Zh1, Diff.Egy., 2013.04.02. NEPTUN: Gyak.Vez.:

Név: Aláírás:

1. (3+4+3 pont)
a)

$$y' = f(x,y) = 2y + yx;$$

Mennyi y'' ? Ird fel y masodrendu Talor polinomjat az $x = 0$ pont korul, ha $y(0) = 3$!

$y(0) = 3$

$y'(0) = 2\cdot 3 + 3\cdot 0 = 6$

$$\boxed{y(x) \approx T_2(x) = 3 + 6x + \frac{15}{2!}x^2}$$

$$y''(x) = \frac{d}{dx}y'(x) = \left(\frac{\partial}{\partial x} + (2y+yx)\frac{\partial}{\partial y}\right)(2y+yx)$$

$$= y + (2y+yx)(2+x), \text{ tehát } y''(0) = 3 + (2\cdot 3 + 3\cdot 0)(2+0) = 15$$

b) Alkalmazd az Euler, illetve a Heun modszert a kovetkezo DE-re $\Delta x = 0.1$ lepeskozzel az $y(2) = 3$ kezdeti feltetel mellett!

$$y' = x^2 - y^3;$$

~~Mit josol a ket modszer $y(2.1)$-re?~~

Euler:

$$y(2.1) \approx y(2) + y'(2,3)\cdot 0.1 = 3 + (2^2 - 3^3)\cdot 0.1 = 3 + (-2.3) = 0.7$$

Heun: $k_1 = y'(2,3) = -23, \quad k_2 = y'(2+0.1;\ 3 + (-23)\cdot 0.1) = 2.1^2 - 0.7^3$

$$y(2.1) \approx 3 + \frac{1}{2}\left((-23)^2 + (2.1^2 - 0.7^3)\right)\cdot 0.1$$

$\sqrt[3]{x}$ $\quad$ 27

c) Legyen $f(x) = \cancel{\sqrt[3]{27}}$, $x_0 = \cancel{x}$ Ird fel f-nek a linearis $f(x_0 + \Delta x) \approx T_1(x_0 + \Delta x)$ kozeliteset, ha $\Delta x = 0.1$! Mennyi $\max_{z\in[x_0,x_0+\Delta x]}|f''(z)|$? Adj nemtrivialis felso korlatot a kozelites $|hiba(\Delta x)| = |f(x_0 + \Delta x) - T_1(x_0 + \Delta x)|$ hibajara!

$f(27) = \sqrt[3]{27} = 3$

$f'(x) = \frac{1}{3}x^{-2/3}$

$f'(27) = \frac{1}{3}\frac{1}{\sqrt[3]{27^2}} = \frac{1}{27}$

$f(27.1) \approx T_1(27.1) = 3 + \frac{1}{27}\cdot 0.1$

$f''(x) = \frac{1}{3}\cdot\left(-\frac{2}{3}\right)x^{-5/3}$

$\max\limits_{z\in[27,27.1]}\left|-\frac{2}{9}z^{-5/3}\right| = \frac{2}{9}\cdot 27^{-5/3} = \frac{2}{3^7}$

hiba:

$|f(27.1) - T_1(27.1)| \leq$

$\leq \frac{1}{2}\cdot\frac{2}{3^7}\cdot 0.1^2$

2. (5+2+3 pont)

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} -y_1 \\ y_1 - 2y_2 \end{pmatrix} = A \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \qquad \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

Keresd meg A sajatertekeit es sajatvektorait!

$$A = \begin{pmatrix} -1 & 0 \\ 1 & -2 \end{pmatrix} \qquad \begin{vmatrix} -1-\lambda & 0 \\ 1 & -2-\lambda \end{vmatrix} = 0 = (-1-\lambda)(-2-\lambda) - 0 \cdot 1$$

$$\lambda_1 = -1 \qquad \lambda_2 = -2$$

$$\begin{pmatrix} -1-(-1) & 0 \\ 1 & -2-(-1) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} -1-(-2) & 0 \\ 1 & -2-(-2) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$x = y \qquad x = 0$$

$$V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad V_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Ird fel a DE altalanos megoldasat!

$$y_{ált} = C_1 e^{-1 \cdot x} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^{-2 \cdot x} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Szamold ki a DE partikularis megoldasat!

$$c_1 \cdot e^{-1 \cdot 0} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-2 \cdot 0} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$c_1 = 1 \qquad c_2 = 2$$

$$y_{part} = 1 \cdot e^{-1 \cdot x} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + 2 \cdot e^{-2x} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

(5 × 2 pont)
Ird fel, hogy milyen osszefugges van A es a sajatertekeket tartalmazo diagonalis D matrixok kozott!

