Differential equations. I. Exercise set

1. Review exercises.

I.) Compute the following indefinite integrals!

$$a)\ x\ln(3x), \qquad b)\ \sin(3x)\sqrt{\cos(3x)}, \qquad c)\ \frac{1}{(x-3)x}$$

Solution:

$$a)\ \frac{1}{2}x^2\log(3x) - \frac{x^2}{4} + C$$
$$b)\ -\frac{2}{9}\cos^{\frac{3}{2}}(3x) + C$$
$$c)\ \frac{1}{3}(\log(x-3) - \log(x)) + C$$

....

II.) Compute the Taylor series of the following functions around $x = x_0$!

$$a)\ e^{3x},\ x_0 = 0; \qquad b)\ \sin(3x),\ x_0 = 0; \qquad c)\ \log(x),\ x_0 = 1; \qquad d)\ \frac{1}{1-x},\ x_0 = 0; \qquad e)\ \frac{1}{x^2+1},\ x_0 = 0.$$

Solution:

$$a)\ 1 + 3x + \frac{3^2x^2}{2!} + \frac{3^3x^3}{3!} + O\left(x^4\right) = 1 + 3x + \frac{9x^2}{2} + \frac{9x^3}{2} + O\left(x^4\right)$$
$$b)\ 3x - \frac{3^3x^3}{3!} + \frac{3^5x^5}{5!} - \frac{3^7x^7}{7!} + O\left(x^9\right) = 3x - \frac{9x^3}{2} + \frac{81x^5}{40} - \frac{243x^7}{560} + O\left(x^9\right)$$
$$c)\ (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + O\left((x-1)^5\right)$$
$$d)\ 1 + x + x^2 + x^3 + x^4 + O\left(x^5\right)$$
$$e)\ 1 - x^2 + x^4 - x^6 + O\left(x^7\right)$$

....

III.) Let $f(x)$ equal to

$$a)\ e^{x+y^2}, \qquad b)\ x\sin(y^2).$$

Compute $f'_x, f'_y, f''_{xx}, f''_{xy}, f''_{yx}, f''_{yy}$! Compute $\frac{d}{dx}f(x, \ln(x))$!

Solution:

$$a)\ e^{x+y^2}, 2ye^{x+y^2}, e^{x+y^2}, 2ye^{x+y^2}, 2ye^{x+y^2}, 4y^2e^{x+y^2} + 2e^{x+y^2}, e^{x+\log^2(x)}\left(\frac{2\log(x)}{x} + 1\right)$$
$$b)\ \sin\left(y^2\right), 2xy\cos\left(y^2\right), 0, 2y\cos\left(y^2\right), 2y\cos\left(y^2\right), 2x\cos\left(y^2\right) - 4xy^2\sin\left(y^2\right), \sin\left(\log^2(x)\right) + 2\log(x)\cos\left(\log^2(x)\right)$$

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2. Transform the following DE into time independent systems!

$$a)\ y' = xy^2 + x; \qquad b)\ y' = x - y; \qquad c)\ \frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} xy_1 + y_2 \\ y_1y_2 + x \end{pmatrix}$$

Solution:

$$a)\ \frac{d}{dx}\begin{pmatrix} y \\ t \end{pmatrix} = \begin{pmatrix} ty^2 + t \\ 1 \end{pmatrix}; \qquad a)\ \frac{d}{dx}\begin{pmatrix} y \\ t \end{pmatrix} = \begin{pmatrix} t - y \\ 1 \end{pmatrix}; \qquad c)\ \frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \\ t \end{pmatrix} = \begin{pmatrix} ty_1 + y_2 \\ y_1y_2 + t \\ 1 \end{pmatrix}$$

;

3. Express the following DE as first order systems!

$$a)\ \ y'' = -y' - 2y; \qquad b)\ \ y''' = y + x; \qquad c)\ \ \frac{d^2}{dx^2}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} y_1' - y_2 \\ y_2' y_1 \end{pmatrix}$$

Solution:

$$a)\ \ \frac{d}{dx}\begin{pmatrix} y \\ v \end{pmatrix} = \begin{pmatrix} v \\ -v - 2y \end{pmatrix}; \qquad b)\ \ \frac{d}{dx}\begin{pmatrix} y \\ v \\ a \end{pmatrix} = \begin{pmatrix} v \\ a \\ y + x \end{pmatrix}; \qquad c)\ \ \frac{d}{dx}\begin{pmatrix} y_1 \\ v_1 \\ y_2 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_1 - y_2 \\ v_2 \\ v_2 y_1 \end{pmatrix}$$

4.

$$a)\ \ y' = f(x, y) = x - y; \qquad b)\ \ y' = f(x, y) = y^2 + yx;$$

How much are y'' and y''' ? Write down y's third order Taylor polynom around $x = 0$, if $y(0) = 5$!

