A. Zh2, Diff.Egy., 2014.05.06.   NEPTUN: Gyak.Vez.:

Név: Aláírás:

1. (3+2+2+3 pont)
Ird fel a $y'' = \lambda y$ DE megoldasait az $y(0) = 0$, $y(L) = 0$ feltetelek mellett y-ra es λ-ra!

λ valós, és $\lambda \le 0$. $\Lambda = \sqrt{-\lambda}$, $y = C_1 \cos \Lambda x + C_2 \sin \Lambda x$

$y(0) \Rightarrow C_1 = 0$, $y(L) = 0 \Rightarrow \Lambda L = k\pi$, $k \in \mathbb{Z}$. $\Rightarrow \Lambda = \frac{k\pi}{L}$

$k = 1, 2, 3 \dots$ $\lambda_k = -\left(\frac{k\pi}{L}\right)^2$, $y_k = \sin \frac{k\pi x}{L}$ vagy $\frac{\sqrt{2}}{\sqrt{L}} \sin \frac{k\pi x}{L}$

normalizálás

Legyen $f(x) = 1$, ha $x \in [0, 1]$, maskulonben $f = 0$. Ha $f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(p) e^{ipx}\, dp$, akkor mennyi $\hat{f}(3)$?

$$\tilde{f}(3) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i3x} \cdot f(x)\, dx = \frac{1}{\sqrt{2\pi}} \int_0^1 e^{-i3x}\, dx = \frac{1}{\sqrt{2\pi}} \left. \frac{e^{-i3x}}{-3i} \right|_0^1$$

$$= \frac{1}{\sqrt{2\pi}} \frac{i}{3} \left(e^{-3i} - 1 \right)$$

$\phi_t(t, x, y) = \phi_{xx}(t, x, y) + \phi_{yy}(t, x, y)$, $y(0, x) = \exp(i(5x + 7y)) + \sin(2x)$. Mennyi $\varphi(t, x, y)$?

Mivel $(\partial_{xx} + \partial_{yy})\, e^{i(5x+7y)} = -(5^2 + 7^2)\, e^{i(5x+7y)}$

$(\partial_{xx} + \partial_{yy}) \sin 2x = -2^2 \sin 2x$, így

$$\varphi(t, x, y) = e^{-\underset{74}{\underbrace{(5^2+7^2)}}t}\, e^{i(5x+7y)} + e^{-2^2} \sin 2x$$

$\phi_{tt}(t, x) = \phi_{xx}(t, x)$, $\phi(0, x) = \sin(3x)$, $\varphi_t(0,x) = \sin(4x)$. Mennyi $\phi(t, x)$?

$$\varphi(t, x) = \sin(3x) \cos(3t) + \sin(4x) \cdot \frac{\sin(4t)}{4}$$

Ha $\varphi = \sin(3x) \cdot C_1(t)$, akkor $\varphi_{tt} = C_1'' \cdot \sin 3x = \varphi_{xx} = -9 \sin 3x \cdot C_1$

tehát $C_1'' = -9C_1$, $C_1(0) = 1$, $C_1'(0) = 0 \Rightarrow C_1 = \cos(3t)$

Ha $\varphi = \sin(4x) \cdot C_2(t)$, akkor $\varphi_{tt} = C_2'' \cdot \sin 4x = \varphi_{xx} = -16 \sin 4x \cdot C_2$

tehát $C_2'' = -16 C_2$, $C_2(0) = 0$, $C_2'(0) = 1 \Rightarrow C_2 = \frac{1}{4} \sin(4t)$

2. (2+1+1+4+2 pont)
Keresd meg a kovetkezo DE altalanos megoldasat! $y' = -\delta(x) - 1$.

$$y = -x + C - H(x) = \begin{cases} -x + C & \text{ha } x < 0 \\ -x + C - 1 & \text{ha } x > 0 \end{cases}$$

Keresd meg a kovetkezo DE megoldasat az ($y(x) = 0$, ha $x < 0$) feltetel mellett! $y'' = 2\delta(x)$.

$$y(x) = \begin{cases} 0, & \text{ha } x < 0 \\ 2x, & \text{ha } x > 0 \end{cases}$$

Keresd meg a kovetkezo DE megoldasat a ($G(x) = 0$, ha $x < 0$) feltetel mellett!
$G'' + 16G = \delta(x)$.
Add meg a kovetkezo mennyisegeket!
$G(0^+) - G(0^-) =$ 0 $G'(0^+) - G'(0^-) =$ 1

Mennyi $G(x)$? $G(x) = 0$, ha $x < 0$

$G(0^+) = 0$, $G'(0^+) = 1$, $G'' + 16G = 0$, ha $x > 0$

Az $y(0) = 0$, $y'(0) = 1$, $y'' + 16y = 0$ megoldása: $y = \frac{1}{4}\sin 4x$.