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} -1 & 0 \\ 1 & -2 \end{pmatrix} \left(\underset{v_1}{\boxed{\begin{matrix} 1 \\ 1 \end{matrix}}} \; \underset{v_2}{\boxed{\begin{matrix} 0 \\ 1 \end{matrix}}} \right) = \begin{pmatrix} \underset{\lambda_1}{\boxed{-1}} & 0 \\ 0 & \underset{\lambda_2}{\boxed{-2}} \end{pmatrix}$$

$D = S^{-1} A S$

vagy

$A = S D S^{-1}$

Mennyi e^{xA}? (Elegendo az, hogy kifejezed az eredmenyt D, illetve egy S matrix es annak inverze szorzatakent!)

$$e^{xA} = S e^{xD} S^{-1} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} e^{-1 \cdot x} & 0 \\ 0 & e^{-2x} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^{-1}$$

Ird fel a partikularis megoldast e^{xA} segitsegevel!

$$y_{part} = e^{xA} \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

2b) Ird at a kovetkezo DE rendszert elsorendu DE rendszerre!

$$\frac{d^2}{dx^2} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 4y_1' - y_2^2 \\ y_2' - y_1' \end{pmatrix}$$

$$\frac{d}{dx} \begin{pmatrix} y_1 \\ v_1 \\ y_2 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ 4v_1 - y_2^2 \\ v_2 \\ v_2 - v_1 \end{pmatrix}$$

Jegy:

1	0-15
2	16-22
3	23-28
4	29-34
5	35-40

Ird at a kovetkezo DE-ket idofuggetlen DE rendszerre!

$$\frac{d}{dx} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x + y_2 \\ y_1 - y_2 + x^5 \end{pmatrix}$$

$$\frac{d}{dx} \begin{pmatrix} y_1 \\ y_2 \\ t \end{pmatrix} = \begin{pmatrix} t + y_2 \\ y_1 - y_2 + t^5 \\ 1 \end{pmatrix}$$

3a. (1+1+1+2 pont)

$y' = 1 - y^6. = f(y)$

Keresd meg a DE fixpontjait!

$$f(y) = 0 = 1 - y^6 \longrightarrow \quad y_1 = -1 \qquad y_2 = 1$$

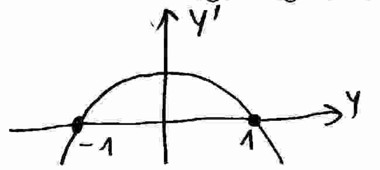
Ird fel a fixpontok koruli linearizalt kozelito DE-t!

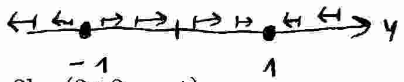
$$f' = \frac{df(y)}{dy} = -6y^5 \quad \Big| \quad f'(-1) = 6, \quad \frac{d}{dx}(y-(-1)) = \frac{d}{dx}\Delta y_1 = 6\Delta y_1 \quad \Big| \quad f'(1) = -6, \quad \frac{d}{dx}(y-1) = \frac{d}{dx}\Delta y_2 = -6\Delta y_2$$

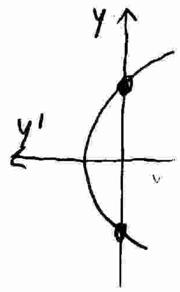
Ha $y(0) = 0$, mennyi

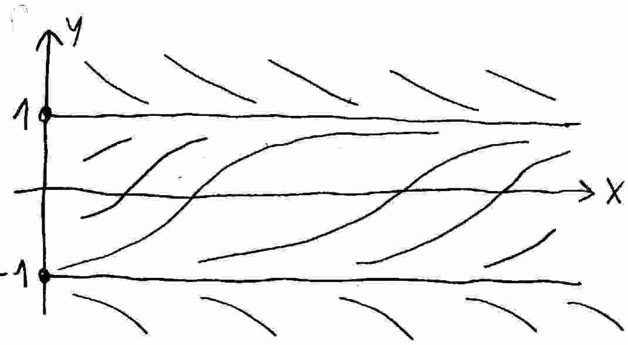
$\lim_{x\to\infty} y(x) = 1$ $\qquad$ $\lim_{x\to-\infty} y(x) = -1$

Vazold a DE megoldasgorbeit!