Solution:

$$y'' = \left(\frac{\partial}{\partial x} + f\frac{\partial}{\partial y}\right) f; \qquad y''' = \left(\frac{\partial}{\partial x} + f\frac{\partial}{\partial y}\right)^2 f,$$

so

. . . .

$$a)\ \ y'' = -x + y + 1, \quad y''' = x - y - 1,$$
$$y(0) = 5, \ \ y'(0) = f(0, 5) = 0 - 5, \ \ y''(0) = -0 + 5 + 1 = 6, \ \ y'''(0) = 0 - 5 - 1 = -6,$$
$$y(x) \approx 5 - 5x + \frac{6}{2!}x^2 + \frac{-6}{3!}x^3$$
$$b)\ \ y'' = 2xy + 2y^3 + 1, \quad y''' = 2x^2 + 8xy^2 + 6y^4 + 2y,$$
$$y(0) = 5, \ \ y'(0) = f(0, 5) = 25, \ \ y''(0) = 251, \ \ y'''(0) = 3760,$$
$$y(x) \approx 5 + 25x + \frac{251}{2!}x^2 + \frac{3760}{3!}x^3$$

5.

$$a)\ \ f(x) = \sin x, \ \ x_0 = \pi/2; \qquad b)\ \ f(x) = \sqrt{x}, \ \ x_0 = 9; \qquad c)\ \ f(x) = 1/x, \ \ x_0 = 2;$$

Compute f's linear approximation $f(x_0 + \Delta x) \approx T_1(x_0 + \Delta x)$ when $\Delta x = 0.1$! Compute $\max_{z\in[x_0, x_0+\Delta x]}|f''(z)|$?! Give a nontrivial upper bound for the error $|\mathrm{err}(\Delta x)| = |f(x_0 + \Delta x) - T_1(x_0 + \Delta x)|$!

. . . .

Solution:

$$a)\ \ \sin(\pi/2 + 0.1) = \sin(\pi/2) + \cos(\pi/2) \cdot 0.1 + \mathrm{err}(0.1)$$
$$|\mathrm{err}(0.1)| \le \frac{1}{2}\Delta x^2 \max_{z\in[x_0, x_0+\Delta x]}|f''(z)| = \frac{1}{2}0.1^2 \max_{z\in[\pi/2, \pi/2+0.1]}|\cos(z)| \le \frac{1}{2}0.1^2 \cdot 1$$

$$c)\ \ 1/(2 + 0.1) = 1/2 - 1/2^2 \cdot 0.1 + \mathrm{err}(0.1)$$
$$|\mathrm{err}(0.1)| \le \frac{1}{2}\Delta x^2 \max_{z\in[x_0, x_0+\Delta x]}|f''(z)| = \frac{1}{2}0.1^2 \max_{z\in[2, 2+0.1]}|-1/z^2| = \frac{1}{2}0.1^2 \cdot 1/4$$

6. Use the Euler and the Heun methods for the following DE with $\Delta x = 0.1$ time step and $y(2) = 3$ initial condition!

$$a)\ \ y' = f(x,y) = x - y; \qquad b)\ \ y' = x - y^2;$$

Do the same for

$$c)\ \ \frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} y_2 \\ -y_1 \end{pmatrix}; \qquad d)\ \ \frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} y_1 - y_2 \\ y_1^2 + x \end{pmatrix}$$

with initial condition: $\begin{pmatrix} y_1(2) \\ y_2(2) \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

What are the predictions of these methods for $y(2.1)$?

Solution:

$$a)\quad Euler:\ \ y(2.1) \approx y(2) + (2-3)\cdot 0.1 = 3 + (-1)\cdot 0.1 = 2.9,$$
$$y(2.2) \approx y(2.1) + (2.1 - 2.9)\cdot 0.1$$
$$Heun:\ \ k_1 = f(2,3) = 2 - 3 = -1, \quad k_2 = f(2+0.1, 3 + f(2,3)\cdot 0.1) = 2.1 - 2.9 = -0.8,$$
$$y(2.1) \approx y(2) + \frac{1}{2}(k_1 + k_2)\cdot 0.1 = 3 + \frac{1}{2}(-1 - 0.8)\cdot 0.1.$$

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7. Solve the DE with $y(0) = 1$ initial codition! Study the unicity of the solutions!

$$a)\ \ y' = y, \qquad b)\ \ y' = y^2, \qquad c)\ \ y' = y^{11/10}, \qquad d)\ \ y' = \sqrt{|y|}, \quad y \geq 0 \quad e)\ \ y' = |y|^{9/10},$$

Solution:

$$a)\ \ y(x) = e^x$$

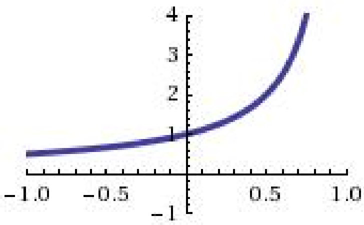
$$b)\ \ \frac{dy}{dx} = y^2, \quad \Longrightarrow \quad \frac{dy}{y^2} = dx, \quad \Longrightarrow \quad \int \frac{dy}{y^2} = \int dx, \quad \Longrightarrow \quad -\frac{1}{y} = x + C,$$
$$\left(y(0) = 1 \quad \Longrightarrow \quad -\frac{1}{1} = 0 + C, \quad \Longrightarrow \quad C = -1 \right), \quad \Longrightarrow \quad y(x) = \frac{1}{1-x}$$

$$c)\ \ y(x) = \frac{1 \times 10^{10}}{(x-10)^{10}}$$

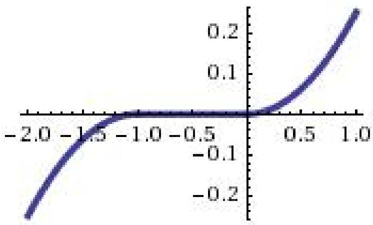
$$d)\ \ y(x) = \begin{cases} \frac{1}{4}(x+2)^2, & ha\ x > -2, \\ 0, & ha\ C \leq x \leq -2 \\ -\frac{1}{4}(x+C)^2, & ha\ x \leq C \end{cases}$$

