 2 \cdot \frac{7}{2} - 7\right)$

$e^{2(x+\tilde{c})}$: autonóm DE.

$K \cdot e^{2x}$: a lin. $y'=2y$ DE megoldása

③ $y'' - 3y' + 2y = 0$, $y(0)=4$, $y'(0)=5$. Mennyi $y(1)$?

$$y = e^{\lambda x} \longrightarrow \lambda^2 - 3\lambda + 2 = 0 \longrightarrow \lambda_1 = 1, \ \lambda_2 = 2$$

$$y_{ált} = C_1 e^{1\cdot x} + C_2^{2x} \longrightarrow y'_{ált} = C_1 \cdot 1 \cdot e^x + C_2 \cdot 2 \cdot e^x$$

$$\left.\begin{matrix} y(0)=4 \longrightarrow C_1 e^{1\cdot 0} + C_2 e^{2\cdot 0} = 4 \longrightarrow C_1 + C_2 = 4 \\ y'(0)=5 \longrightarrow C_1 \cdot 1 \cdot e^{1\cdot 0} + C_2 \cdot 2 \cdot e^{2\cdot 0} = 5 \longrightarrow C_1 + 2C_2 = 5 \end{matrix}\right\} \longrightarrow C_1 = 3, \ C_2 = 1$$

$$y_{part} = 3e^x + e^{2x}$$

④ $y'' - 4y' + 4y = 0$, $y(0)=3$, $y'(0)=1$. Mennyi $y(1)$?

$$y = e^{\lambda x} \longrightarrow \lambda^2 - 4\lambda + 4 = 0 \longrightarrow \lambda_1 = \lambda_2 = 2$$

$$y_{ált} = C_1 \cdot e^{2x} + C_2 x e^{2x} \longrightarrow y'_{ált} = C_1 \cdot 2 \cdot e^{2x} + C_2 e^{2x} + C_2 x \cdot 2 e^{2x}$$

$$\left.\begin{matrix} y(0)=3 \longrightarrow C_1 \cdot e^{2\cdot 0} + C_2 \cdot 0 \cdot e^{2\cdot 0} = 3 \\ y'(0)=1 \longrightarrow C_1 \cdot 2 \cdot e^{2\cdot 0} + C_2 \cdot e^{2\cdot 0} + C_2 \cdot 0 \cdot 2 \cdot e^{2\cdot 0} = 1 \end{matrix}\right\} \longrightarrow \left.\begin{matrix} C_1 = 3 \\ 2C_1 + C_2 = 1 \end{matrix}\right\} \longrightarrow \begin{matrix} C_1 = 3 \\ C_2 = -5 \end{matrix}$$

$$y_{part} = 3 \cdot e^{2x} - 5xe^{2x}$$

$$y(1) = 3 \cdot e^2 - 5 \cdot 1 e^2$$

⑤ $y'' - 2y' + 5y = 0$, $y(0) = 1$, $y(0) = 2$. Mennyi $y(1)$?

$y = e^{\lambda x} \to \lambda^2 - 2\lambda + 5 = 0 \to \lambda_1 = 1 + 2i$, $\lambda_2 = 1 - 2i$

$y_{ált} = C_1 e^{(1+2i)x} + C_2 e^{(1-2i)x} \longrightarrow y'_{ált} = C_1(1+2i)e^{(1+2i)x} + C_2(1-2i)e^{(1+2i)x}$

(vagy $y_{ált} = e^{1\cdot x}(\tilde{C}_1 \cos(2x) + \tilde{C}_2 \sin(2x))$)

$$\left.\begin{array}{l} y(0) = 1 \longrightarrow C_1 e^{(1+2i)\cdot 0} + C_2 e^{(1-2i)\cdot 0} = C_1 + C_2 = 1 \\ y'(0) = 2 \longrightarrow C_1(1+2i)e^0 + C_2(1-2i)e^0 = C_1(1+2i) + C_2(1-2i) = 2 \end{array}\right\} \longrightarrow$$