3b. (2+3 pont)

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} (y_2^3+8)(y_1-3) \\ (y_1^3-1)y_2. \end{pmatrix}$$

Keresd meg a DE fixpontjat!

$$\left.\begin{array}{l} (y_2^3+8)(y_1-3) = 0 \\ (y_1^3-1)y_2 = 0 \end{array}\right\} \Longrightarrow \quad y_2 = -2,\ y_1 = 1 \quad P_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

vagy

$$y_1 = 3,\ y_2 = 0 \quad P_2 = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$$

Ird fel a fixpont koruli linearizalt kozelito DE-t!

Jacobi mátrix:

$$J = \begin{pmatrix} \frac{\partial}{\partial y_1}(y_2^3+8)(y_1-3) & \frac{\partial}{\partial y_2}(y_2^3+8)(y_1-3) \\ \frac{\partial}{\partial y_1}(y_1^3-1)y_2 & \frac{\partial}{\partial y_2}(y_1^3-1)y_2 \end{pmatrix} = \begin{pmatrix} y_2^3+8 & 3y_2^2(y_1-3) \\ 3y_1^2 y_2 & (y_1^3-1) \end{pmatrix}$$

$$J(P_1) = \begin{pmatrix} 0 & -24 \\ -6 & 0 \end{pmatrix} \qquad \Bigg| \qquad J(P_2) = \begin{pmatrix} 8 & 0 \\ 0 & 26 \end{pmatrix}$$

$$\frac{d}{dx}\begin{pmatrix} y_1-1 \\ y_2-(-2) \end{pmatrix} = \frac{d}{dx}\begin{pmatrix} \Delta y_1 \\ \Delta y_2 \end{pmatrix} = \begin{pmatrix} 0 & -24 \\ -6 & 0 \end{pmatrix}\begin{pmatrix} \Delta y_1 \\ \Delta y_2 \end{pmatrix} \qquad \Bigg| \qquad \frac{d}{dx}\begin{pmatrix} y_1-3 \\ y_2-0 \end{pmatrix} = \frac{d}{dx}\begin{pmatrix} \Delta y_1 \\ \Delta y_2 \end{pmatrix} = \begin{pmatrix} 8 & 0 \\ 0 & 26 \end{pmatrix}\begin{pmatrix} \Delta y_1 \\ \Delta y_2 \end{pmatrix}$$

Pót
Zh1, Diff.Egy., 2013.~~04.02.~~ 05.15 NEPTUN: Gyak.Vez.:

Név: Aláírás:

1. (4+(3+3) pont)

a)

$$y' = f(x, y) = x^2 + 5y - 2;$$

Mennyi y'' ? Ird fel y masodrendu Talor polinomjat az $x = 0$ pont korul, ha $y(0) = 3$!

$$y'' = \left(f(x,y)\frac{\partial}{\partial y} + \frac{\partial}{\partial x}\right)(x^2+5y-2) = (x^2+5y-2)(5) + 2x$$

$$y'(0) = 0^2 + 5\cdot 3 - 2 = 13 \qquad y''(0) = (0^2 + 5\cdot 3 - 2)\cdot 5 + 2\cdot 0 = 65$$

$$y(x) \approx T_2(x) = 3 + 13x + \frac{65}{2!}x^2$$

b) Alkalmazd az Euler, illetve a Heun modszert a kovetkezo DE-re $\Delta x = 0.01$ lepeskozzel! az $\bar{y}(2) =$ kezdeti feltetel mellett!

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} y_1 - y_2 \\ 3y_2^2 + x. \end{pmatrix}, \qquad \bar{y}(2) = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

Mit josol a ket modszer $\bar{y}(2.01)$-re?