(where $C \leq -2$ is an arbitrary constant).

b)



d)



The solution is defined on:
a) and d): $(-\infty, \infty)$,
b): $(-\infty, 1)$,
c): $(-\infty, 10)$.
The solution is not unique for case d), as the Lipsitz condition is violated at $y = 0$.

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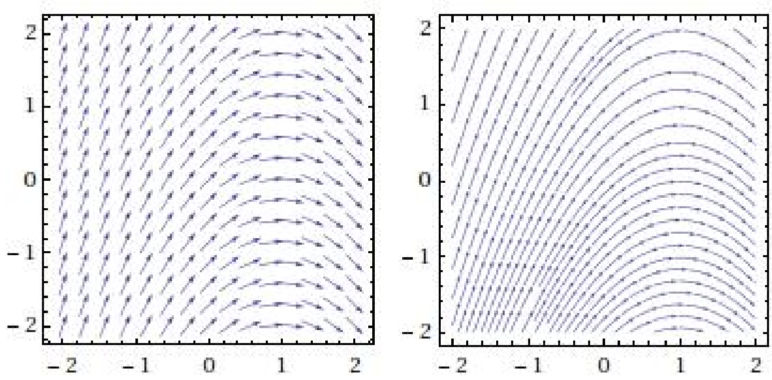
8. Draw the velocity field and solution curves of the $y' = f(x)$ DE!

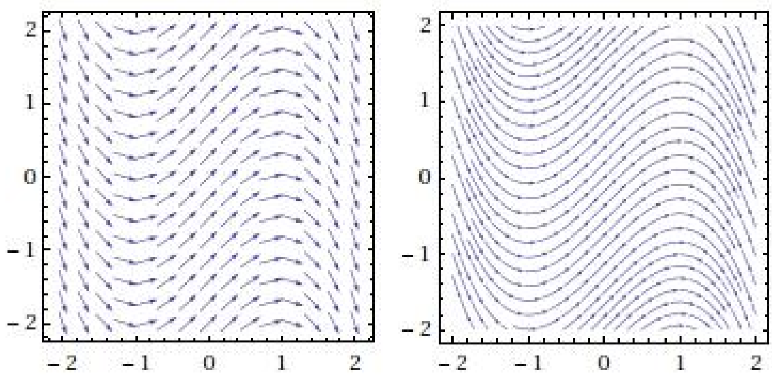
$$a)\ y' = 1, \qquad b)\ y' = x, \qquad c)\ y' = 1 - x, \qquad d)\ y' = x^2, \qquad e)\ y' = 1 - x^2,$$

Solution:

c)



e)



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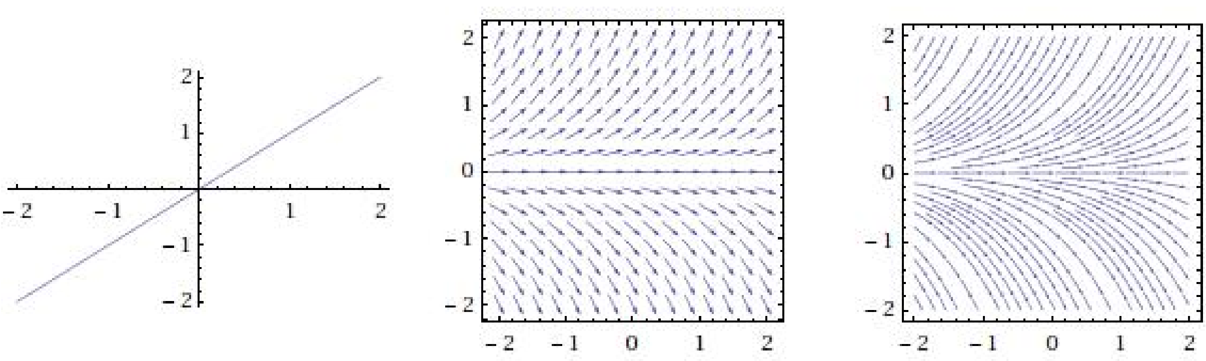
9. Draw the velocity field and solution curves of the $y' = f(y)$ DE! Find the fixpoints of the dynamics and write down the linearized DE around the fixpoints! Study the stability of the fixpoints!

