Differencialegyenletek. I. Feladatsor

1. Ismetlo feldatok.
   I.) Szamold ki a kovetkezo fuggvenyek hatarozatlan integraljait!

$$a)\ \ x\ln(3x), \qquad b)\ \ \sin(3x)\sqrt{\cos(3x)}, \qquad c)\ \ \frac{1}{(x-3)x}$$

Megoldas:

$$\begin{aligned} a)&\ \ \frac{1}{2}x^2\log(3x)-\frac{x^2}{4}+C\\ b)&\ \ -\frac{2}{9}\cos^{\frac{3}{2}}(3x)+C\\ c)&\ \ \frac{1}{3}(\log(x-3)-\log(x))+C \end{aligned}$$

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II.) Ird fel a kovetkezo fuggvenyek Taylor sorat az $x = x_0$ pont korul!

$$a)\ \ e^{3x},\ x_0=0; \qquad b)\ \ \sin(3x),\ x_0=0; \qquad c)\ \ \log(x),\ x_0=1; \qquad d)\ \ \frac{1}{1-x},\ x_0=0; \qquad e)\ \ \frac{1}{x^2+1},\ x_0=0.$$

Megoldas:

$$\begin{aligned} a)&\ \ 1+3x+\frac{3^2x^2}{2!}+\frac{3^3x^3}{3!}+O\left(x^4\right)=1+3x+\frac{9x^2}{2}+\frac{9x^3}{2}+O\left(x^4\right)\\ b)&\ \ 3x-\frac{3^3x^3}{3!}+\frac{3^5x^5}{5!}-\frac{3^7x^7}{7!}+O\left(x^9\right)=3x-\frac{9x^3}{2}+\frac{81x^5}{40}-\frac{243x^7}{560}+O\left(x^9\right)\\ c)&\ \ (x-1)-\frac{1}{2}(x-1)^2+\frac{1}{3}(x-1)^3-\frac{1}{4}(x-1)^4+O\left((x-1)^5\right)\\ d)&\ \ 1+x+x^2+x^3+x^4+O\left(x^5\right)\\ e)&\ \ 1-x^2+x^4-x^6+O\left(x^7\right) \end{aligned}$$

....

III.) Legyen $f(x)$

$$a)\ \ e^{x+y^2}, \qquad b)\ \ x\sin(y^2).$$

Mennyi $f'_x, f'_y, f''_{xx}, f''_{xy}, f''_{yx}, f''_{yy}$? Mennyi $\frac{d}{dx}f(x,\ln(x))$?

Megoldas:

$$a)\ \ e^{x+y^2}, 2ye^{x+y^2}, e^{x+y^2}, 2ye^{x+y^2}, 2ye^{x+y^2}, 4y^2e^{x+y^2}+2e^{x+y^2}, e^{x+\log^2(x)}\left(\frac{2\log(x)}{x}+1\right)$$

$$b)\ \ \sin\left(y^2\right), 2xy\cos\left(y^2\right), 0, 2y\cos\left(y^2\right), 2y\cos\left(y^2\right), 2x\cos\left(y^2\right)-4xy^2\sin\left(y^2\right), \sin\left(\log^2(x)\right)+2\log(x)\cos\left(\log^2(x)\right)$$

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2. Ird at a kovetkezo DE-ket idofuggetlen DE rendszerre!

$$a)\ \ y'=xy^2+x; \qquad b)\ \ y'=x-y; \qquad c)\ \ \frac{d}{dx}\begin{pmatrix}y_1\\ y_2\end{pmatrix}=\begin{pmatrix}xy_1+y_2\\ y_1y_2+x\end{pmatrix}$$

Megoldas:

$$a)\ \ \frac{d}{dx}\begin{pmatrix}y\\ t\end{pmatrix}=\begin{pmatrix}ty^2+t\\ 1\end{pmatrix}; \qquad a)\ \ \frac{d}{dx}\begin{pmatrix}y\\ t\end{pmatrix}=\begin{pmatrix}t-y\\ 1\end{pmatrix}; \qquad c)\ \ \frac{d}{dx}\begin{pmatrix}y_1\\ y_2\\ t\end{pmatrix}=\begin{pmatrix}ty_1+y_2\\ y_1y_2+t\\ 1\end{pmatrix}$$

;

3. Ird at a kovetkezo egyenleteket elsorende DE rendszerre!

$$a)\ \ y'' = -y' - 2y; \qquad b)\ \ y''' = y + x; \qquad c)\ \ \frac{d^2}{dx^2}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} y_1' - y_2 \\ y_2' y_1 \end{pmatrix}$$

Megoldas:

$$a)\ \ \frac{d}{dx}\begin{pmatrix} y \\ v \end{pmatrix} = \begin{pmatrix} v \\ -v - 2y \end{pmatrix}; \qquad b)\ \ \frac{d}{dx}\begin{pmatrix} y \\ v \\ a \end{pmatrix} = \begin{pmatrix} v \\ a \\ y + x \end{pmatrix}; \qquad c)\ \ \frac{d}{dx}\begin{pmatrix} y_1 \\ v_1 \\ y_2 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_1 - y_2 \\ v_2 \\ v_2 y_1 \end{pmatrix}$$

4.

$$a)\ \ y' = f(x,y) = x - y; \qquad b)\ \ y' = f(x,y) = y^2 + yx;$$

Mennyi y'' es y''' ? Ird fel y harmadrendu Talor polinomjat az $x = 0$ pont korul, ha $y(0) = 5$!