Euler:

$$\bar{y}(2.01) \approx \begin{pmatrix} 2 \\ 3 \end{pmatrix} + 0.01\cdot\begin{pmatrix} 2-3 \\ 3\cdot 3^2 + 2 \end{pmatrix} = \begin{pmatrix} 1.99 \\ 3.29 \end{pmatrix}$$

Heun:

$$\bar{y}(2.01) \approx \begin{pmatrix} 2 \\ 3 \end{pmatrix} + \frac{1}{2}\left[\begin{pmatrix} 2-3 \\ 3\cdot 3^2 + 2 \end{pmatrix} + \begin{pmatrix} 1.99 - 3.29 \\ 3\cdot 3.29^2 + 2.01 \end{pmatrix}\right]$$

2. (5+2+3 pont)

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} -y_1 - 3y_2 \\ 3y_1 - y_2 \end{pmatrix} = A \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \qquad \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

Keresd meg A sajatertekeit es sajatvektorait!

$$0 = \det(A - \lambda E) = \begin{vmatrix} -1-\lambda & -3 \\ 3 & -1-\lambda \end{vmatrix} = (-1-\lambda)^2 - (-3)\cdot 3 = \lambda^2 + 2\lambda + 10$$

$$\lambda_1 = -1+3i \qquad \lambda_2 = -1-3i$$

① $\begin{pmatrix} -1 & -3 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = (-1+3i)\begin{pmatrix} x \\ y \end{pmatrix}$,

vagy

$$\begin{pmatrix} -1-(-1+3i) & 3 \\ -3 & -1-(-1+3i) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{array}{l} -3ix + 3y = 0 \\ -3x - 3iy = 0 \end{array}\right\} \rightarrow y = ix \qquad V_1 = \begin{pmatrix} 1 \\ i \end{pmatrix}$$

② $$\begin{pmatrix} -1-(-1-3i) & 3 \\ -3 & -1-(-1-3i) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{array}{l} 3ix + 3y = 0 \\ -3x + 3iy = 0 \end{array}\right\} \rightarrow y = -ix \qquad V_2 = \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

Ird fel a DE altalanos megoldasat!

$$\overline{y}_{ált}(x) = C_1 e^{(-1+3i)x}\begin{pmatrix} 1 \\ i \end{pmatrix} + C_2 e^{(-1-3i)x}\begin{pmatrix} 1 \\ -i \end{pmatrix}$$

Szamold ki a DE partikularis megoldasat!

$$C_1\begin{pmatrix} 1 \\ i \end{pmatrix} + C_2\begin{pmatrix} 1 \\ -i \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \Rightarrow \begin{array}{l} c_1 + c_2 = 1 \\ ic_1 - ic_2 = 3 \\ \hline c_1 - c_2 = -3i \end{array}$$

$$\Downarrow$$

$$c_1 = \frac{1-3i}{2}, \quad c_2 = \frac{1+3i}{2}$$

$$\overline{y}_{part}(x) = \frac{1-3i}{2} e^{(-1+3i)x}\begin{pmatrix} 1 \\ i \end{pmatrix} + \frac{1+3i}{2} e^{(-1-3i)x}\begin{pmatrix} 1 \\ -i \end{pmatrix}$$

0

(5 × 2 pont)

$$A = \begin{pmatrix} 5 & 0 \\ 6 & 5 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 6x & 0 \end{pmatrix} + \frac{1}{2!}\underbrace{\begin{pmatrix} 0 & 0 \\ 6x & 0 \end{pmatrix}^2}_{0} +$$

Mennyi e^{xA}?

$$\exp(xA) = \exp\left[x \cdot \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 6x & 0 \end{pmatrix}\right] = \exp\left[x \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}\right] \cdot \exp\begin{pmatrix} 0 & 0 \\ 6x & 0 \end{pmatrix}$$

$$= \begin{pmatrix} e^{5x} & 0 \\ 0 & e^{5x} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 6x & 1 \end{pmatrix} = \begin{pmatrix} e^{5x} & 0 \\ 6xe^{5x} & e^{5x} \end{pmatrix}$$

Ird fel a

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 5y_1 \\ 6y_1 + 5y_2 \end{pmatrix}, \qquad \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

DE partikularis megoldasat e^{xA} segitsegevel!

$$\bar{y}(x) = e^{xA}\,\bar{y}(0) = \begin{pmatrix} e^{5x} & 0 \\ 6xe^{5x} & e^{5x} \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

2b) Ird at a kovetkezo DE rendszert elsorendu DE rendszerre!

