LevPZh2, Diff.Egy., 2018.05.30. NEPTUN: :

Név: Aláírás:

1.(2+2+4+2 pont)

1a. Mi az $y'(t) = 3 - 4\delta(t-1)$, $y(0) = 3$ DE megoldasa?

$$y' = 3 - 4\delta(t-1) \longrightarrow y(t) = 3t - 4H(t-1) + C \quad , \text{ahol } H(t-1) = \begin{cases} 0, & \text{ha } t < 1 \\ 1, & \text{ha } t > 1 \end{cases}$$

$$y(0) = 3 = 3\cdot 0 - 4H(0-1) + C = C \rightarrow C = 3$$

$$y(t) = 3t - 4H(t-1) + 3$$

1b. Legyen $f(t) = t$ es $g(t) = t^2 - 1$. Mennyi a $h = f * g$ fuggveny $H(s)$ Laplace transzformaltja? ($\mathcal{L}(t^n) = n!/s^{n+1}$)

$$\mathcal{L}(f) = F(s) = \frac{1!}{s^2} \quad , \quad \mathcal{L}(g) = G(s) = \frac{2!}{s^3} - \frac{1}{s}$$

$$\mathcal{L}(f * g) = \mathcal{L}(f)\cdot\mathcal{L}(g) = \frac{1}{s^2}\cdot\left(\frac{2}{s^3} - \frac{1}{s}\right) = \frac{2}{s^5} - \frac{1}{s^3}$$

1c1. Mi a $G''(t) - 4G(t) = \delta(t)$ DE retardalt fundamentalis megoldasa?

$t < 0$: $G(t) = 0$

$t > 0$: $G''(t) - 4G(t) = 0 \longrightarrow G(t) = C_1 e^{2t} + C_2 e^{-2t}$, mivel $\lambda^2 - 4 = 0 \rightarrow$
$\rightarrow \lambda_{1,2} = \pm 2$

$t \approx 0$: $G(0^+) - \underbrace{G(0^-)}_{=0} = 0$, $G'(0^+) - \underbrace{G'(0^-)}_{=0} = 1$

$$\text{tehát } G(0^+) = 0,\ G'(0^+) = 1 \longrightarrow \left.\begin{array}{l} C_1 + C_2 = 0 \\ 2C_1 - 2C_2 = 1 \end{array}\right\} \longrightarrow C_1 = -C_2 = \frac{1}{4}$$

$$G(t) = \begin{cases} 0 & , \text{ha } t < 0 \\ \frac{1}{4}e^{2t} - \frac{1}{4}e^{-2t} & , \text{ha } t > 0 \end{cases}$$

1c2. Mi az $y''(t) - 4y(t) = f(t)$, $y(t) = f(t) = 0$, ha $t << 0$ DE megoldasa?

$$y(t) = \int_{-\infty}^{\infty} G(t-s) f(s)\, ds = \int_{-\infty}^{t} \frac{1}{4}\left(e^{2(t-s)} - e^{-2(t-s)}\right) f(s)\, ds$$

$3+2+3+2$ pont

2. (~~4+2+4 pont~~) Legyen

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 2y_1+3y_2 \\ 3y_1+2y_2 \end{pmatrix} = A\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \qquad \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{illetve} \quad \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

2a) Keresd meg A sajatertekeit es sajatvektorait!

$$0 = \det(A-\lambda E) = \begin{vmatrix} 2-\lambda & 3 \\ 3 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 - 3^2 = \lambda^2 - 4\lambda - 5 \longrightarrow \lambda_1 = 5, \ \lambda_2 = -1$$

$\lambda_1 = 5$:

$$\begin{bmatrix} 2-5 & 3 \\ 3 & 2-5 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-3u+3v=0 \longrightarrow u=v$$

$$\overline{V}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$\lambda_2 = -1$

$$\begin{bmatrix} 2-(-1) & 3 \\ 3 & 2-(-1) \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$3u+3v=0 \longrightarrow u=-v$$

$$\overline{V}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

2b) Ird fel a DE altalanos megoldasat!

$$\overline{y}_{ált}(t) = C_1 e^{5t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^{-1\cdot t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

2c) Szamold ki a DE partikularis megoldasait!

