Differential equations. I. Exercise set

1. Review exercises.
   I.) Compute the following indefinite integrals!

$$a)\ x\ln(3x), \qquad b)\ \sin(3x)\sqrt{\cos(3x)}, \qquad c)\ \frac{1}{(x-3)x}$$

....

II.) Compute the Taylor series of the following functions around $x = x_0$!

$$a)\ e^{3x},\ x_0 = 0; \qquad b)\ \sin(3x),\ x_0 = 0; \qquad c)\ \log(x),\ x_0 = 1; \qquad d)\ \frac{1}{1-x},\ x_0 = 0; \qquad e)\ \frac{1}{x^2+1},\ x_0 = 0.$$

....

III.) Let $f(x)$ equal to

$$a)\ e^{x+y^2}, \qquad b)\ x\sin(y^2).$$

Compute $f'_x, f'_y, f''_{xx}, f''_{xy}, f''_{yx}, f''_{yy}$! Compute $\frac{d}{dx}f(x,\ln(x))$!

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2. Transform the following DE into time independent systems!

$$a)\ y' = xy^2 + x; \qquad b)\ y' = x - y; \qquad c)\ \frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} xy_1 + y_2 \\ y_1y_2 + x \end{pmatrix}$$

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3. Express the following DE as first order systems!

$$a)\ y'' = -y' - 2y; \qquad b)\ y''' = y + x; \qquad c)\ \frac{d^2}{dx^2}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} y_1' - y_2 \\ y_2'y_1 \end{pmatrix}$$

4.

$$a)\ y' = f(x,y) = x - y; \qquad b)\ y' = f(x,y) = y^2 + yx;$$

How much are y'' and y''' ? Write down y's third order Taylor polynom around $x = 0$, if $y(0) = 5$!

5.

$$a)\ f(x) = \sin x,\ \ x_0 = \pi/2; \qquad b)\ f(x) = \sqrt{x},\ \ x_0 = 9; \qquad c)\ f(x) = 1/x,\ \ x_0 = 2;$$

Compute f's linear approximation $f(x_0 + \Delta x) \approx T_1(x_0 + \Delta x)$ when $\Delta x = 0.1$! Compute $\max_{z\in[x_0,x_0+\Delta x]}|f''(z)|$?! Give a nontrivial upper bound for the error $|\mathrm{err}(\Delta x)| = |f(x_0 + \Delta x) - T_1(x_0 + \Delta x)|$!

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6. Use the Euler and the Heun methods for the following DE with $\Delta x = 0.1$ time step and $y(2) = 3$ initial condition!

$$a)\ \ y' = f(x,y) = x - y; \qquad b)\ \ y' = x - y^2;$$

Do the same for

$$c)\ \ \frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} y_2 \\ -y_1 \end{pmatrix}; \qquad d)\ \ \frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} y_1 - y_2 \\ y_1^2 + x \end{pmatrix}$$

with initial condition: $\begin{pmatrix} y_1(2) \\ y_2(2) \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

What are the predictions of these methods for $y(2.1)$?

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7. Solve the DE with $y(0) = 1$ initial codition! Study the unicity of the solutions!

$$a)\ \ y' = y, \qquad b)\ \ y' = y^2, \qquad c)\ \ y' = y^{11/10}, \qquad d)\ \ y' = \sqrt{|y|}, \quad y \geq 0 \quad e)\ \ y' = |y|^{9/10},$$

....

8. Draw the velocity field and solution curves of the $y' = f(x)$ DE!

$$a)\ y' = 1, \qquad b)\ y' = x, \qquad c)\ y' = 1 - x, \qquad d)\ y' = x^2, \qquad e)\ y' = 1 - x^2,$$

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9. Draw the velocity field and solution curves of the $y' = f(y)$ DE! Find the fixpoints of the dynamics and write down the linearized DE around the fixpoints! Study the stability of the fixpoints!