$$\frac{d^2}{dx^2}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} y_1'^2 - y_2 \\ 2y_2' - 3y_1' \end{pmatrix}$$

$$\frac{d}{dx}\begin{pmatrix} y_1 \\ v_1 \\ y_2 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_1^2 - y_2 \\ v_2 \\ 2v_2 - 3v_1 \end{pmatrix}$$

Ird at a kovetkezo DE-ket idofuggetlen DE rendszerre!

$$\frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x + x^5 \\ y_1 y_2 + x^5 \end{pmatrix}$$

$$\frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \\ t \end{pmatrix} = \begin{pmatrix} t + t^5 \\ y_1 y_2 + t^5 \\ 1 \end{pmatrix}$$

3a. (1+1+1+2 pont)

$y' = (y^2-4)y. = y^3 - 4y$

Keresd meg a DE fixpontjait!

$$(y^2-4)y = (y+2)(y-2)y = 0 \rightarrow y_1 = -2,\ y_2 = 2,\ y_3 = 0$$

Ird fel a fixpontok koruli linearizalt kozelito DE-t! $\frac{d}{dy}(y^3-4y) = 3y^2-4$

$$\frac{d}{dx}(y-(-2)) = \frac{d}{dx}\Delta y_1 = 8\Delta y_1,\quad \frac{d}{dx}(y-2) = \frac{d}{dx}\Delta y_2 = 8\Delta y_2,\quad \frac{d}{dx}(y-0) = \frac{d}{dx}\Delta y_3 = -4\Delta y_3$$

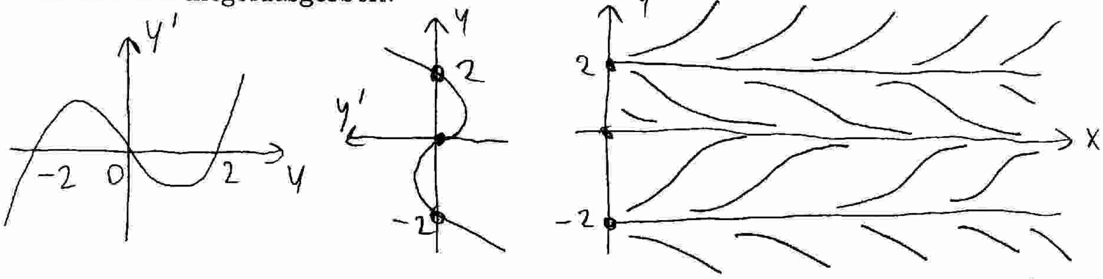
$$3\cdot(-2)^2-4 \qquad 3\cdot 2^2-4 \qquad 3\cdot 0^2-4$$

Ha $y(0) = 0$, mennyi

$\lim_{x\to\infty} y(x) =$ 0 $\qquad \lim_{x\to-\infty} y(x) =$ 0

Mivel $y=0$ fixpont (egyensúlyi állapot)

Vazold a DE megoldasgorbeit!



3b. (2+3 pont)

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} y_1 - y_2 + 1 \\ (3y_1-2y_2)(5y_1-4y_2). \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

Keresd meg a DE fixpontjait!

$$y_1 - y_2 + 1 = 0$$
$$(3y_1-2y_2)(5y_1-4y_2) = 0$$

$y_2 = y_1 + 1$, tehát

$(3y_1 - 2(y_1+1))$, vagy $(5y_1 - 4(y_1+1)) = 0$

$$\Rightarrow \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} = P_1 \text{ vagy } \begin{pmatrix} 4 \\ 5 \end{pmatrix} = P_2$$

Ird fel a fixpont koruli linearizalt kozelito DE-t!

$$\text{Jac} = \begin{pmatrix} \frac{\partial f_1}{\partial y_1} & \frac{\partial f_1}{\partial y_2} \\ \frac{\partial f_2}{\partial y_1} & \frac{\partial f_2}{\partial y_2} \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 3(5y_1-4y_2)+(3y_1-2y_2)\cdot 5 & * \end{pmatrix}$$

$$* = (-2)\cdot(5y_1-4y_2) + (3y_1-2y_2)\cdot(-4)$$

$$\text{Jac}(P_1) = \begin{pmatrix} 1 & -1 \\ -6 & 4 \end{pmatrix} \qquad \text{Jac}(P_2) = \begin{pmatrix} 1 & -1 \\ 10 & -8 \end{pmatrix}$$