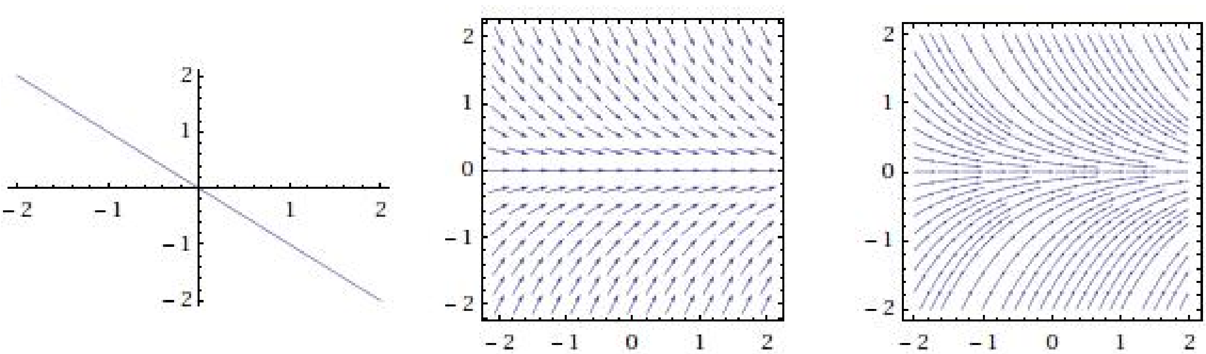
$$a)\ y' = 1, \qquad b)\ y' = y, \qquad c)\ y' = -y, \qquad d)\ y' = y + 1,$$
$$e)\ y' = -1 + y^2, \qquad f)\ y' = y(1 - y), \qquad g)\ y' = y(1 - y)(1 + y).$$

Solution:

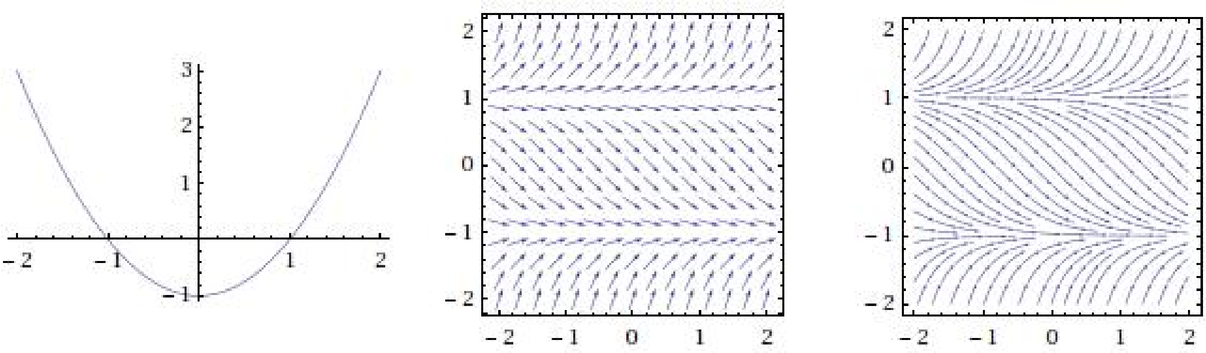
b) $y' = y$



c) $y' = -y$



e) $y' = y^2 - 1$



g) $y' = f(y) = y(1-y)(1+y) = +y - y^3$

$f'(y) = \frac{d}{dy} f(y) = 1 - 3y^2$. The fixpoints:

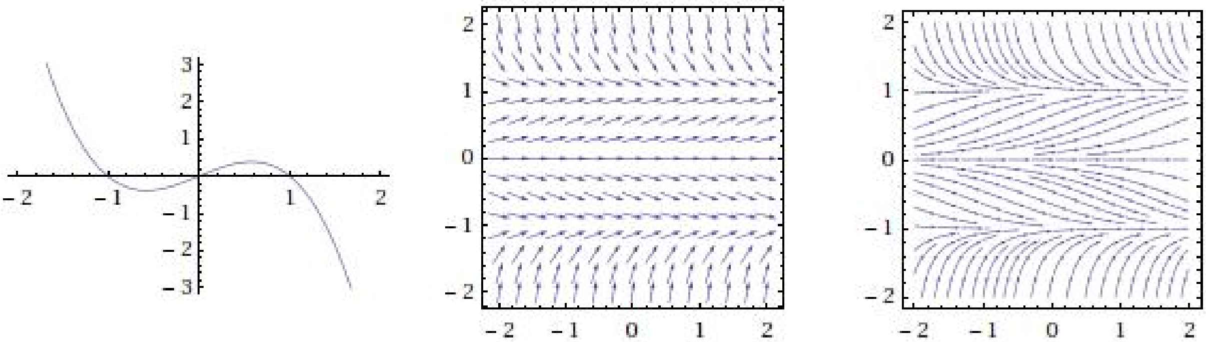
$$y_1 = 0, \qquad y_2 = 1, \qquad y_3 = -1,$$

$$f'(y_1) = f'(0) = 1 - 3 \cdot 0^2 = 1 > 0, \qquad f'(1) = -2 < 0, \qquad f'(-1) = -2 < 0.$$

The sign of f' determines the stability of the fixpoints $y_1,\ y_2,\ y_3$: *unstable, stable, stable.*

The linearized equations:

$$\frac{d}{dx}(y-0) = \frac{d}{dx}\Delta y_1 = 1 \cdot \Delta y_1, \qquad \frac{d}{dx}(y-1) = \frac{d}{dx}\Delta y_2 = -2 \cdot \Delta y_2, \qquad \frac{d}{dx}(y-(-1)) = \frac{d}{dx}\Delta y_3 = -2 \cdot \Delta y_3,$$



