Zh1, Diff.Egy., 2014.03.25. NEPTUN: Gyak.Vez.:

Név: Aláírás:

1. ($5 \times 2$ pont)

1. Legyen $u = (1-2i, 4i)^T$, $v = (5+6i, 7)^T$. Mennyi az $(u, v)$ belso szorzat?

$$= \overline{(1-2i)}(5+6i) + \overline{4i}\cdot 7 = (1+2i)(5+6i) - 4i\cdot 7 = (-7+16i) - 28i$$
$$= -7 - 12i$$

2. Legyen $f_1 = (\cos 45°, \sin, 45°)^T$, $f_2 = (\cos 135°, \sin, 135°)^T$ egy ortonormalt bazis. Fejezd ki $v = (7, 8)^T$ vektort az $f$-ek linearis $\alpha f_1 + \beta f_2$ kombinaciojakent! Mennyi $\alpha$ es $\beta$ ?

$$\alpha = (f_1, v) = \frac{1}{\sqrt{2}}\cdot 7 + \frac{1}{\sqrt{2}}\cdot 8 = \frac{15}{\sqrt{2}}$$

$$\beta = (f_2, v) = \left(-\frac{1}{\sqrt{2}}\right)\cdot 7 + \frac{1}{\sqrt{2}} 8 = \frac{1}{\sqrt{2}}$$

3. $y' = \sin(t^2)$, $y(3) = 5$. Fejezd ki $y(7)$-et a hatarozott integralas segitsegevel!

$$y(7) = 5 + \int_3^7 \sin(t^2)\,dt$$

4. Adott $f(x+\Delta x)$, $f(x)$, $f(x-\Delta x)$. Ird fel $f''(x)$ numerikus approximaciojat!

$$f''(x) = \frac{\frac{f(x+\Delta x) - f(x)}{\Delta x} - \frac{f(x) - f(x-\Delta x)}{\Delta x}}{\Delta x} = \frac{1}{\Delta x^2}\left(f(x+\Delta x) - 2f(x) + f(x-\Delta x)\right)$$

5. Legyen $f(x) = H(x) = \sum_{n\in\mathbb{Z}} \hat{f}_n \frac{e^{inx}}{\sqrt{2\pi}}$ , ha $x \in (-\pi, \pi)$ Mennyi $\hat{f}_{-5}$ es $\hat{f}_4$?

$$\hat{f}_{-5} = \left(\frac{e^{-i5x}}{\sqrt{2\pi}}, H(x)\right) = \int_{-\pi}^{\pi} \frac{e^{i5x}}{\sqrt{2\pi}} H(x)\,dx = \int_0^{\pi} \frac{e^{i5x}}{\sqrt{2\pi}}\,dx$$
$$= \frac{e^{i5x}}{\sqrt{2\pi}\cdot i\cdot 5}\Bigg|_0^{\pi} = \frac{1}{\sqrt{2\pi}\cdot i\cdot 5}\left((-1) - 1\right) = +i\frac{\sqrt{2}}{\sqrt{\pi}\cdot 5}$$

$$\hat{f}_4 = \frac{e^{-i4x}}{-\sqrt{2\pi}\cdot i\cdot 4}\Bigg|_0^{\pi} = 0$$

2. (5+2+3 pont)

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} y_1 \\ y_1 + 2y_2 \end{pmatrix} = A \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \qquad \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

Keresd meg $A$ sajatertekeit es sajatvektorait!

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} \quad \det(A - \lambda E) = \begin{vmatrix} 1-\lambda & 0 \\ 1 & 2-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda) - 0 \cdot 1 = 0$$

$\lambda_1 = 1$

$1-1$

$$\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$x = -y$

$$v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$\lambda_2 = 2$

$$\begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$x = 0$

$$v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Ird fel a DE altalanos megoldasat!

$$y(t) = c_1 \cdot e^{t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Szamold ki a DE partikularis megoldasat!

$$c_1 e^{1 \cdot 0} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{2 \cdot 0} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$c_1 = 1 \qquad c_2 = 4$$

$$y_{part}(t) = e^{t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + 4 e^{2t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