$\overline{y}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = C_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\left.\begin{aligned} 1 &= C_1 + C_2 \\ 0 &= C_1 - C_2 \end{aligned}\right\} \longrightarrow C_1 = C_2 = \frac{1}{2}$$

$$\overline{y}^{I}_{part}(t) = \frac{1}{2}e^{5t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{2}e^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} e^{5t}+e^{-t} \\ e^{5t}-e^{-t} \end{pmatrix}$$

$\overline{y}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = C_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\left.\begin{aligned} 0 &= C_1 + C_2 \\ 1 &= C_1 - C_2 \end{aligned}\right\} \longrightarrow C_1 = \frac{1}{2}, \ C_2 = -\frac{1}{2}$$

$$\overline{y}^{II}_{part}(t) = \frac{1}{2}e^{5t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{1}{2}e^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} e^{5t}-e^{t} \\ e^{5t}+e^{t} \end{pmatrix}$$

2d) Mennyi e^{tA} ?

$$e^{tA} = \begin{pmatrix} | & | \\ \overline{y}^{I}_{part}(t) & \overline{y}^{II}_{part}(t) \\ | & | \end{pmatrix} = \frac{1}{2}\begin{pmatrix} e^{5t}+e^{-t} & e^{5t}-e^{t} \\ e^{5t}-e^{-t} & e^{5t}+e^{-t} \end{pmatrix}$$

vagy

$$e^{tA} = \begin{pmatrix} \boxed{\begin{matrix} 1 \\ 1 \end{matrix}} & \boxed{\begin{matrix} 1 \\ -1 \end{matrix}} \end{pmatrix}\begin{pmatrix} e^{5t} & 0 \\ 0 & e^{-t} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1}$$

$\overline{V}_1$ $\overline{V}_2$

2 + 2 + 4 + 2 pont

2e) Mi az $\frac{d}{dt}\bar{y}(t) = A\bar{y}(t) + \bar{f}(t)$, $\bar{y}(t) = \bar{f}(t) = 0$, ha $t << 0$, DE megoldasa?

$$\bar{y}(t) = \int_{-\infty}^{t} e^{(t-s)A} \bar{f}(s)\, ds =$$

$$= \int_{-\infty}^{t} \frac{1}{2} \begin{bmatrix} e^{5(t-s)} + e^{-(t-s)} & e^{5(t-s)} - e^{-(t-s)} \\ e^{5(t-s)} - e^{-(t-s)} & e^{5(t-s)} + e^{-(t-s)} \end{bmatrix} \begin{bmatrix} f_1(s) \\ f_2(s) \end{bmatrix} ds$$

3. Legyen

$$\partial_t \phi(t,x) = \partial_x^2 \phi(t,x), \quad \phi(t, x+2\pi) = \phi(t,x), \quad \phi(0,x) = f(x),$$

ahol $f(x) = 2$, ha $x \in [0,\pi]$, amugy -2 az intervallum tobbi reszen.

3a. Ird fel egy ortonormalt bazist $L^2([-\pi,\pi], dx)$-nek!

$$\vec{e}_n = e_n(x) = \frac{e^{inx}}{\sqrt{2\pi}}, \quad n \in \mathbb{Z}$$

3b. Szamold ki f ezen bazis szerinti kifejteset!

$$\vec{f} = \sum_{n \in \mathbb{Z}} \hat{f}_n \vec{e}_n = \sum_{n \in \mathbb{Z}} \hat{f}_n \cdot \frac{e^{inx}}{\sqrt{2\pi}}, \quad \text{ahol}$$

$$\hat{f}_n = (\vec{e}_n, \vec{f}) = \int_{-\pi}^{\pi} \frac{e^{-inx}}{\sqrt{2\pi}} \cdot f(x)\, dx = \int_{-\pi}^{0} \frac{e^{-inx}}{\sqrt{2\pi}} \cdot (-2)\, dx + \int_{0}^{\pi} \frac{e^{-inx}}{\sqrt{2\pi}} \cdot 2\, dx$$

$$= \frac{-2}{\sqrt{2\pi}} \left[\frac{e^{-inx}}{-in} \right]_{-\pi}^{0} + \frac{2}{\sqrt{2\pi}} \left[\frac{e^{-inx}}{-in} \right]_{0}^{\pi} = \sqrt{\frac{2}{\pi}} \cdot \frac{i}{n} \left[-\left(1 - e^{in\pi}\right) + \left(e^{-in\pi} - 1\right) \right]$$

$$\left(e^{in\pi} = \begin{cases} +1, \text{ ha } n \text{ páros} \\ -1, \text{ ha } n \text{ páratlan} \end{cases}, \quad \text{így } \hat{f}_n = \begin{cases} 0, \text{ ha } n \text{ páros} \\ \sqrt{\frac{2}{\pi}} \cdot \frac{i}{n} \cdot (-4) \end{cases} \right)$$