10. Find the eigenvectors and eigenvalues of A ! Find the similarity transformation S which diagonalize A, i.e. $D = S^{-1}AS$ where D is diagonal! Express v as the linear combination of the eigenvectors! Compute $A^{13}v$!

$$a)\ (7) \qquad b)\ \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \qquad c)\ \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix} \qquad d)\ \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix} \qquad e)\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad f)\ \begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix}$$

$$g)\ \begin{pmatrix} 2 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 7 \end{pmatrix} \qquad h)\ \begin{pmatrix} 2 & -3 & 0 \\ 3 & 2 & 0 \\ 0 & 0 & 7 \end{pmatrix}$$

Here v is:

$$a)\ v = (8); \qquad b-f)\ v = \begin{pmatrix} 3 \\ 4 \end{pmatrix}; \qquad g-h)\ v = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

Solution:

b)

$$A = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$$

Eigenvalues: $\lambda_1 = 3, \qquad \lambda_2 = 2$,
eigenvectors:

$$v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$D = S^{-1}AS = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$$

As A is a diagonal matrix, these results are trivial.

d)

$$A = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}$$

Charactristic equations for the eignvalues:

$$\det(A - \lambda E) = \begin{vmatrix} 2-\lambda & 1 \\ 0 & 3-\lambda \end{vmatrix} = (2-\lambda)(3-\lambda) - 1 \cdot 0 = 0$$

Eigenvalues: $\lambda_1 = 3, \qquad \lambda_2 = 2$.

Eigenvectors (for $\lambda_1 = 3$):

$$\begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 3\begin{pmatrix} x \\ y \end{pmatrix}, \qquad \text{vagy} \qquad \begin{pmatrix} 2-3 & 1 \\ 0 & 3-3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

So $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ x \end{pmatrix}$. Choose a nonzero vector, eg. $x = 1$.
Eigenvectors:

$$v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

The similarity transformation that diagonalize A:

$$D = S^{-1}AS = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$$

Here S consists of the v_1 and v_2 column vectors.

How much is $A^{13}v$?

$$A^{13}v = (SDS^{-1})^{13}v = SD^{13}S^{-1}v = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 3^{13} & 0 \\ 0 & 2^{13} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

or

$$\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \alpha v_1 + \beta v_2,$$

where

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = S^{-1}\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \end{pmatrix},$$

consequently

$$A^{13}v = A^{13}(\alpha v_1 + \beta v_2) = \alpha\lambda_1^{13}v_1 + \beta\lambda_2^{13}v_2 = 4 \cdot 3^{13}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + (-1) \cdot 2^{13}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

f)

$$A = \begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix}$$

Eigenvalues: $\lambda_1 = 2 + 3i, \qquad \lambda_2 = 2 - 3i$,
Eigenvectors:

$$v_1 = \begin{pmatrix} i \\ 1 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} -i \\ 1 \end{pmatrix}$$

$$D = S^{-1}AS = \begin{pmatrix} -\frac{i}{2} & \frac{1}{2} \\ \frac{i}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2+3i & 0 \\ 0 & 2-3i \end{pmatrix}$$

g) As A is the union of the d) and a) blocks, we get that

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 7 \end{pmatrix}$$

Eigenvalues: $\lambda_1 = 3, \qquad \lambda_2 = 2, \qquad \lambda_3 = 7, \qquad ,$
Eigenvectors:

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$D = S^{-1}AS = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 7 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 7 \end{pmatrix}$$

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11. Solve the following DE for the A matrices of the previous exercises!

$$\frac{d}{dx}y = Ay, \quad y(0) = v$$

Write down the general and the particular solutions.
Compute $\exp(xA)$! Express the particular solution with the help of $\exp(xA)$! Study the stability of the $y = 0$ fixpoint!

. . . .

Solution:

d)

$$A = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}$$

Eigenvalues: $\lambda_1 = 3, \quad \lambda_2 = 2$. Eigenvectors:

$$v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

The general solution:

$$y_{alt}(x) = \sum_i C_i e^{\lambda_i x} v_i = C_1 e^{3x} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^{2x} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

If

$$y(0) = \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} = 4 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + (-1) \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

then the particular slution is

$$y_{part}(x) = 4e^{3x} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + (-1)e^{2x} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

As the real parts of the eigenvalues are positive, the $y = 0$ fixpoint is unstable.

$$e^{xA} = e^{xSDS^{-1}} = Se^{xD}S^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} e^{3x} & 0 \\ 0 & e^{2x} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$$

So the particular solution is

$$y_{part}(x) = e^{xA} \begin{pmatrix} 3 \\ 4 \end{pmatrix}.$$

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12. $y'' = -y$. Write down the characteristic equation and the general solution of the DE! Write the DE as a first order sysyte, solve it and compare the solutions!

Solution:

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The characteristic equation:

$$y'' = -y \quad \Longrightarrow \quad \lambda^2 = -1 \quad \Longrightarrow \quad \lambda_1 = 0 + 1 \cdot i, \ \ \lambda_2 = 0 - 1 \cdot i,$$

general solution:

$$y = C_1 e^{(0+i)x} + C_2 e^{(0-i)x} = e^{0 \cdot x} \left(\widetilde{C_1} \cos(1 \cdot x) + \widetilde{C_2} \sin(1 \cdot x) \right)$$

Here $C_1 = \widetilde{C_1}/2 + \widetilde{C_2}/(2i), \ C_2 = \widetilde{C_1}/2 - \widetilde{C_2}/(2i)$.