(5 × 2 pont)

Ird fel, hogy milyen osszefugges van $A$ es a sajatertekeket tartalmazo diagonalis $D$ matrixok kozott!

$$D = S^{-1} A S$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} \left( \underset{V_1}{\boxed{\begin{matrix} 1 \\ -1 \end{matrix}}} \underset{V_2}{\boxed{\begin{matrix} 0 \\ 1 \end{matrix}}} \right)$$

Vagy

$$\begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}^{-1} \qquad A = S D S^{-1}$$

Mennyi $e^{xA}$? (Elegendo az, hogy kifejezed az eredmenyt $D$, illetve egy $S$ matrix es annak inverze szorzatakent!)

$$e^{xA} = S e^{xD} S^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} e^{x} & 0 \\ 0 & e^{2x} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}^{-1}$$

Ird fel a partikularis megoldast $e^{xA}$ segitsegevel!

$$y_{part}(x) = e^{xA} y(0) = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} e^{x} & 0 \\ 0 & e^{2x} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

2b) Ird at a kovetkezo DE rendszert elsorendu DE rendszerre!

$$\frac{d^2}{dx^2} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 4y_1' - y_2^2 \\ y_2' - y_1' \end{pmatrix}$$

$$V_1 = y_1' \qquad V_2 = y_2'$$

$$\frac{d}{dx} \begin{pmatrix} y_1 \\ y_2 \\ V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} V_1 \\ V_2 \\ 4V_1 - y_2^2 \\ V_2 - V_1 \end{pmatrix}$$

Ird fel $e^{2-i\pi/4}$ algebrai alakjat!

$$e^{2 - i\frac{\pi}{4}} = e^{2} e^{-\frac{\pi}{4} i} = e^{2}\left( \cos\left(-\frac{\pi}{4}\right) + i \sin\left(-\frac{\pi}{4}\right) \right) =$$

$$= e^{2}\left( \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} i \right)$$

3a. (1+1+1+2 pont)
$y' = (y^2 - 1)(y - 2)$. $= y^3 - 2y^2 - y + 2$
Keresd meg a DE fixpontjait!

$y' = 0 \longrightarrow y_1 = 1 \qquad y_2 = -1 \qquad y_3 = 2$

Ird fel a fixpontok koruli linearizalt kozelito DE-t!

$$\frac{d(y^2-1)(y-2)}{dy} = 3y^2 - 4y - 1$$

$\Delta y_1 = y - 1$

$(\Delta y_1)' = (3 \cdot 1^2 - 4 \cdot 1 - 1)\Delta y_1 = -2\Delta y_1$

$\Delta y_2 = y - (-1)$

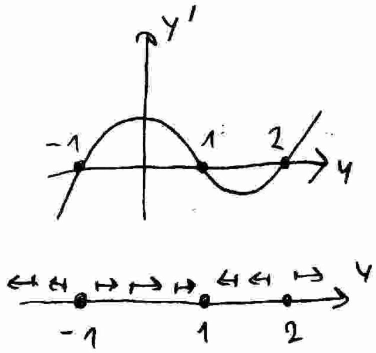
$(\Delta y_2)' = 6\Delta y_2$

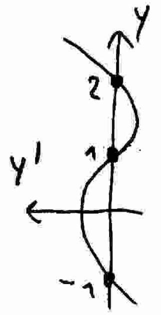
$\Delta y_3 = y - 2$

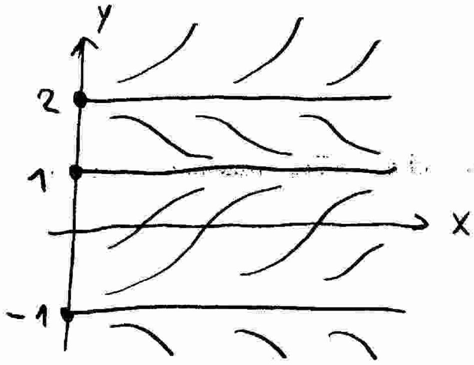
$(\Delta y_3)' = +3\Delta y_3$

Ha $y(0) = 0$, mennyi
$\lim_{x\to\infty} y(x) =$ $1$ $\qquad \lim_{x\to-\infty} y(x) =$ $-1$

Vazold a DE megoldasgorbeit!