$$\frac{d}{dx}\begin{pmatrix} y_1 - 2 \\ y_2 - 3 \end{pmatrix} = \frac{d}{dx}\begin{pmatrix} \Delta y_1 \\ \Delta y_2 \end{pmatrix} \approx \begin{pmatrix} 1 & -1 \\ -6 & 4 \end{pmatrix}\begin{pmatrix} \Delta y_1 \\ \Delta y_2 \end{pmatrix} \quad\Bigg|\quad \frac{d}{dx}\begin{pmatrix} y_1 - 4 \\ y_2 - 5 \end{pmatrix} = \frac{d}{dx}\begin{pmatrix} \Delta y_1 \\ \Delta y_2 \end{pmatrix} \approx \begin{pmatrix} 1 & -1 \\ 10 & -8 \end{pmatrix}\begin{pmatrix} \Delta y_1 \\ \Delta y_2 \end{pmatrix}$$

4. (3+2+3+2 pont)

a)

$$y' = f(x,y) = 1 + 5xy;$$

Mennyi y'' ? Ird fel y masodrendu Talor polinomjat az $x = 0$ pont korul, ha $y(0) = 3$!

$$y'' = \frac{d}{dx} y' = \frac{d}{dx} f(x, y(x)) = \left(\frac{\partial}{\partial x} + y' \frac{\partial}{\partial y}\right) f = \left(\frac{\partial}{\partial x} + f \frac{\partial}{\partial y}\right) f$$

$$= \left(\partial_x + (1+5xy)\partial_y\right)(1+5xy) = 5y + (1+5xy)\cdot 5x = 5(x+y) + 25x^2y$$

$y(0) = 3$

$y'(0) = 1 + 5\cdot 0\cdot 3 = 1$

$y''(0) = 5(0+3) + 25\cdot 0^2\cdot 3 = 15$

$x \approx 0 \qquad y(x) \approx 3 + 1x + \frac{15}{2} x^2$

b) Alkalmazd az Euler, illetve a Heun modszert a kovetkezo DE-re $\Delta x = 0.01$ lepeskozzel!

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} y_2 \\ x + y_1 y_2. \end{pmatrix}, \qquad \bar{y}(2) = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

Mit josol a ket modszer $\bar{y}(2.01)$-re?

Euler:

$$y(2.01) \approx \begin{pmatrix} 2 \\ 3 \end{pmatrix} + 0.01 \begin{pmatrix} 3 \\ 2+2\cdot 3 \end{pmatrix} = \begin{pmatrix} 2.03 \\ 3.08 \end{pmatrix}$$

Heun:

$$y(2.01) \approx \begin{pmatrix} 2 \\ 3 \end{pmatrix} + \frac{1}{2}\left[\begin{pmatrix} 3 \\ 2+2\cdot 3 \end{pmatrix} + \begin{pmatrix} 3.08 \\ 2.01 + 2.03\cdot 3.08 \end{pmatrix}\right]\cdot 0.01$$

c) $x_0 = 5, x_{n+1} = 3x_n + 6$, Mennyi x_n?

fixpont: $x_f = 3x_f + 6 \Rightarrow x_f = -3$

$x_n = 3^n(5 - (-3)) + (-3) = 3^n \cdot 8 - 3$

2. (5+2+3 pont)

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 2y_1 - 3y_2 \\ 3y_1 + 2y_2 \end{pmatrix} = A \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \quad \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

Keresd meg A sajatertekeit es sajatvektorait!

$$A = \begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix}, \quad \det(A - \lambda E) = \begin{vmatrix} 2-\lambda & -3 \\ 3 & 2-\lambda \end{vmatrix} = 0 = (2-\lambda)^2 - (-3)\cdot 3$$

$$\lambda_1 = 2+3i \quad \lambda_2 = 2-3i$$

$$\begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = (2+3i)\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -ix \end{pmatrix}$$

$$\lambda_1 = 2+3i, \quad V_1 = \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

$$\begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = (2-3i)\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ ix \end{pmatrix}$$

$$\lambda_2 = 2-3i, \quad V_2 = \begin{pmatrix} 1 \\ i \end{pmatrix}$$

Ird fel a DE altalanos megoldasat!