Megoldas:

$$y'' = \left(\frac{\partial}{\partial x} + f\frac{\partial}{\partial y}\right) f; \qquad y''' = \left(\frac{\partial}{\partial x} + f\frac{\partial}{\partial y}\right)^2 f,$$

tehat

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$$a)\ \ y'' = -x + y + 1, \quad y''' = x - y - 1,$$
$$y(0) = 5,\ \ y'(0) = f(0,5) = 0 - 5,\ \ y''(0) = -0 + 5 + 1 = 6,\ \ y'''(0) = 0 - 5 - 1 = -6,$$
$$y(x) \approx 5 - 5x + \frac{6}{2!}x^2 + \frac{-6}{3!}x^3$$
$$b)\ \ y'' = 2xy + 2y^3 + 1, \quad y''' = 2x^2 + 8xy^2 + 6y^4 + 2y,$$
$$y(0) = 5,\ \ y'(0) = f(0,5) = 25,\ \ y''(0) = 251,\ \ y'''(0) = 3760,$$
$$y(x) \approx 5 + 25x + \frac{251}{2!}x^2 + \frac{3760}{3!}x^3$$

5.

$$a)\ \ f(x) = \sin x,\ \ x_0 = \pi/2; \qquad b)\ \ f(x) = \sqrt{x},\ \ x_0 = 9; \qquad c)\ \ f(x) = 1/x,\ \ x_0 = 2;$$

Ird fel f-nek a linearis $f(x_0 + \Delta x) \approx T_1(x_0 + \Delta x)$ kozeliteset, ha $\Delta x = 0.1$! Mennyi $\max_{z\in[x_0, x_0+\Delta x]}|f''(z)|$? Adj felso korlatot a kozelites $|\text{hiba}(\Delta x)| = |f(x_0 + \Delta x) - T_1(x_0 + \Delta x)|$ hibajara!

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Megoldas:

$$a)\ \ \sin(\pi/2 + 0.1) = \sin(\pi/2) + \cos(\pi/2)\cdot 0.1 + \text{hiba}(0.1)$$
$$|\text{hiba}(0.1)| \le \frac{1}{2}\Delta x^2 \max_{z\in[x_0, x_0+\Delta x]}|f''(z)| = \frac{1}{2}0.1^2 \max_{z\in[\pi/2, \pi/2+0.1]}|\cos(z)| \le \frac{1}{2}0.1^2 \cdot 1$$

$$c)\ \ 1/(2 + 0.1) = 1/2 - 1/2^2 \cdot 0.1 + \text{hiba}(0.1)$$
$$|\text{hiba}(0.1)| \le \frac{1}{2}\Delta x^2 \max_{z\in[x_0, x_0+\Delta x]}|f''(z)| = \frac{1}{2}0.1^2 \max_{z\in[2, 2+0.1]}|-1/z^2| = \frac{1}{2}0.1^2 \cdot 1/4$$

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6. Alkalmazd az Euler, illetve a Heun modszert a kovetkezo DE-ekre $\Delta x = 0.1$ lepeskozzel az $y(2) = 3$ kezdeti feltetel mellett!

$$a)\ \ y' = f(x, y) = x - y; \qquad b)\ \ y' = x - y^2;$$

Vegezd el ugyanezt az

$$c)\ \ \frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} y_2 \\ -y_1 \end{pmatrix}; \qquad d)\ \ \frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} y_1 - y_2 \\ y_1^2 + x \end{pmatrix}$$

egyenletekre az $\begin{pmatrix} y_1(2) \\ y_2(2) \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ kezdeti feltetel mellett!

Mit josolnak ezek a modszerek y(2.1)-re?

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Megoldas:

$$a)\quad Euler:\ \ y(2.1) \approx y(2) + (2-3)\cdot 0.1 = 3 + (-1)\cdot 0.1 = 2.9,$$
$$y(2.2) \approx y(2.1) + (2.1 - 2.9)\cdot 0.1$$
$$Heun:\ \ k_1 = f(2,3) = 2 - 3 = -1, \quad k_2 = f(2 + 0.1, 3 + f(2,3)\cdot 0.1) = 2.1 - 2.9 = -0.8,$$
$$y(2.1) \approx y(2) + \frac{1}{2}(k_1 + k_2)\cdot 0.1 = 3 + \frac{1}{2}(-1 - 0.8)\cdot 0.1.$$

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7. Oldd meg a kovetkezo DE-ket az $y(0) = 1$ kezdeti feltetel mellett, vizsgald meg a megoldasok egyertelmuseget, illetve hatarozd meg, hogy milyen intervalumon vannak azok ertelmezve! Magyarazd meg a felmerulo problemakat!

$$a)\ \ y' = y, \qquad b)\ \ y' = y^2, \qquad c)\ \ y' = y^{11/10}, \qquad d)\ \ y' = \sqrt{|y|},\quad y \ge 0 \quad e)\ \ y' = |y|^{9/10},$$