3b. (2+3 pont) Mennyi

$$\exp\left[t\begin{pmatrix}5 & 6\\0 & 5\end{pmatrix}\right]$$

Mivel $\begin{pmatrix}5t & 0\\0 & 5t\end{pmatrix}\begin{pmatrix}0 & 6t\\0 & 0\end{pmatrix} = \begin{pmatrix}0 & 6t\\0 & 0\end{pmatrix}\begin{pmatrix}5t & 0\\0 & 5t\end{pmatrix}$

$$= \exp\left[\begin{pmatrix}5t & 0\\0 & 5t\end{pmatrix} + \begin{pmatrix}0 & 6t\\0 & 0\end{pmatrix}\right] = \exp\begin{pmatrix}5t & 0\\0 & 5t\end{pmatrix} \cdot \exp\begin{pmatrix}0 & 6t\\0 & 0\end{pmatrix} =$$

$$= \begin{pmatrix}e^{5t} & 0\\0 & e^{5t}\end{pmatrix}\left[\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} + \begin{pmatrix}0 & 6t\\0 & 0\end{pmatrix} + \frac{1}{2!}\underbrace{\begin{pmatrix}0 & 6t\\0 & 0\end{pmatrix}^2}_{0} + \ldots\right] =$$

$$= \begin{pmatrix}e^{5t} & 0\\0 & e^{5t}\end{pmatrix}\begin{pmatrix}1 & 6t\\0 & 0\end{pmatrix} = \begin{pmatrix}e^{5t} & 6te^{5t}\\0 & e^{5t}\end{pmatrix}$$

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1. (5 × 2 pont)

1. Legyen $u = (1-2i, 2-4i)^T$, $v = (5-6i, 7)^T$. Mennyi az $(u, v)$ belso szorzat?

$$= \overline{(1-2i)}(5-6i) + \overline{(2-4i)}\cdot 7 = (1+2i)(5-6i) + (2+4i)\cdot 7$$
$$= (17+4i) + (14+28i) = 31 + 32i$$

2. Legyen $f_1 = (\cos 30°, \sin 30°)^T$, $f_2 = (\cos 120°, \sin 120°)^T$ egy ortonormalt bazis. Fejezd ki $v = (7, 8)^T$ vektort az $f$-ek linearis $\alpha f_1 + \beta f_2$ kombinaciojakent! Mennyi $\alpha$ es $\beta$ ?

$$\alpha = (f_1, v) = \frac{\sqrt{3}}{2}\cdot 7 + \frac{1}{2}\cdot 8$$
$$\beta = (f_2, v) = \left(-\frac{1}{2}\right)\cdot 7 + \frac{\sqrt{3}}{2}\cdot 8$$

3. $y' = \cos(t^2)$, $y(3) = 5$. Fejezd ki $y(7)$-et a hatarozott integralas segitsegevel!

$$y(7) = 5 + \int_3^7 \cos(t^2)\,dt$$

4. Adott $f(x + \Delta x)$, $f(x)$, $f(x - \Delta x)$. Ird fel $f''(x)$ numerikus approximaciojat!

$$f''(x) \approx \frac{\frac{f(x+\Delta x) - f(x)}{\Delta x} - \frac{f(x) - f(x-\Delta x)}{\Delta x}}{\Delta x} = \frac{1}{\Delta x^2}\left(f(x+\Delta x) - 2f(x) + f(x-\Delta x)\right)$$

5. Legyen $f(x) = H(-x) = \sum_{n\in\mathbb{Z}} \hat{f}_n \frac{e^{inx}}{\sqrt{2\pi}}$ , ha $x \in (-\pi, \pi)$ Mennyi $\hat{f}_3$ es $\hat{f}_4$?