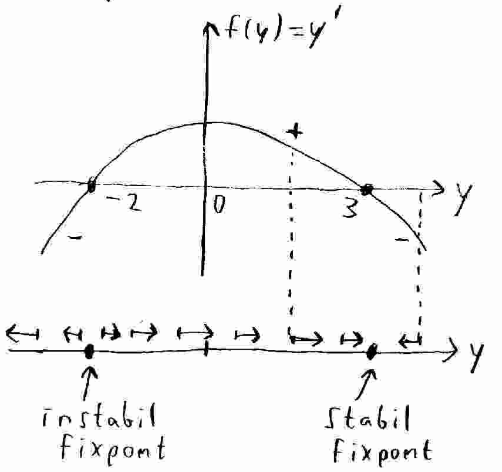
$$\longrightarrow \begin{array}{l} C_1 = \frac{1}{2} - \frac{1}{4}i \\ C_2 = \frac{1}{2} + \frac{1}{4}i \end{array} \longrightarrow y_{part} = \left(\tfrac{1}{2} - \tfrac{1}{4}i\right) e^{(1+2i)x} + \left(\tfrac{1}{2} + \tfrac{1}{4}i\right) e^{(1-2i)x}$$

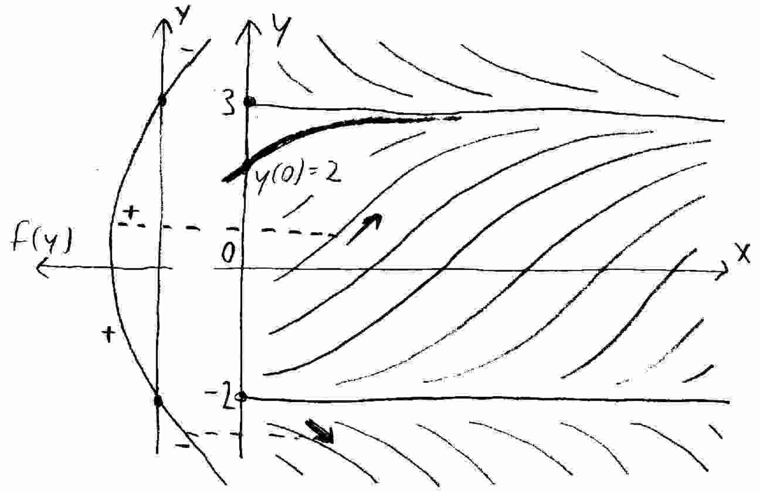
Mivel $e^{(1+2i)x} = e^x(\cos(2x) + i\sin(2x))$, stb.

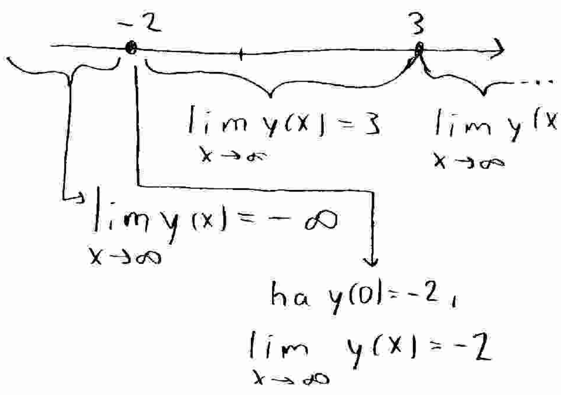
$y_{part} = e^x\left[\cos(2x) + \frac{1}{2}\sin(2x)\right]$

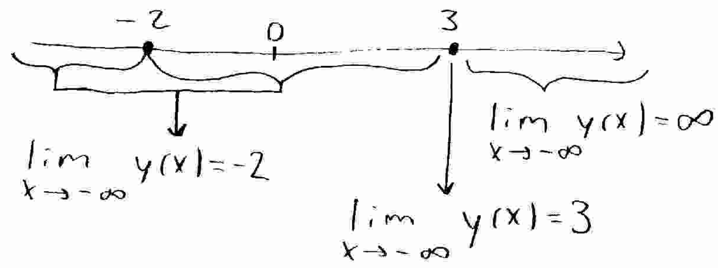
⑥ $y' = -(y+2)(y-3)$, $y(0) = 2$. Mennyi $\lim_{x\to\infty} y(x)$?

$y' = f(y) = -(y+2)(y-3) \to$ fixpontok: $y' = f(y) = 0 \longrightarrow y_1 = -2$, $y_2 = 3$









Ha $y(0) = 2$, akkor $\lim_{x\to\infty} y(x) = 3$, mivel $y = 2$ az $y_F = -2$ instabil fixpont után, illetve az $y_F = 3$ stabil fixpont előtt helyezkedik el.