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Megoldas:

$$a)\ \ y(x) = e^x$$

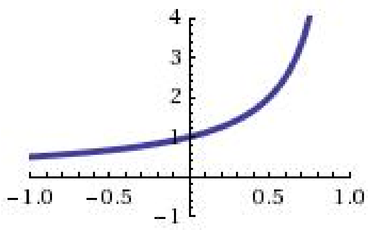
$$b)\ \ \frac{dy}{dx} = y^2, \quad \Longrightarrow \quad \frac{dy}{y^2} = dx, \quad \Longrightarrow \quad \int \frac{dy}{y^2} = \int dx, \quad \Longrightarrow \quad -\frac{1}{y} = x + C,$$
$$\left(y(0) = 1 \quad \Longrightarrow \quad -\frac{1}{1} = 0 + C, \quad \Longrightarrow \quad C = -1\right), \quad \Longrightarrow \quad y(x) = \frac{1}{1-x}$$

$$c)\ \ y(x) = \frac{1\times 10^{10}}{(x-10)^{10}}$$

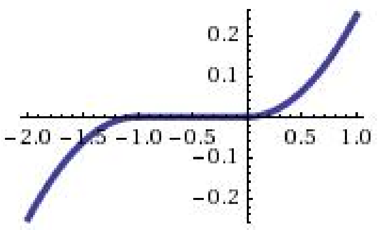
$$d)\ \ y(x) = \begin{cases} \frac{1}{4}(x+2)^2, & ha\ x > -2, \\ 0, & ha\ C \le x \le -2 \\ -\frac{1}{4}(x+C)^2, & ha\ x \le C \end{cases}$$

(ahol $C \le -2$ valamilyen tetszoleges allando).

b)



d)



A megoldas az a) es a d) estekben $(-\infty, \infty)$-en ertelmezett, b) esetben $(-\infty, 1)$-en ertelmezett, mig a c) esetben $(-\infty, 10)$-en.
A megoldas a d) esetben nem egyertelmu, mivel $y = 0$ eseten nem teljesul a lokalis Lipsitz feltetel.
(Pl. ha $y > 0$, akkor $\frac{d}{dy}\sqrt{|y|} = \frac{1}{2\sqrt{y}} \to \infty$, ha $y \to 0^+$.)

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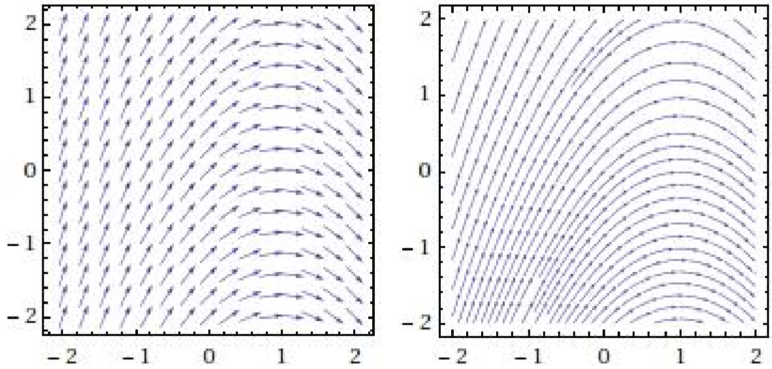
8. Rajzold le az $y' = f(x)$ DE iranymezojet es a megoldasgorbeit!

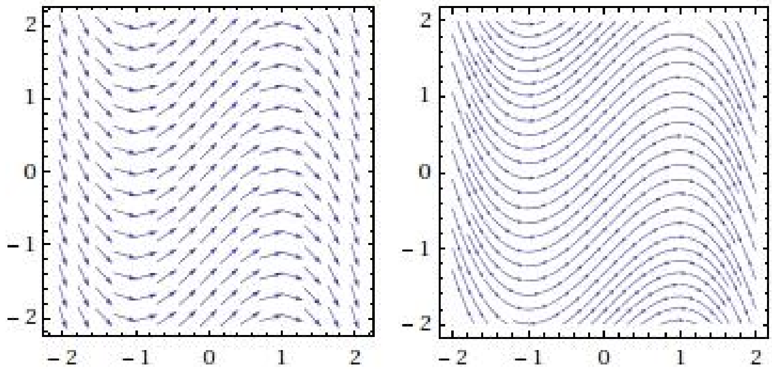
$$a)\ y' = 1, \qquad b)\ y' = x, \qquad c)\ y' = 1 - x, \qquad d)\ y' = x^2, \qquad e)\ y' = 1 - x^2,$$

Megoldas:

c)



e)



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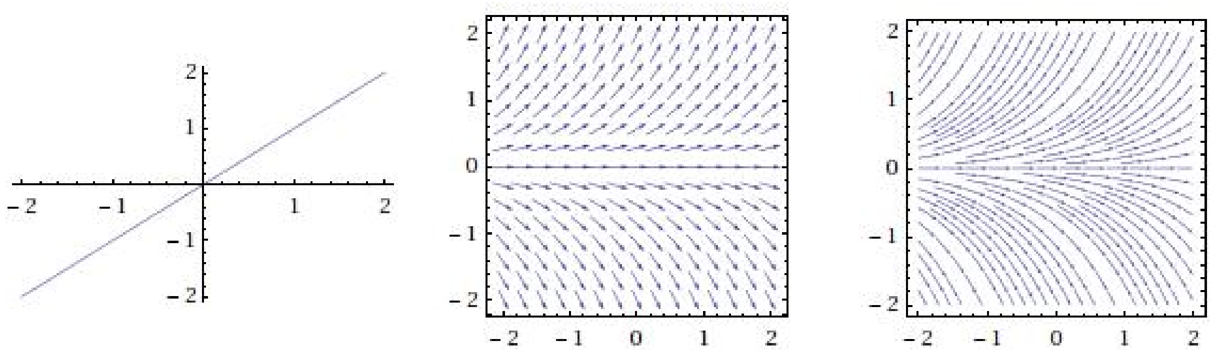
9. Rajzold le az $y' = f(y)$ DE iranymezojet es a megoldasgorbeit! Keresd meg a DE fixpontjait es ird fel a fixponttol valo elteresre vonatkozo DE linearizalt alakjat! Vizsgald meg a fixpontok stabilitasat!