3c. Mennyi $\phi(t,x)$? Hasznalj Fourier sort ϕ kifejezesere!

$$\varphi(t,x) = \sum_{n \in \mathbb{Z}} \hat{f}_n \cdot e^{-n^2 t} \vec{e}_n = \sum_{n \in \mathbb{Z}} \hat{f}_n \cdot e^{-n^2 t} \frac{e^{inx}}{\sqrt{2\pi}},$$

mivel $\partial_x^2 e_n(x) = \partial_x^2 \frac{e^{inx}}{\sqrt{2\pi}} = (in)^2 \frac{e^{inx}}{\sqrt{2\pi}} = -n^2 \frac{e^{inx}}{\sqrt{2\pi}}$

4. (2+2 pont)

4a. a) Szamitsd ki a Laplace tr. definicioja alapjan: $F(s) = \mathcal{L}(f(t)) = \mathcal{L}(\cos(-3t+2))$.

$F(s)=$ $$\int_0^\infty e^{-st}\cdot\frac{e^{i(-3t+2)}+e^{-i(-3t+2)}}{2}\,dt = \int_0^\infty \frac{e^{2i}}{2}e^{-(s+3i)t}+\frac{e^{-2i}}{2}e^{-(s-3i)t}\,dt$$

$$= \frac{e^{2i}}{2}\left[\frac{e^{-(s+3i)t}}{-(s+3i)}\right]_{t=0}^{\infty}+\frac{e^{2i}}{2}\left[\frac{e^{-(s-3i)t}}{-(s-3i)}\right]_{t=0}^{\infty} =$$

$$= \frac{e^{2i}}{2}\cdot\frac{1}{s+3i}+\frac{e^{-2i}}{2}\cdot\frac{1}{s-3i}$$

Legyen $f(t) = t$. Mennyi $(f\star f)(t)$?

$$(f*f)(t) = \int_0^t f(t-\tau)\cdot f(\tau)\,d\tau = \int_0^t (t-\tau)\tau\,d\tau = \int_0^t t\tau-\tau^2\,d\tau$$

$$= \left[t\frac{\tau^2}{2}\right]_{\tau=0}^{t}-\left[\frac{\tau^3}{3}\right]_{\tau=0}^{t} = \frac{t^3}{2}-\frac{t^3}{3} = \frac{1}{6}t^3$$

4b. (2+4 pont)

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} (y_1-1)(y_2-2) \\ (y_1-3)(y_2-4). \end{pmatrix}$$

Keresd meg a DE fixpontjat!

$(y_1-1)(y_2-2)=0 \longrightarrow y_1=1$, vagy $y_2=2$

$[y_1=1$ és $(1-3)(y_2-4)=0] \to y_2=4$ tehát $P_1=\begin{pmatrix}1\\4\end{pmatrix}$, $P_2=\begin{pmatrix}3\\2\end{pmatrix}$

$[y_2=2$ és $(y_1-3)(2-4)=0] \longrightarrow y_1=3$

Ird fel a fixpont koruli linearizalt kozelito DE-t!

$$\mathrm{Jac} = \begin{bmatrix} \partial_{y_1}[(y_1-1)(y_2-2)] & \partial_{y_2}[(y_1-1)(y_2-3)] \\ \partial_{y_1}[(y_1-3)(y_2-4)] & \partial_{y_2}[(y_1-3)(y_2-4)] \end{bmatrix} = \begin{pmatrix} y_2-2 & y_1-1 \\ y_2-4 & y_1-3 \end{pmatrix}$$

$$\mathrm{Jac}(P_1) = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} \qquad \mathrm{Jac}(P_2) = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$$

Linearizált DE:

$$\frac{d}{dt}\begin{pmatrix} y_1-1 \\ y_2-4 \end{pmatrix} = \frac{d}{dt}\begin{pmatrix} \Delta y_1 \\ \Delta y_2 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} \Delta y_1 \\ \Delta y_2 \end{pmatrix} \qquad \frac{d}{dt}\begin{pmatrix} y_1-3 \\ y_2-2 \end{pmatrix} = \frac{d}{dt}\begin{pmatrix} \Delta y_1 \\ \Delta y_2 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}\begin{pmatrix} \Delta y_1 \\ \Delta y_2 \end{pmatrix}$$