⑦ $\dfrac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$, $\begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$. Mennyi $y_1(1)$?

$A = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$.

Sajátértékek: $\begin{vmatrix} 1-\lambda & 2 \\ 0 & 3-\lambda \end{vmatrix} = (1-\lambda)(3-\lambda) - 2\cdot 0 = 0 \longrightarrow \lambda_1 = 1,\ \lambda_2 = 3.$

Sajátvektorok:

$\lambda_1 = 1: \quad \begin{pmatrix} 1-1 & 2 \\ 0 & 3-1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \longrightarrow \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ 0 \end{pmatrix} \longrightarrow v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\lambda_2 = 3: \quad \begin{pmatrix} 1-3 & 2 \\ 0 & 3-3 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \longrightarrow \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_1 \end{pmatrix} \longrightarrow v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\longrightarrow y_{ált}: C_1 \cdot e^{1\cdot x}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2\, e^{3\cdot x}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \end{pmatrix} \longrightarrow C_1 e^{1\cdot 0}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 e^{3\cdot 0}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \end{pmatrix} \longrightarrow \begin{matrix} C_1 = 2 \\ C_2 = 3 \end{matrix}$

$\longrightarrow y_{part} = 2\cdot e^{x}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 3e^{3x}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2e^{x} + 3e^{3x} \\ 3e^{3x} \end{pmatrix}$

vagy:

$\bar{y}(x) = e^{xA}\,\bar{y}(0)$

$S^{-1}AS = D: \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{-1}\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}\left(\overset{v_1}{\boxed{\begin{matrix} 1 \\ 0 \end{matrix}}}\ \overset{v_2}{\boxed{\begin{matrix} 1 \\ 1 \end{matrix}}}\right) = \begin{pmatrix} \overset{\lambda_1}{1} & 0 \\ 0 & \overset{\lambda_2}{3} \end{pmatrix}$

$SDS^{-1} = A: \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$

$e^{xA} = \sum_{n=0}^{\infty} \frac{x^n}{n!} A^n = \sum_{n=0}^{\infty} \frac{x^n}{n!}(SDS^{-1})^n = \sum_{n=0}^{\infty} \frac{x^n}{n!} S D^n S^{-1} =$

$= S\left(\sum_{n=0}^{\infty} \frac{x^n}{n!} D^n\right) S^{-1} = S e^{xD} S^{-1}$

$e^{x\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}} = e^{\begin{pmatrix} 1x & 0 \\ 0 & 3x \end{pmatrix}} = \begin{pmatrix} e^{1x} & 0 \\ 0 & e^{3x} \end{pmatrix} = e^{xD}$

$e^{xA} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} e^{x} & 0 \\ 0 & e^{3x} \end{pmatrix}\underbrace{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{-1}}_{\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}} = \begin{pmatrix} e^{x} & -e^{x} + e^{3x} \\ 0 & e^{3x} \end{pmatrix}$

$\bar{y}(x) = e^{xA}\,\bar{y}(0) = \begin{pmatrix} e^{x} & -e^{x} + e^{3x} \\ 0 & e^{3x} \end{pmatrix}\begin{pmatrix} 5 \\ 3 \end{pmatrix} = \begin{pmatrix} 2e^{x} + 3e^{3x} \\ 3e^{3x} \end{pmatrix}$

$y_1(1) = 2\cdot e^{1} + 3e^{3\cdot 1} = 2e + 3e^{3}$

⑧ $\frac{d}{dx}\begin{pmatrix}y_1\\y_2\end{pmatrix} = \begin{pmatrix}2&1\\0&2\end{pmatrix}\begin{pmatrix}y_1\\y_2\end{pmatrix}$, $\begin{pmatrix}y_1(0)\\y_2(0)\end{pmatrix} = \begin{pmatrix}3\\4\end{pmatrix}$, Mennyi $y_1(1)$?