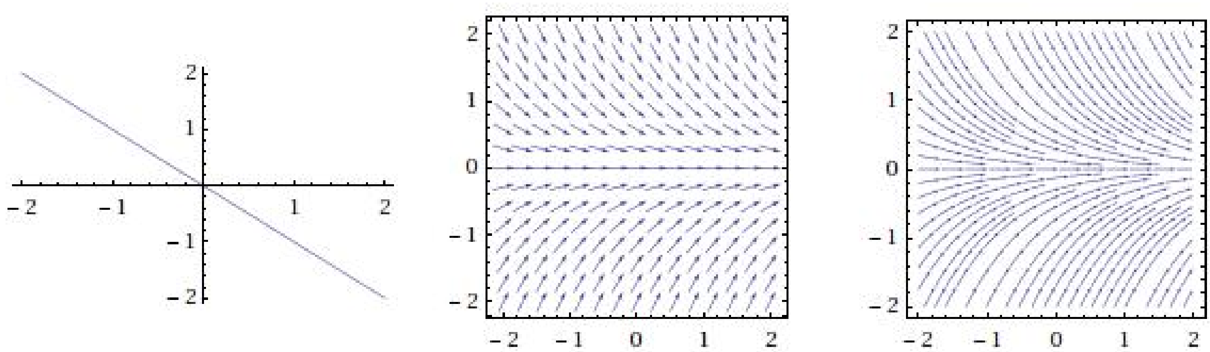
$$a)\ y' = 1, \qquad b)\ y' = y, \qquad c)\ y' = -y, \qquad d)\ y' = y + 1,$$
$$e)\ y' = -1 + y^2, \qquad f)\ y' = y(1 - y), \qquad g)\ y' = y(1 - y)(1 + y).$$

Megoldas:

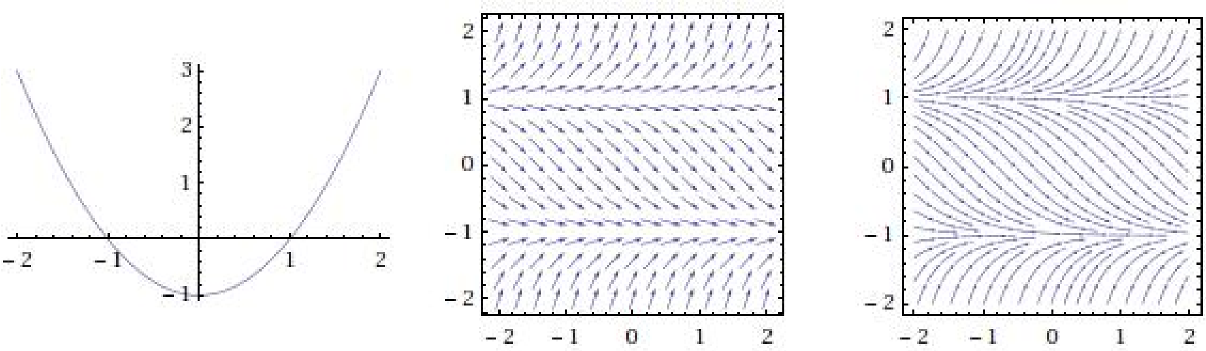
b) $y' = y$



c) $y' = -y$



e) $y' = y^2 - 1$



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g) $y' = f(y) = y(1-y)(1+y) = +y - y^3$
$f'(y) = \frac{d}{dy} f(y) = 1 - 3y^2$. A fixpontok:

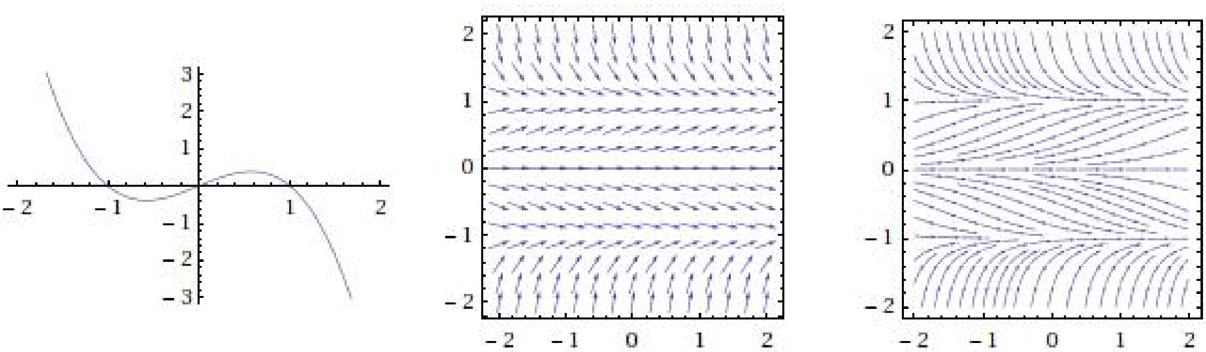
$$y_1 = 0, \qquad\qquad y_2 = 1, \qquad\qquad y_3 = -1,$$

$$f'(y_1) = f'(0) = 1 - 3 \cdot 0^2 = 1 > 0, \qquad f'(1) = -2 < 0, \qquad f'(-1) = -2 < 0.$$

f' elojele alapjan az $y_1,\ y_2,\ y_3$ fixpontok stabilitasa: *instabil, stabil, stabil.*