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = C_1 e^{(2+3i)t}\begin{pmatrix} 1 \\ -i \end{pmatrix} + C_2 e^{(2-3i)t}\begin{pmatrix} 1 \\ i \end{pmatrix}$$

Szamold ki a DE partikularis megoldasat!

$$\begin{pmatrix} 1 \\ 3 \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ -i \end{pmatrix} + C_2 \begin{pmatrix} 1 \\ i \end{pmatrix} = \begin{pmatrix} C_1 + C_2 \\ i(-C_1 + C_2) \end{pmatrix}$$

$$\left.\begin{aligned} C_1 + C_2 &= 1 \\ -C_1 + C_2 &= 3/i = -3i \end{aligned}\right\} \longrightarrow C_2 = \frac{1-3i}{2}, \quad C_1 = \frac{1+3i}{2}$$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \frac{1-3i}{2} e^{(2+3i)t}\begin{pmatrix} 1 \\ -i \end{pmatrix} + \frac{1+3i}{2} e^{(2-3i)t}\begin{pmatrix} 1 \\ i \end{pmatrix}$$

3. (5×2 pont)
Ird at az $y'' + y' + y = 0$ DE-t az uj

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} y \\ y' \end{pmatrix}$$

valtozok segitsegevel egy elsorendu $\frac{d}{dt}\bar{y} = A\bar{y}$ DE rendszerre! Mennyi A ?

$$\frac{d}{dt}\begin{pmatrix} y \\ y' \end{pmatrix} = \begin{pmatrix} y' \\ y'' \end{pmatrix} = \begin{pmatrix} y' \\ -y'-y \end{pmatrix} \Longleftrightarrow \frac{d}{dt}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

Melyek A sajatertekeik?

$$0 = \begin{vmatrix} 0-\lambda & 1 \\ -1 & -1-\lambda \end{vmatrix} = (0-\lambda)(-1-\lambda)+1 \Longrightarrow \lambda_{1,2} = \frac{-1 \pm \sqrt{1-4}}{2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$$

Ird fel az eredeti masodrendu DE altalanos megoldasat!

$$y = C_1 e^{\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)t} + C_2 e^{\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)t} = e^{-\frac{1}{2}t}\left(\tilde{C}_1 \cos\frac{\sqrt{3}}{2}t + \tilde{C}_2 \sin\frac{\sqrt{3}}{2}t\right)$$

Ird at a kovetkezo DE rendszert elsorendu DE rendszerre!

$$\frac{d^2}{dx^2}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} y_1^2 \\ y_2' - y_1' \end{pmatrix}$$

$$\frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \\ v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ y_1^2 \\ v_2 - v_1 \end{pmatrix}$$

Ird fel $e^{2-i\pi/6}$ algebrai alakjat!

$$e^{2-i\pi/6} = e^2\left(\cos\left(-\frac{\pi}{6}\right) + i \sin\left(-\frac{\pi}{6}\right)\right)$$

$$= e^2\left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right)$$

4.a (1+1+1+2 pont)
$y' = \cancel{\frac{1}{1+x^2}} - 1/2.$ $\quad y' = \frac{1}{1+y^2} - \frac{1}{2}$
Keresd meg a DE fixpontjait!

$$y' = 0 = \frac{1}{1+y^2} - \frac{1}{2} \implies y_1 = 1, \quad y_2 = -1$$

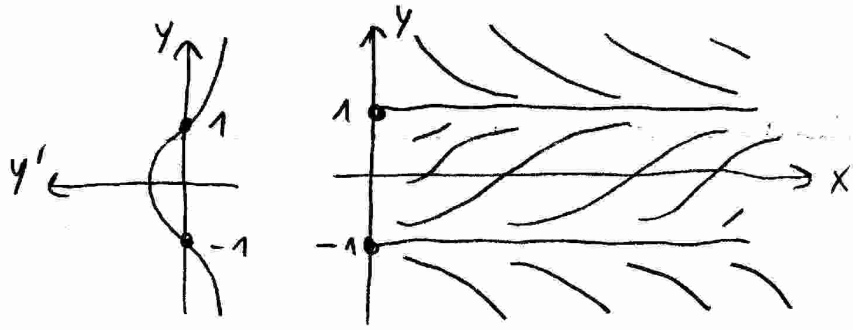
Ird fel a fixpontok koruli linearizalt kozelito DE-t!