. . . .

First order DE system:

$$\begin{pmatrix} y \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} y \\ v \end{pmatrix} = A \begin{pmatrix} y \\ v \end{pmatrix}$$

Eigenvalues and eigenvectors of A:

$$\lambda_1 = i, \;\; v_1 = \begin{pmatrix} 1 \\ i \end{pmatrix}, \qquad \lambda_1 = -i, \;\; v_1 = \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

General solution:

$$\begin{pmatrix} y \\ v \end{pmatrix} = C_1 e^{ix} \begin{pmatrix} 1 \\ i \end{pmatrix} + C_2 e^{-ix} \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

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13. Find the eigenvalues and eigenvectors of A !

$$a) \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad b) \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \qquad c) \begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix} \qquad d) \begin{pmatrix} 7 & 0 & 0 \\ 0 & 2 & 3 \\ 0 & 0 & 2 \end{pmatrix}$$

Compute $\exp(xA)$!

. . . .

Solution:

c)

Eigenvalue:

$$0 = \det(A - \lambda E) = \begin{vmatrix} 2-\lambda & 3 \\ 0 & 2-\lambda \end{vmatrix} \quad \Longrightarrow \quad \lambda_1 = 2.$$

The only eigenvector:

$$\begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 2 \begin{pmatrix} x \\ y \end{pmatrix} \quad \Longrightarrow \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix}, \qquad \text{vagyis} \quad v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$\exp(xA)$:

$$\exp(xA) = \exp\left[\begin{pmatrix} 2x & 0 \\ 0 & 2x \end{pmatrix} + \begin{pmatrix} 0 & 3x \\ 0 & 0 \end{pmatrix}\right] = \exp\begin{pmatrix} 2x & 0 \\ 0 & 2x \end{pmatrix} \cdot \exp\begin{pmatrix} 0 & 3x \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} e^{2x} & 0 \\ 0 & e^{2x} \end{pmatrix} \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 3x \\ 0 & 0 \end{pmatrix} + \frac{1}{2!}\begin{pmatrix} 0 & 3x \\ 0 & 0 \end{pmatrix}^2 + \cdots\right] = \begin{pmatrix} e^{2x} & 0 \\ 0 & e^{2x} \end{pmatrix} \begin{pmatrix} 1 & 3x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e^{2x} & 3xe^{2x} \\ 0 & e^{2x} \end{pmatrix}$$

The first, $\exp(C + D) = \exp(C) \cdot \exp(D)$ type rewriting is possibble since $[C, D] = CD - DC = 0$:

$$\begin{pmatrix} e^{2x} & 0 \\ 0 & e^{2x} \end{pmatrix} \begin{pmatrix} 0 & 3x \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 3x \\ 0 & 0 \end{pmatrix} \begin{pmatrix} e^{2x} & 0 \\ 0 & e^{2x} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0$$

At last we used that

$$\begin{pmatrix} 0 & 3x \\ 0 & 0 \end{pmatrix}^2 = (3x)^2 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0$$

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14. Solve the following DE for the A matrices of the previous exercise

$$\frac{d}{dx} y = Ay, \;\; y(0) = v$$

write down the particular solution with the help of e^{xA}, if v is:

$$a - c) \;\; v = \begin{pmatrix} 3 \\ 4 \end{pmatrix}; \qquad d) \;\; v = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

. . . .

Solution:

c)

$$y_{part}(x) = e^{xA} y(0) = \exp\left[x \begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix}\right] \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} e^{2x} & 3xe^{2x} \\ 0 & e^{2x} \end{pmatrix} \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

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15. Damped oscillator: $y'' = -y - ky'$. Find the general solution! How much is k if the char.eq. has only one solution? In that case write the DE as a first order system, and study the coefficient matrix' Jordan normal form.

Solution:

Char.eq.:

$$y'' = -y - ky' \quad \Longrightarrow \quad \lambda^2 = -1 - k\lambda \quad \Longrightarrow \quad \lambda_{1,2} = \frac{-k \pm \sqrt{k^2-4}}{2}$$

There is only one root, if $k = \pm 2$. We stick to the $k = 2$ case, so $\lambda = -1$.
General solution:

$$y_{alt} = C_1 e^{-x} + C_2 x e^{-x}.$$

The same as a first order system:

$$\begin{pmatrix} y \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} y \\ v \end{pmatrix} = A \begin{pmatrix} y \\ v \end{pmatrix}$$

A's eigenvectors and eigenvalues:

$$\lambda = -1, \qquad v_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

Jordan normal form:

$$Av_1 = \lambda v_1,$$
$$Av_2 = \lambda v_2 + v_1$$
$$\text{tehat}$$
$$v_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \qquad S = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix},$$
$$J = S^{-1}AS = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$$

So

$$\exp(xA) = \exp(xSJS^{-1}) = S\exp(xJ)S^{-1} = S \begin{pmatrix} e^{-x} & xe^{-x} \\ 0 & e^{-x} \end{pmatrix} S^{-1}$$

---

16. $y'' = y - y^3$. introduce $p = y'$. Show that the DE can be written in the following Hamiltonian form:

$$y' = \frac{\partial H}{\partial p}, \qquad p' = -\frac{\partial H}{\partial y}.$$

Compute H ! Show that $H' = 0$!