A linearizalt egyenletek a fixpontok korul:

$$\frac{d}{dx}(y-0) = \frac{d}{dx}\Delta y_1 = 1 \cdot \Delta y_1, \qquad \frac{d}{dx}(y-1) = \frac{d}{dx}\Delta y_2 = -2 \cdot \Delta y_2, \qquad \frac{d}{dx}(y-(-1)) = \frac{d}{dx}\Delta y_3 = -2 \cdot \Delta y_3,$$



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10. Keresd mag az A matrix sajatertekeit es sajatvektorait! Keresd meg azt as S hasonlolosagi transzformaciot, ami dagonalizalja A-t, vagyis $D = S^{-1}AS$, ahol D diagonalis! Ird fel a v vektort a sajatvektorok linearis kombinaciojakent! Mennyi $A^{13}v$?

$$a)\ \ (7) \qquad b)\ \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \qquad c)\ \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix} \qquad d)\ \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix} \qquad e)\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad f)\ \begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix}$$

$$g)\ \begin{pmatrix} 2 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 7 \end{pmatrix} \qquad h)\ \begin{pmatrix} 2 & -3 & 0 \\ 3 & 2 & 0 \\ 0 & 0 & 7 \end{pmatrix}$$

Itt a v vektor erteke:

$$a)\ \ v = (8); \qquad b-f)\ \ v = \begin{pmatrix} 3 \\ 4 \end{pmatrix}; \qquad g-h)\ \ v = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

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Megoldas:

b)

$$A = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$$

Sajatertekek: $\lambda_1 = 3, \qquad \lambda_2 = 2$,
Sajatvektorok:

$$v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad\qquad v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$D = S^{-1}AS = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$$

Mivel A eleve diagonalis volt, ez a feladat trivialis, a sajatertekek a diagonalis elemek, a sajatvektorok pedig a standard bazis.

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d)

$$A = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}$$

Sajatertekek egyenlete:

$$\det(A - \lambda E) = \begin{vmatrix} 2-\lambda & 1 \\ 0 & 3-\lambda \end{vmatrix} = (2-\lambda)(3-\lambda) - 1 \cdot 0 = 0$$

Sajatertekek: $\lambda_1 = 3, \qquad \lambda_2 = 2$.

Sajatvektorok egyenlete ($\lambda_1 = 3$-ra):

$$\begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 3 \begin{pmatrix} x \\ y \end{pmatrix}, \qquad \text{vagy} \qquad \begin{pmatrix} 2-3 & 1 \\ 0 & 3-3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Innen $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ x \end{pmatrix}$. Ezek kozul valasztunk egy nem nulla vektort, pl. legyen $x = 1$.
Sajatvektorok:

$$v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Az A matrixot diagonalizalo hasonlosagi transzformacio:

$$D = S^{-1}AS = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$$

Itt S a v_1 es a v_2 oszlopvektorokbol allo matrix.

Mennyi $A^{13}v$?

$$A^{13}v = (SDS^{-1})^{13}v = SD^{13}S^{-1}v = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 3^{13} & 0 \\ 0 & 2^{13} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

vagy

$$\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \alpha v_1 + \beta v_2,$$

ahol

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = S^{-1} \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \end{pmatrix},$$

tehat

$$A^{13}v = A^{13}(\alpha v_1 + \beta v_2) = \alpha\lambda_1^{13}v_1 + \beta\lambda_2^{13}v_2 = 4 \cdot 3^{13} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + (-1) \cdot 2^{13} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

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f)

$$A = \begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix}$$

Sajatertekek: $\lambda_1 = 2 + 3i, \qquad \lambda_2 = 2 - 3i$,
Sajatvektorok:

$$v_1 = \begin{pmatrix} i \\ 1 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} -i \\ 1 \end{pmatrix}$$

$$D = S^{-1}AS = \begin{pmatrix} -\frac{i}{2} & \frac{1}{2} \\ \frac{i}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2+3i & 0 \\ 0 & 2-3i \end{pmatrix}$$

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g) Mivel A a d) es az a) blokkok kombinacioja, igy ezen ket feladat eredmenyeit felhasznalva a kovetkezoket kapjuk:

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 7 \end{pmatrix}$$

Sajatertekek: $\lambda_1 = 3, \qquad \lambda_2 = 2, \qquad \lambda_3 = 7, \qquad ,$
Sajatvektorok:

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$D = S^{-1}AS = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 7 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 7 \end{pmatrix}$$

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11. Oldd meg az elozo feladatban szereplo A matrixokra az

$$\frac{d}{dx} y = Ay, \quad y(0) = v$$

DE-t! Ird fel az altalanos, illetve a partikuralis megoldast! Vizsgald meg az $y = 0$ fixpont stabilitasat! Mennyi $\exp(xA)$? Ird fel a partikularis megoldast e^{xA} segitsegevel!