$$\frac{d}{dy}\left(\frac{1}{1+y^2} - \frac{1}{2}\right) = -\frac{2y}{(1+y^2)^2} = f'(y)$$

$$f'(1) = -\frac{1}{2}, \quad \frac{d}{dt}(y-1) = \frac{d}{dt}\Delta y \approx -\frac{1}{2}\Delta y \quad \Big| \quad f'(-1) = \frac{1}{2}, \quad \frac{d}{dt}(y-(-1)) = \frac{d}{dt}\Delta y \approx \frac{1}{2}\Delta y$$

Ha $y(0) = 0$, mennyi
$\lim_{x\to\infty} y(x) =$ 1 $\qquad \lim_{x\to-\infty} y(x) =$ -1

Vazold a DE megoldasgorbeit!



4b. (2+3 pont) Mennyi

$$\exp\left[t\begin{pmatrix}-5 & 3\\ 0 & -5\end{pmatrix}\right]$$

$$= e^{t\begin{pmatrix}-5 & 0\\ 0 & -5\end{pmatrix} + t\begin{pmatrix}0 & 3\\ 0 & 0\end{pmatrix}} = e^{t\begin{pmatrix}-5 & 0\\ 0 & -5\end{pmatrix}} \cdot e^{t\begin{pmatrix}0 & 3\\ 0 & 0\end{pmatrix}}$$

nulla

$$= \begin{pmatrix}e^{-5t} & 0\\ 0 & e^{-5t}\end{pmatrix} \cdot \left[\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \begin{pmatrix}0 & 3t\\ 0 & 0\end{pmatrix} + \frac{1}{2!}\begin{pmatrix}0 & 3t\\ 0 & 0\end{pmatrix}^2 + \dots\right]$$

$$= \begin{pmatrix}e^{-5t} & 0\\ 0 & e^{-5t}\end{pmatrix}\begin{pmatrix}1 & 3t\\ 0 & 1\end{pmatrix} = \begin{pmatrix}e^{-5t} & 3te^{-5t}\\ 0 & e^{-5t}\end{pmatrix}$$

4. (3+2+3+2 pont)
a)

$$y'(x) = f(x, y) = 5 - x - y;$$

Mennyi y'' ? Ird fel y masodrendu Talor polinomjat az $x = 0$ pont korul, ha $y(0) = 3$!

$$y'' = \left(\frac{\partial}{\partial x} + f \cdot \frac{\partial}{\partial y}\right) f = \left(\frac{\partial}{\partial x} + (5-x-y)\frac{\partial}{\partial y}\right)(5-x-y) =$$

$$= -1 + (5-x-y)(-1) = -6 + x + y$$

$$y(0) = 3, \quad y'(0) = 5 - 0 - 3 = 2, \quad y''(0) = -6 + 0 + 3 = -3$$

$$y(x) \approx 3 + 2x - \frac{3}{2!} x^2$$

b) Alkalmazd az Euler, illetve a Heun modszert a kovetkezo DE-re $\Delta x = 0.01$ lepeskozzel!

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} y_1 \\ x + y_2 + 1. \end{pmatrix}, \qquad \bar{y}(2) = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

Mit josol a ket modszer $\bar{y}(2.01)$-re?
Euler:

$$y(2.01) \approx \begin{pmatrix} 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 2 \\ 2 + 3 + 1 \end{pmatrix} \cdot 0.01 = \begin{pmatrix} 2.02 \\ 3.06 \end{pmatrix}$$

Heun:

$$y(2.01) \approx \begin{pmatrix} 2 \\ 3 \end{pmatrix} + \frac{1}{2} \cdot 0.01 \cdot \left(\begin{pmatrix} 2 \\ 2+3+1 \end{pmatrix} + \begin{pmatrix} 2.02 \\ 2.01 + 3.06 + 1 \end{pmatrix} \right)$$

c) $x_0 = 0$, $x_{n+1} = 7x_n + 4$, Mennyi x_n?

fixpont: $7x_f + 4 = x_f \longrightarrow x_f = -\frac{2}{3}$

$$x_n = 7^n \cdot \left(0 - \left(-\frac{2}{3}\right)\right) + \left(-\frac{2}{3}\right) = 7^n \cdot \frac{2}{3} - \frac{2}{3}$$