Write the DE as a first order system, find its fixed points, write down the linearized DE around the fixed points and study the stability of the fixed points!

Solution:

$$y' = p = \frac{\partial H}{\partial p} \quad \Longrightarrow \quad H(y,p) = \frac{p^2}{2} + h(y),$$
$$p' = y'' = y - y^3 = -\frac{\partial H}{\partial y} \quad \Longrightarrow \quad H = \frac{p^2}{2} + \frac{y^4}{4} - \frac{y^2}{2}$$

$$H' = pp' + y^3y' - yy' = p(y - y^3) + (y - y^3)y' = p(y - y^3) + (y - y^3)p = 0$$

The first order DE system:

$$\frac{d}{dx}\begin{pmatrix} y \\ p \end{pmatrix} = \begin{pmatrix} p \\ y - y^3 \end{pmatrix}$$

Equilibrium states (fixedpoints):

$$\begin{pmatrix}0\\0\end{pmatrix} = \frac{d}{dx}\begin{pmatrix}y\\p\end{pmatrix} = \begin{pmatrix}p\\y-y^3\end{pmatrix} \qquad \Longrightarrow \qquad y_1 = \begin{pmatrix}0\\0\end{pmatrix}, \qquad y_2 = \begin{pmatrix}1\\0\end{pmatrix}, \qquad y_3 = \begin{pmatrix}-1\\0\end{pmatrix},$$

A Jacobi matrix:

$$J = \begin{pmatrix} \frac{\partial}{\partial y}p & \frac{\partial}{\partial p}p \\ \frac{\partial}{\partial y}(y-y^3) & \frac{\partial}{\partial p}(y-y^3) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1-3y^2 & 0 \end{pmatrix}$$

The values of the Jacobian at the fixedpoints:

$$J(y_1) = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \qquad J(y_2) = \begin{pmatrix}0 & 1\\ -2 & 0\end{pmatrix}, \qquad J(y_3) = \begin{pmatrix}0 & 1\\ -2 & 0\end{pmatrix}$$

The eigenvalues of the matrices:

$$y_1: \quad (1,-1), \qquad y_2: \quad (0-\sqrt{2}i, 0+\sqrt{2}i), \qquad y_3: \quad (0-\sqrt{2}i, 0+\sqrt{2}i).$$

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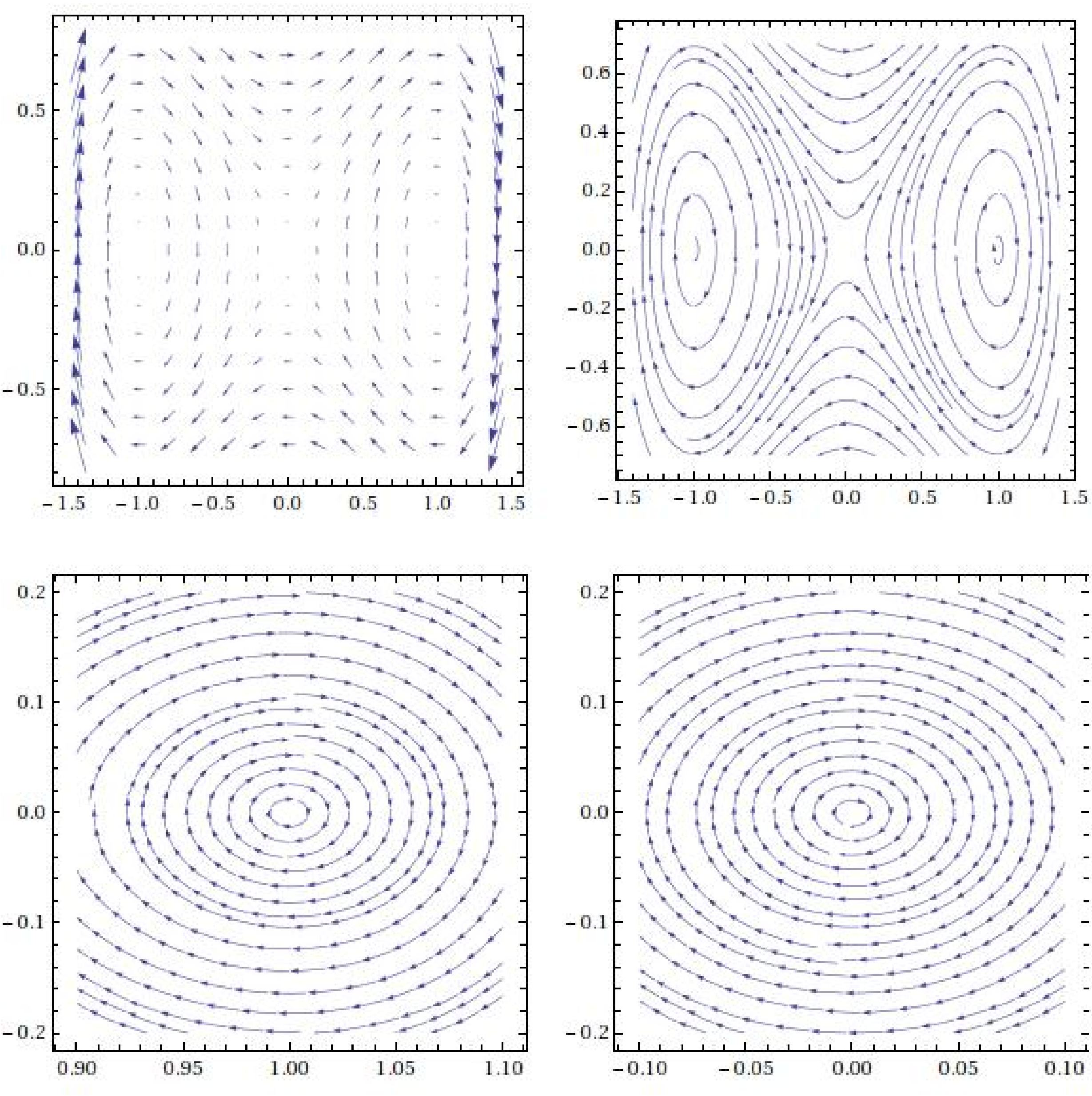
$$y_1: \quad -1 < 0 < 1 \quad \Longrightarrow \quad y_1 \text{ saddle point, unstable}$$
$$y_{2,3}: \Re(0 \pm \sqrt{2}i) = 0,\ \Im(0 \pm \sqrt{2}i) \neq 0 \quad \Longrightarrow \quad y_{2,3} \text{ center, stable},$$