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Megoldas:

d)

$$A = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}$$

Sajatertekek: $\lambda_1 = 3, \quad \lambda_2 = 2$. Sajatvektorok:

$$v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Az altalanos megoldas:

$$y_{alt}(x) = \sum_i C_i e^{\lambda_i x} v_i = C_1 e^{3x} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^{2x} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Ha

$$y(0) = \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} = 4 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + (-1) \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

akkor a partikularis megoldas

$$y_{part}(x) = 4e^{3x} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + (-1)e^{2x} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Mivel mindket sajatertek valos resze pozitiv, igy az $y = 0$ fixpont instabil.
Az xA matrix exponencialis fuggvenye:

$$e^{xA} = e^{xSDS^{-1}} = Se^{xD}S^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} e^{3x} & 0 \\ 0 & e^{2x} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$$

Ennek segitsegevel a partikularis megoldas az

$$y_{part}(x) = e^{xA} \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

alakban irhato fel.

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12. $y'' = -y$. Ird fel a DE karakterisztikus egyenletet, illetve a DE altalanos megoldasat! Ird at a DE-t egy elsorendu DE-rendszerre es oldd meg az elozo feladathoz hasonloan! Hasonlitsd ossze a ket megoldasi modszert!

Megoldas:

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A karakterisztikus egyenlet es annak gyokei:

$$y'' = -y \quad \Longrightarrow \quad \lambda^2 = -1 \quad \Longrightarrow \quad \lambda_1 = 0 + 1 \cdot i, \;\; \lambda_2 = 0 - 1 \cdot i,$$

tehat az altalanos megoldas:

$$y = C_1 e^{(0+i)x} + C_2 e^{(0-i)x} = e^{0 \cdot x} \left(\widetilde{C_1} \cos(1 \cdot x) + \widetilde{C_2} \sin(1 \cdot x) \right)$$

Itt $C_1 = \widetilde{C_1}/2 + \widetilde{C_2}/(2i), \; C_2 = \widetilde{C_1}/2 - \widetilde{C_2}/(2i)$.

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Ugyanez elsorendu DE rendszerkent:

$$\begin{pmatrix} y \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} y \\ v \end{pmatrix} = A \begin{pmatrix} y \\ v \end{pmatrix}$$

A sajatertekei es sajatvektorai:

$$\lambda_1 = i, \;\; v_1 = \begin{pmatrix} 1 \\ i \end{pmatrix}, \qquad \lambda_1 = -i, \;\; v_1 = \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

Tehat az altalanos megoldas:

$$\begin{pmatrix} y \\ v \end{pmatrix} = C_1 e^{ix} \begin{pmatrix} 1 \\ i \end{pmatrix} + C_2 e^{-ix} \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

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13. Keresd mag az A matrix sajatertekeit es sajatvektorait!

$$a) \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad b) \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \qquad c) \begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix} \qquad d) \begin{pmatrix} 7 & 0 & 0 \\ 0 & 2 & 3 \\ 0 & 0 & 2 \end{pmatrix}$$

Mennyi $\exp(xA)$?

. . . .

Megoldas:

c)

Sajatertek:

$$0 = \det(A - \lambda E) = \begin{vmatrix} 2-\lambda & 3 \\ 0 & 2-\lambda \end{vmatrix} \qquad \Longrightarrow \qquad \lambda_1 = 2.$$

Az egyetlen sajatvektor:

$$\begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 2 \begin{pmatrix} x \\ y \end{pmatrix} \qquad \Longrightarrow \qquad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix}, \qquad \text{vagyis} \quad v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$\exp(xA)$:

$$\exp(xA) = \exp\left[\begin{pmatrix} 2x & 0 \\ 0 & 2x \end{pmatrix} + \begin{pmatrix} 0 & 3x \\ 0 & 0 \end{pmatrix} \right] = \exp \begin{pmatrix} 2x & 0 \\ 0 & 2x \end{pmatrix} \cdot \exp \begin{pmatrix} 0 & 3x \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} e^{2x} & 0 \\ 0 & e^{2x} \end{pmatrix} \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 3x \\ 0 & 0 \end{pmatrix} + \frac{1}{2!} \begin{pmatrix} 0 & 3x \\ 0 & 0 \end{pmatrix}^2 + \cdots \right] = \begin{pmatrix} e^{2x} & 0 \\ 0 & e^{2x} \end{pmatrix} \begin{pmatrix} 1 & 3x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e^{2x} & 3xe^{2x} \\ 0 & e^{2x} \end{pmatrix}$$

Itt az elso, $\exp(C + D) = \exp(C) \cdot \exp(D)$ tipusu atalakitast azert lehetett elvegezni, mert esetunkben $[C, D] = CD - DC = 0$ volt:

$$\begin{pmatrix} e^{2x} & 0 \\ 0 & e^{2x} \end{pmatrix} \begin{pmatrix} 0 & 3x \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 3x \\ 0 & 0 \end{pmatrix} \begin{pmatrix} e^{2x} & 0 \\ 0 & e^{2x} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0$$

Az utolso elotti atalakitasnal pedig azt hasznaltuk ki, hogy

$$\begin{pmatrix} 0 & 3x \\ 0 & 0 \end{pmatrix}^2 = (3x)^2 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0$$

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14. Oldd meg az elozo feladatban szereplo A matrixokra az

$$\frac{d}{dx} y = Ay, \;\; y(0) = v$$

DE-t! Ird fel a partikularis megoldast e^{xA} segitsegevel, ha a v vektor erteke:

$$a-c) \;\; v = \begin{pmatrix} 3 \\ 4 \end{pmatrix}; \qquad d) \;\; v = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

. . . .