$$\hat{f}_3 = \left(\frac{e^{i3x}}{\sqrt{2\pi}}, H(-x)\right) = \int_{-\pi}^{\pi} \frac{e^{-i3x}}{\sqrt{2\pi}} H(-x)\,dx = \int_{-\pi}^{0} \frac{e^{-i3x}}{\sqrt{2\pi}}\,dx =$$
$$= \left.\frac{e^{-i3x}}{-\sqrt{2\pi}\cdot i\cdot 3}\right|_{-\pi}^{0} = \frac{\sqrt{2}}{-\sqrt{\pi}\, i\cdot 3} = +i\,\frac{\sqrt{2}}{3\sqrt{\pi}}$$

$$\hat{f}_4 = \left.\frac{e^{-i4x}}{-\sqrt{2\pi}\cdot i\cdot 4}\right|_{-\pi}^{0} = 0$$

2. (5+2+3 pont)

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 2y_1 \\ y_1 + 3y_2 \end{pmatrix} = A \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \quad \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

Keresd meg $A$ sajatertekeit es sajatvektorait!

$$A = \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}, \quad \det(A - \lambda E) = \begin{vmatrix} 2-\lambda & 0 \\ 1 & 3-\lambda \end{vmatrix} = (2-\lambda)(3-\lambda) - 0 \cdot 1 = 0$$

$$\lambda_1 = 2 \qquad\qquad \lambda_2 = 3$$

$$2-2 \quad \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad 2-3 \quad \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$x = -y \qquad\qquad x = 0$$

$$V_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad\qquad V_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Ird fel a DE altalanos megoldasat!

$$y(t) = C_1 e^{2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + C_2 e^{3t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Szamold ki a DE partikularis megoldasat!

$$C_1 e^{2\cdot 0} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + C_2 e^{3\cdot 0} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$C_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + C_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$C_1 = 1, \quad C_2 = 4$$

$$y_{part}(t) = e^{2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + 4 e^{3t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

(5 × 2 pont)
Ird fel, hogy milyen osszefugges van $A$ es a sajatertekeket tartalmazo diagonalis $D$ matrixok kozott!

$$\cdot \quad D = S^{-1} A S \qquad V_1 \quad V_2$$

$$\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} \boxed{\begin{matrix} 1 \\ -1 \end{matrix}} & \boxed{\begin{matrix} 0 \\ 1 \end{matrix}} \end{pmatrix}$$

vagy

$$\begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}^{-1} \qquad A = SDS^{-1}$$

Mennyi $e^{xA}$? (Elegendo az, hogy kifejezed az eredmenyt $D$, illetve egy $S$ matrix es annak inverze szorzatakent!)

$$e^{xA} = S e^{xD} S^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} e^{2x} & 0 \\ 0 & e^{3x} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}^{-1}$$

Ird fel a partikularis megoldast $e^{xA}$ segitsegevel!

$$y_{part}(x) = e^{xA} y(0) = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} e^{2x} & 0 \\ 0 & e^{3x} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

2b) Ird at a kovetkezo DE rendszert elsorendu DE rendszerre!

$$\frac{d^2}{dx^2} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} y_1' + y_2^2 \\ y_2' - y_1' \end{pmatrix}$$

$$V_1 = y_1', \quad V_2 = y_2'$$

$$\frac{d}{dx} \begin{pmatrix} y_1 \\ y_2 \\ V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} V_1 \\ V_2 \\ V_1 + y_2^2 \\ V_2 - V_1 \end{pmatrix}$$

Ird fel $e^{2-i\pi/6}$ algebrai alakjat!

$$e^{2-i\pi/6} = e^2 e^{-\frac{\pi}{6} i} = e^2 \left( \cos\left(-\frac{\pi}{6}\right) + i \cdot \sin\left(-\frac{\pi}{6}\right) \right)$$

$$= e^2 \left( \frac{\sqrt{3}}{2} - \frac{1}{2} i \right)$$

3a. (1+1+1+2 pont)

$y' = (y^2-1)(2-y). = -y^3+2y^2+y-2$

Keresd meg a DE fixpontjait!