$A = \begin{pmatrix}2&1\\0&2\end{pmatrix}$, Jordan-blokk mátrix.

$$e^{xA} = e^{x\left[\begin{pmatrix}2&0\\0&2\end{pmatrix}+\begin{pmatrix}0&1\\0&0\end{pmatrix}\right]} = e^{x\begin{pmatrix}2&0\\0&2\end{pmatrix}+x\begin{pmatrix}0&1\\0&0\end{pmatrix}} = e^{x\begin{pmatrix}2&0\\0&2\end{pmatrix}} \cdot e^{x\begin{pmatrix}0&1\\0&0\end{pmatrix}},$$

Mivel $e^{B+C} = e^B e^C = e^C e^B$, ha $BC = CB$, vagyis ha $[B,C] = BC - CB = 0$.

$$1 + \frac{(B+C)^1}{1!} + \frac{1}{2!}(B+C)^2 + \frac{1}{3!}(B+C)^3 + \dots = \left(1 + \frac{B}{1!} + \frac{B^2}{2!} + \frac{B^3}{3!} + \dots\right)\left(1 + \frac{C}{1!} + \frac{C^2}{2!} + \frac{C^3}{3!} + \dots\right)$$

$$= \dots + \underbrace{\frac{(\dots + BCC + CBC + CCB + \dots)}{3!}}_{\frac{\binom{3}{2}}{3!} B^1 C^2} + \dots \qquad = \dots + \frac{1}{1!}\frac{1}{2!} B^1 \cdot C^2 + \dots$$

$\frac{\binom{3}{2}}{3!} B^1 C^2$, mivel $BCC = CBC = CCB$. $\left(\frac{\frac{3!}{(3-2)!\,2!}}{3!} = \frac{1}{1!\,2!}\right)$

$$e^{x\begin{pmatrix}0&1\\0&0\end{pmatrix}} = 1 + x\begin{pmatrix}0&1\\0&0\end{pmatrix} + \frac{x^2}{2!}\begin{pmatrix}0&1\\0&0\end{pmatrix}^2 + \dots = 1 + \begin{pmatrix}0&x\\0&0\end{pmatrix} = \begin{pmatrix}1&x\\0&1\end{pmatrix}, \text{ mivel } \begin{pmatrix}0&1\\0&0\end{pmatrix}^2 = 0$$

Tehát $e^{xA} = \begin{pmatrix}e^{2x}&0\\0&e^{2x}\end{pmatrix}\begin{pmatrix}1&x\\0&1\end{pmatrix} = \begin{pmatrix}e^{2x}&xe^{2x}\\0&e^{2x}\end{pmatrix}$

$$\bar{y}(1) = \begin{pmatrix}e^{2\cdot1}&1\cdot e^{2\cdot1}\\0&e^{2\cdot1}\end{pmatrix}\begin{pmatrix}3\\4\end{pmatrix} = \begin{pmatrix}7e^2\\4e^2\end{pmatrix} \longrightarrow y_1(1) = 7e^2$$

Ebben a feladatban A az $e_1 = \begin{pmatrix}1\\0\end{pmatrix}$, $e_2 = \begin{pmatrix}0\\1\end{pmatrix}$ bázisban már eleve a Jordan normálformában volt. Mi a tulajdonsága ennek a bázisnak?

$$\begin{pmatrix}2&1\\0&2\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = 2\cdot\begin{pmatrix}1\\0\end{pmatrix}, \quad \left[\begin{pmatrix}2&1\\0&2\end{pmatrix} - 2\cdot\begin{pmatrix}1&0\\0&1\end{pmatrix}\right]\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0&1\\0&0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$$

$$A e_1 = \lambda e_1, \quad (A - \lambda) e_2 = e_1 \qquad (\lambda = 2)$$

Hogyan transzformáljunk egy mátrixot Jordan normálformába?

pl: $A = \begin{pmatrix}4&1\\-1&6\end{pmatrix} \rightarrow \begin{vmatrix}4-\lambda&1\\-1&6-\lambda\end{vmatrix} = (4-\lambda)(6-\lambda)+1 = \lambda^2 - 10\lambda + 25 \rightarrow \lambda_{1,2} = 5$

$$\begin{pmatrix}4-5&1\\-1&6-5\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = 0 \longrightarrow x = y \longrightarrow v_1 = \begin{pmatrix}1\\1\end{pmatrix}$$

$$\begin{pmatrix}4-5&1\\-1&6-5\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix} \longrightarrow x = -1 + y \longrightarrow v_2 = \begin{pmatrix}-1\\0\end{pmatrix}$$