Megoldas:

c)

$$y_{part}(x) = e^{xA} y(0) = \exp\left[x \begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix} \right] \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} e^{2x} & 3xe^{2x} \\ 0 & e^{2x} \end{pmatrix} \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

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15. Csillapitott harmonikus oszcillator: $y'' = -y - ky'$. Ird fel a DE karakterisztikus egyenletet, es hatarozd meg, hogy milyen k ertek eseten esenk egybe a gyokei! Ird fel a DE altalanos megoldasat!
Ird at a DE-t egy elsorendu renszerre, es vizsgald meg az egyutthatomatrix Jordan dekompoziciojat!

....

Megoldas:

Karakterisztikus egyenlet:

$$y'' = -y - ky' \quad \Longrightarrow \quad \lambda^2 = -1 - k\lambda \quad \Longrightarrow \quad \lambda_{1,2} = \frac{-k \pm \sqrt{k^2 - 4}}{2}$$

Egy gyok van, ha $k = \pm 2$. Mi a $k = 2$ esetet vizsgaljuk, ekkor $\lambda = -1$.
Az altalanos megoldas:

$$y_{alt} = C_1 e^{-x} + C_2 x e^{-x}.$$

Ugyanez elsorendu DE rendszerkent:

$$\begin{pmatrix} y \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} y \\ v \end{pmatrix} = A \begin{pmatrix} y \\ v \end{pmatrix}$$

A sajaterteke, sajatvektora:

$$\lambda = -1, \qquad v_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

A Jordan normalforma:

$$Av_1 = \lambda v_1,$$
$$Av_2 = \lambda v_2 + v_1$$
$$\text{tehat}$$
$$v_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \qquad S = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix},$$
$$J = S^{-1} A S = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$$

Ennek alapjan

$$\exp(xA) = \exp(xSJS^{-1}) = S\exp(xJ)S^{-1} = S \begin{pmatrix} e^{-x} & xe^{-x} \\ 0 & e^{-x} \end{pmatrix} S^{-1}$$

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16. $y'' = y - y^3$. Legyen $p = y'$. Mutasd meg, hogy a DE a kovetkezo (Hamilton fele) alakban is felirhato:

$$y' = \frac{\partial H}{\partial p}, \qquad p' = -\frac{\partial H}{\partial y}.$$

Mennyi H ? Mutasd meg, hogy $H' = 0$!

Ird at a DE-t egy elsorendu DE rendszerre, keresd meg annak a fixpontjait, ird fel a fixpontoktol valo elteresre vonatkozo liearizalt DE-ket! Vizsgald meg a fixpontok stabilitasat!

....

Megoldas:

$$y' = p = \frac{\partial H}{\partial p} \quad \Longrightarrow \quad H(y,p) = \frac{p^2}{2} + h(y),$$
$$p' = y'' = y - y^3 = -\frac{\partial H}{\partial y} \quad \Longrightarrow \quad H = \frac{p^2}{2} + \frac{y^4}{4} - \frac{y^2}{2}$$

$$H' = pp' + y^3 y' - yy' = p(y - y^3) + (y - y^3)y' = p(y - y^3) + (y - y^3)p = 0$$

Elsorendu DE-rendszer:

$$\frac{d}{dx}\begin{pmatrix} y \\ p \end{pmatrix} = \begin{pmatrix} p \\ y - y^3 \end{pmatrix}$$

Egyensulyi allapotok (fixpontok):

$$\begin{pmatrix}0\\0\end{pmatrix} = \frac{d}{dx}\begin{pmatrix}y\\p\end{pmatrix} = \begin{pmatrix}p\\y-y^3\end{pmatrix} \qquad \Longrightarrow \qquad y_1 = \begin{pmatrix}0\\0\end{pmatrix}, \qquad y_2 = \begin{pmatrix}1\\0\end{pmatrix}, \qquad y_3 = \begin{pmatrix}-1\\0\end{pmatrix},$$

A Jacobi matrix:

$$J = \begin{pmatrix} \frac{\partial}{\partial y}p & \frac{\partial}{\partial p}p \\ \frac{\partial}{\partial y}(y-y^3) & \frac{\partial}{\partial p}(y-y^3) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1-3y^2 & 0 \end{pmatrix}$$

A Jacobi matrix erteke a fixpontokban:

$$J(y_1) = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}, \qquad J(y_2) = \begin{pmatrix}0 & 1\\-2 & 0\end{pmatrix}, \qquad J(y_3) = \begin{pmatrix}0 & 1\\-2 & 0\end{pmatrix}$$

A matrixok sajatertekei:

$$y_1: \quad (1,-1), \qquad y_2: \quad (0-\sqrt{2}i, 0+\sqrt{2}i), \qquad y_3: \quad (0-\sqrt{2}i, 0+\sqrt{2}i).$$