$y'=0 \longrightarrow y_1 = 1 \qquad y_2 = -1, \qquad y_3 = 2$

Ird fel a fixpontok koruli linearizalt kozelito DE-t!

$$\frac{d(y^2-1)(2-y)}{dy} = -3y^2+4y+1$$

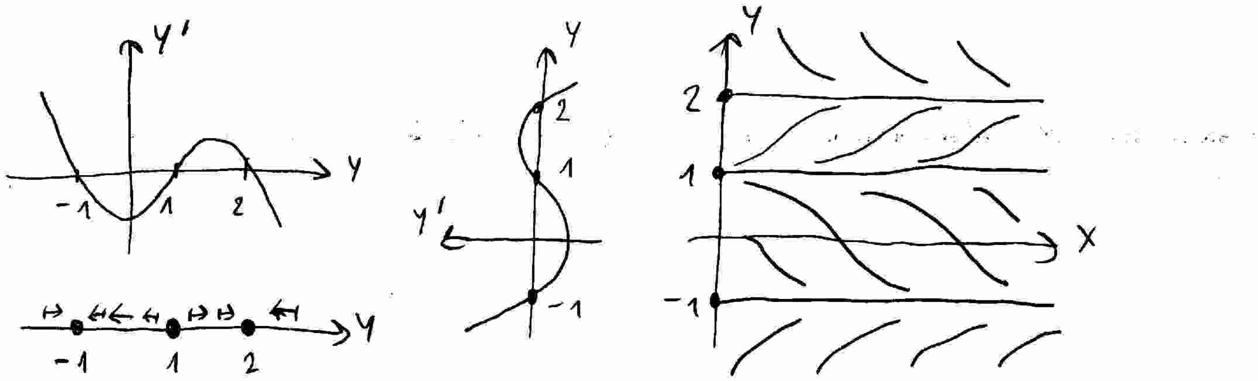
$\Delta y_1 = y-1$ | $\Delta y_2 = y-(-1)$ | $\Delta y_3 = y-2$

$(\Delta y_1)' = (-3\cdot 1^2+4\cdot 1+1)\Delta y_1 = 2\Delta y_1$ | $(\Delta y_2)' = -6\Delta y_1$ | $(\Delta y_3)' = -3\Delta y_3$

Ha $y(0) = 0$, mennyi

$\lim_{x\to\infty} y(x) = -1$ $\qquad$ $\lim_{x\to-\infty} y(x) = 1$

Vazold a DE megoldasgorbeit!




3b. (2+3 pont) Mennyi

$$\exp\left[t\begin{pmatrix}5 & 0\\ 3 & 5\end{pmatrix}\right]$$

Mivel $\begin{pmatrix}5t & 0\\ 0 & 5t\end{pmatrix}\begin{pmatrix}0 & 0\\ 3t & 0\end{pmatrix} = \begin{pmatrix}0 & 0\\ 3t & 0\end{pmatrix}\begin{pmatrix}5t & 0\\ 0 & 5t\end{pmatrix}$

$$= \exp\left[\begin{pmatrix}5t & 0\\ 0 & 5t\end{pmatrix} + \begin{pmatrix}0 & 0\\ 3t & 0\end{pmatrix}\right] = \exp\begin{pmatrix}5t & 0\\ 0 & 5t\end{pmatrix}\cdot\exp\begin{pmatrix}0 & 0\\ 3t & 0\end{pmatrix}$$

$$= \begin{pmatrix}e^{5t} & 0\\ 0 & e^{5t}\end{pmatrix}\left[\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \begin{pmatrix}0 & 0\\ 3t & 0\end{pmatrix} + \frac{1}{2!}\underbrace{\begin{pmatrix}0 & 0\\ 3t & 0\end{pmatrix}}_{=0} + \dots\right]$$

$$= \begin{pmatrix}e^{5t} & 0\\ 0 & e^{5t}\end{pmatrix}\begin{pmatrix}1 & 0\\ 3t & 1\end{pmatrix} = \begin{pmatrix}e^{5t} & 0\\ 3te^{5t} & e^{5t}\end{pmatrix}$$