The linearized DE around for example around y_2:

$$\frac{d}{dx}\begin{pmatrix}y-1\\p-0\end{pmatrix} = \frac{d}{dx}\begin{pmatrix}\Delta y\\ \Delta p\end{pmatrix} = \begin{pmatrix}0 & 1\\ -2 & 0\end{pmatrix}\begin{pmatrix}\Delta y\\ \Delta p\end{pmatrix}$$



The first row shows the velocity field and solution curves of the DE. The first figure of the second row presents the solution curves of the nonlinear DE around the y_2 fixedpoint, while the second (almost identical) figure presents the solution curves of the linearized DE.

17. Write down the Euler-Lagrange equations for the Lagrangians L and M !

$$(y')^2 - y^2, \quad y' + 8, \quad (y')^2 + y', \quad L = (y')^4 + (y-1)^2,$$
$$M = \left((y_1')^2 + (y_2')^2\right)/2 - V(y_1, y_2),$$
$$\left((y_1')^2 + (y_2')^2\right)/2 + A_1(y_1, y_2)y_1' + A_2(y_1, y_2)y_2'$$

Solution:

L:

$$\frac{\partial L}{\partial y} = 2(y-1), \qquad \frac{\partial L}{\partial y'} = 4(y')^3,$$
$$\frac{d}{dx}\left(4(y')^3\right) - 2(y-1) = 0.$$

M:

$$\frac{\partial M}{\partial y_1'} = y_1', \qquad \frac{\partial M}{\partial y_2'} = y_2', \qquad \frac{\partial M}{\partial y_1} = -\frac{\partial V}{\partial y_1}, \qquad \frac{\partial M}{\partial y_2} = -\frac{\partial V}{\partial y_2},$$
$$\frac{d}{dx}y_1' - (-\frac{\partial V}{\partial y_1}) = 0, \qquad \frac{d}{dx}y_2' - (-\frac{\partial V}{\partial y_2}) = 0,$$

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$$y_1'' = -\frac{\partial V}{\partial y_1}, \qquad y_2'' = -\frac{\partial V}{\partial y_2}$$

18. Let $S[u] = \int_0^1 (y'(x))^4 + xy(x)\ dx$ where u is defined on $[0,1]$ and vanishes at the endpoints. Let V be defined on $[0,1]$, assume that it vanishes at the endpoints and is continuous. Assume also that elements of V are piecewise affine on the $[0,1/3]$, $[1/3,2/3]$, $[2/3,1]$ intervals. Let ϕ_1 and ϕ_2 be a basis of V, such that $\phi_1(1/3) = \phi_2(2/3) = 1$ and $\phi_2(1/3) = \phi_1(2/3) = 0$. Let $u_h = c_1\phi_1 + c_2\phi_2$. Compute the $S[u_h] = s(c_1, c_2)$ two variable function! (For the computation of the $xy(x)$ term in the integral use some approximate method!)

Solution:

$$S[u_h]$$
$$\approx \left[(3c_1)^4 \cdot \frac{1}{3} + (3(c_2 - c_1))^4 \cdot \frac{1}{3} + (-3c_2)^4 \cdot \frac{1}{3}\right]$$
$$+\left[\frac{1}{2}\Big(0 \cdot 0 + \frac{1}{3} \cdot c_1\Big) \cdot \frac{1}{3} + \frac{1}{2}\Big(\frac{1}{3} \cdot c_1 + \frac{2}{3} \cdot c_2\Big) \cdot \frac{1}{3} + \frac{1}{2}\Big(\frac{2}{3} \cdot c_2 + 1 \cdot 0\Big) \cdot \frac{1}{3}\right]$$