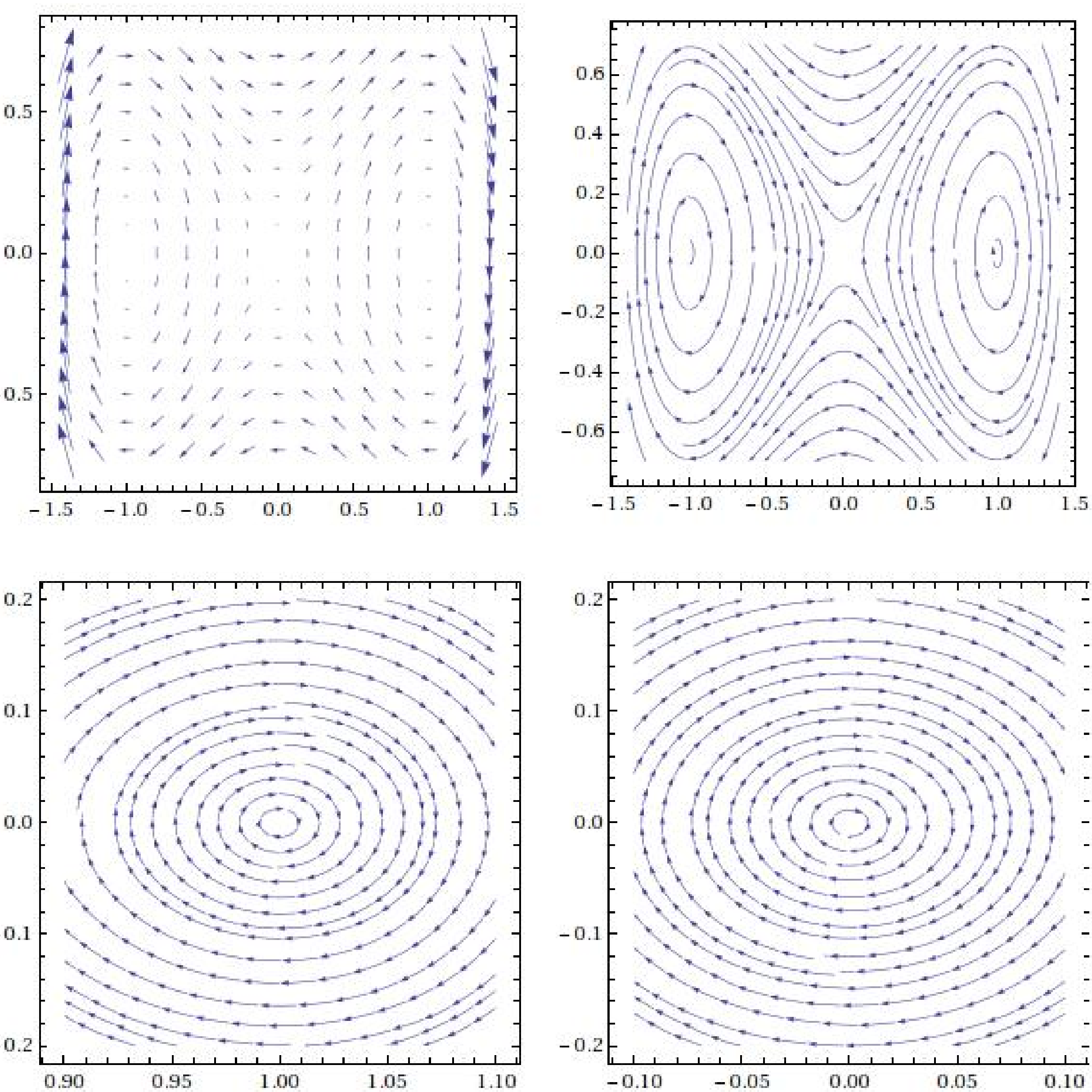
Mivel

$$y_1: \quad -1 < 0 < 1 \quad \Longrightarrow \quad y_1 \text{ instabil nyeregpont}$$

$$y_{2,3}: \Re(0 \pm \sqrt{2}i) = 0,\ \Im(0 \pm \sqrt{2}i) \neq 0 \quad \Longrightarrow \quad y_{2,3} \text{ centrum, stabil, de nem aszimptotikusan}$$

A linearizalt DE pl. az y_2 fixpont korul:

$$\frac{d}{dx}\begin{pmatrix}y-1\\p-0\end{pmatrix} = \frac{d}{dx}\begin{pmatrix}\Delta y\\\Delta p\end{pmatrix} = \begin{pmatrix}0 & 1\\-2 & 0\end{pmatrix}\begin{pmatrix}\Delta y\\\Delta p\end{pmatrix}$$



Az felso sor ket abraja a DE vektormezojet, illetve annak megoldasgorbeit mutatja. A masodik sor elso abraja a megoldasgorbeket abrazolja az y_2 fixpont korul, mig a masodik abra a linearizalt, kozelito egyenlet megoldasgorbeit tartalmazza.

17. $y'' = y - y^3 - y'$. Legyen $p = y'$ Ird at a DE-t egy elsorendu DE rendszerre, keresd meg annak a fixpontjait, ird fel a fixpontoktol valo elteresre vonatkozo liearizalt DE-ket! Vizsgald meg a fixpontok stabilitasat!

Legyen $H = p^2/2 - y^2/2 + y^4/4$. Mutasd meg, hogy $H' \leq 0$ a harom kozul ket fixpontban ! Bizonyitsd be ennek alapjan ezen fixpontok stabilitasat!