Tehát $\begin{pmatrix}1&-1\\1&0\end{pmatrix}^{-1}\begin{pmatrix}4&1\\-1&6\end{pmatrix}\left(\boxed{\begin{matrix}1\\1\end{matrix}}\ \boxed{\begin{matrix}-1\\0\end{matrix}}\right) = \begin{pmatrix}5&1\\0&5\end{pmatrix}$

$$\begin{pmatrix}1&-1\\1&0\end{pmatrix}\begin{pmatrix}5&1\\0&5\end{pmatrix}\begin{pmatrix}1&-1\\1&0\end{pmatrix}^{-1} = \begin{pmatrix}4&1\\-1&6\end{pmatrix}$$

⑨ $\frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ -1 & -2 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$, $\begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$. Mennyi $y_1(1)$?

saját értékek: $\begin{vmatrix} -2-\lambda & 1 \\ -1 & -2-\lambda \end{vmatrix} = \lambda^2+4\lambda+5=0 \longrightarrow \lambda_1 = -2+i,\ \lambda_2 = -2-i$

Sajátvektorok:

$\begin{pmatrix} -2-(-2+i) & 1 \\ -1 & -2-(-2+i) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0 \longrightarrow x = -iy \longrightarrow v_1 = \begin{pmatrix} -i \\ 1 \end{pmatrix}$

$\begin{pmatrix} -2-(-2-i) & 1 \\ -1 & -2-(-2-i) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0 \longrightarrow x = iy \longrightarrow v_2 = \begin{pmatrix} i \\ 1 \end{pmatrix}$

$$y_{\text{ált}} = c_1 e^{(-2+i)x}\begin{pmatrix} -i \\ 1 \end{pmatrix} + c_2 e^{(-2-i)x}\begin{pmatrix} i \\ 1 \end{pmatrix}$$

Kezdeti érték: $\begin{pmatrix} 3 \\ 4 \end{pmatrix} = c_1\begin{pmatrix} -i \\ 1 \end{pmatrix} + c_2\begin{pmatrix} i \\ 1 \end{pmatrix} \longrightarrow c_1 = 2+\frac{3}{2}i,\ c_2 = 2-\frac{3}{2}i$

$$y_{\text{part}} = \left(2+\frac{3}{2}i\right)e^{(-2+i)x} + \left(2-\frac{3}{2}\right)i\, e^{(-2-i)x}$$

$$= e^{-2x}\left(4\cos(2x) - 3\sin(2x)\right)$$

⑩ a) $y' = y^2-4$. Fixpontok: $y^2-4=0 \longrightarrow y_1=-2,\ y_2=2$

Linearizált egyenlet a fixpontok körül:

① $(y-(-2))' = (\Delta y)' = \underbrace{2\cdot(-2)}_{f'(-2)}\Delta y$ ② $(y-2)' = (\Delta y)' = \underbrace{2\cdot 2}_{f'(2)}\Delta y$

$y' = f(y) = y^2-4$

b) $\frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} (y_1-2)(y_2-3) \\ (y_1-4)(y_2-5) \end{pmatrix} = \begin{pmatrix} f_1(y_1,y_2) \\ f_2(y_1,y_2) \end{pmatrix}$

$$F(y_1,y_2) = \begin{pmatrix} \frac{\partial f_1}{\partial y_1} & \frac{\partial f_1}{\partial y_2} \\ \frac{\partial f_2}{\partial y_1} & \frac{\partial f_2}{\partial y_2} \end{pmatrix} = \begin{pmatrix} y_2-3 & y_1-2 \\ y_2-5 & y_1-4 \end{pmatrix}$$

Fixpontok: $\left.\begin{matrix} (y_1-2)(y_2-3)=0 \\ (y_1-4)(y_2-5)=0 \end{matrix}\right\} \to$ a) $\begin{matrix} y_1=2 \\ y_2=5 \end{matrix}$ vagy b) $\begin{matrix} y_1=4 \\ y_2=3 \end{matrix}$

a) $F(2,5) = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$ b) $F(4,3) = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$

$$\frac{d}{dx}\begin{pmatrix} y_1-2 \\ y_2-5 \end{pmatrix} = \frac{d}{dx}\begin{pmatrix} \Delta y_1 \\ \Delta y_2 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} \Delta y_1 \\ \Delta y_2 \end{pmatrix}$$

$$\frac{d}{dx}\begin{pmatrix} y_1-4 \\ y_2-3 \end{pmatrix} = \frac{d}{dx}\begin{pmatrix} \Delta y_1 \\ \Delta y_2 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}\begin{pmatrix} \Delta y_1 \\ \Delta y_2 \end{pmatrix}$$