$$a)\ y' = 1, \qquad b)\ y' = y, \qquad c)\ y' = -y, \qquad d)\ y' = y + 1,$$
$$e)\ y' = -1 + y^2, \qquad f)\ y' = y(1-y), \qquad g)\ y' = y(1-y)(1+y).$$

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10. Find the eigenvectors and eigenvalues of A ! Find the similarity transformation S which diagonalize A, i.e. $D = S^{-1}AS$ where D is diagonal! Express v as the linear combination of the eigenvectors! Compute $A^{13}v$!

$$a)\ (7) \qquad b)\ \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \qquad c)\ \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix} \qquad d)\ \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix} \qquad e)\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad f)\ \begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix}$$
$$g)\ \begin{pmatrix} 2 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 7 \end{pmatrix} \qquad h)\ \begin{pmatrix} 2 & -3 & 0 \\ 3 & 2 & 0 \\ 0 & 0 & 7 \end{pmatrix}$$

Here v is:

$$a)\ v = (8); \qquad b-f)\ v = \begin{pmatrix} 3 \\ 4 \end{pmatrix}; \qquad g-h)\ v = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

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11. Solve the following DE for the A matrices of the previous exercises!

$$\frac{d}{dx}y = Ay, \quad y(0) = v$$

Write down the general and the particular solutions.
Compute $\exp(xA)$! Express the particular solution with the help of $\exp(xA)$! Study the stability of the $y = 0$ fixpoint!

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12. $y'' = -y$. Write down the characteristic equation and the general solution of the DE! Write the DE as a first order sysyte, solve it and compare the solutions!

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13. Find the eigenvalues and eigenvectors of A !

$$a)\ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad b)\ \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \qquad c)\ \begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix} \qquad d)\ \begin{pmatrix} 7 & 0 & 0 \\ 0 & 2 & 3 \\ 0 & 0 & 2 \end{pmatrix}$$

Compute $\exp(xA)$!

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14. Solve the following DE for the A matrices of the previous exercise

$$\frac{d}{dx}y = Ay, \quad y(0) = v$$

write down the particular solution with the help of e^{xA}, if v is:

$$a-c)\ v = \begin{pmatrix} 3 \\ 4 \end{pmatrix}; \qquad d)\ v = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

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15. Damped oscillator: $y'' = -y - ky'$. Find the general solution! How much is k if the char.eq. has only one solution? In that case write the DE as a first order system, and study the coefficient matrix' Jordan normal form.

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16. $y'' = y - y^3$. introduce $p = y'$. Show that the DE can be written in the following Hamiltonian form:

$$y' = \frac{\partial H}{\partial p}, \qquad p' = -\frac{\partial H}{\partial y}.$$

Compute H ! Show that $H' = 0$!

Write the DE as a first order system, find its fixed points, write down the linearized DE around the fixed points and study the stability of the fixed points!

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17. Write down the Euler-Lagrange equations for the Lagrangians L and M !

$$(y')^2 - y^2, \quad y' + 8, \quad (y')^2 + y', \quad L = (y')^4 + (y-1)^2,$$
$$M = \left((y_1')^2 + (y_2')^2\right)/2 - V(y_1, y_2),$$
$$\left((y_1')^2 + (y_2')^2\right)/2 + A_1(y_1, y_2)y_1' + A_2(y_1, y_2)y_2'$$

18. Let $S[u] = \int_0^1 (y'(x))^4 + xy(x)\, dx$ where u is defined on $[0,1]$ and vanishes at the endpoints. Let V be defined on $[0,1]$, assume that it vanishes at the endpoints and is continuous. Assume also that elements of V are piecewise affine on the $[0,1/3]$, $[1/3,2/3]$, $[2/3,1]$ intervals. Let ϕ_1 and ϕ_2 be a basis of V, such that $\phi_1(1/3) = \phi_2(2/3) = 1$ and $\phi_2(1/3) = \phi_1(2/3) = 0$. Let $u_h = c_1\phi_1 + c_2\phi_2$. Compute the $S[u_h] = s(c_1, c_2)$ two variable function! (For the computation of the $xy(x)$ term in the integral use some approximate method!)