Tehát

$$G(x) = \begin{cases} 0, & \text{ha } x < 0 \\ \frac{1}{4}\sin 4x, & \text{ha } x > 0 \end{cases}$$

Ird fel az $y'' + 16y = f(x)$ DE megoldasat, ha ($y(x) = f(x) = 0$, ha $x \ll 0$).

$$y(x) = \int_{-\infty}^{\infty} G(x-z) f(z)\, dz =$$

$$= \int_{-\infty}^{x} \frac{1}{4} \sin(4(x-z)) f(z)\, dz$$

3. (2+2+3+2+1 pont)
Szamold ki az $f = 3H(t-5)$ fuggveny Laplace transzformaltjat a definicio alapjan!

$$\mathcal{L}(f) = \int_0^\infty e^{-st}\, 3H(t-5)\,dt = \int_5^\infty e^{-st}\cdot 3 = \frac{3e^{-st}}{-s}\Big|_5^\infty =$$

$$= 0 - \left(\frac{3e^{-s\cdot 5}}{-s}\right) = \frac{3\cdot e^{-5s}}{s}$$

a Laplac tr. féle értelemben.

Szamold ki az alabbi fuggvenypar $f*g$ konvoluciojat! $f(t) = 1$, $g(t) = \cos 5t$

$$(f*g)(t) = \int_0^t f(t-\tau)\,g(\tau)\,d\tau = \int_0^t 1\cdot\cos(5\tau)\,d\tau =$$

$$= \frac{\sin(5\tau)}{5}\Big|_{\tau=0}^{t} = \frac{\sin(5t)}{5} - \frac{\sin(5\cdot 0)}{5} = \frac{\sin(5t)}{5}$$

Legyen $y'' + 16y = \underbrace{(t-1)^2}_{t^2-2t+1}$, $y(0) = 3$, $y'(0) = 4$. Mennyi $\mathcal{L}(y(t)) = Y(s)$? ($\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}$.)

$$(s^2Y(s) - s\cdot 3 - 4) + 16Y(s) = \frac{2}{s^3} - 2\cdot\frac{1}{s^2} + \frac{1}{s}$$

$$Y(s) = \frac{1}{\underbrace{s^2+16}_{(s+4i)(s-4i)}}\left(3s+4+\left[\frac{2}{s^3} - \frac{2}{s^2} + \frac{1}{s}\right]\right)$$

Ird fel azt, hogy hogyan nez ki $Y(s)$ parcialis tort felbontasa! (Az egyutthatokat nem kell kiszamolni!)

$$Y(s) = \frac{A}{s+4i} + \frac{B}{s-4i} + \frac{C}{s^3} + \frac{D}{s^2} + \frac{E}{s}$$

Mennyi $y(t)$? ($\mathcal{L}(e^{at}) = \frac{1}{s-a}$.)

$$y(t) = A\,e^{-4it} + B\,e^{4it} + \frac{C}{2}t^2 + Dt + E$$

4. (3+2+3+2 pont)
a)

$$y' = f(x,y) = 1 + 5xy;$$

Mennyi y'' ? Ird fel y masodrendu Talor polinomjat az $x = 0$ pont korul, ha $y(0) = 3$!

$$y'' = \frac{d}{dx}y' = \frac{d}{dx}f(x, y(x)) = \left(\frac{\partial}{\partial x} + y'\frac{\partial}{\partial y}\right)f = \left(\frac{\partial}{\partial x} + f\frac{\partial}{\partial y}\right)f$$

$$= (\partial_x + (1+5xy)\partial_y)(1+5xy) = 5y + (1+5xy)\cdot 5x = 5(x+y) + 25x^2y$$

$y(0) = 3$

$y'(0) = 1 + 5\cdot 0\cdot 3 = 1$

$y''(0) = 5(0+3) + 25\cdot 0^2\cdot 3 = 15$

$x \approx 0 \qquad y(x) \approx 3 + 1x + \frac{15}{2}x^2$

b) Alkalmazd az Euler, illetve a Heun modszert a kovetkezo DE-re $\Delta x = 0.01$ lepeskozzel!

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} y_2 \\ x + y_1 y_2. \end{pmatrix}, \qquad \bar{y}(2) = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

Mit josol a ket modszer $\bar{y}(2.01)$-re?
Euler:

$$y(2.01) \approx \begin{pmatrix} 2 \\ 3 \end{pmatrix} + 0.01\begin{pmatrix} 3 \\ 2+2\cdot 3 \end{pmatrix} = \begin{pmatrix} 2.03 \\ 3.08 \end{pmatrix}$$

Heun:

$$y(2.01) \approx \begin{pmatrix} 2 \\ 3 \end{pmatrix} + \frac{1}{2}\left[\begin{pmatrix} 3 \\ 2+2\cdot 3 \end{pmatrix} + \begin{pmatrix} 3.08 \\ 2.01 + 2.03\cdot 3.08 \end{pmatrix}\right]\cdot 0.01$$

c) $x_0 = 5, x_{n+1} = 3x_n + 6$, Mennyi x_n?

fixpont: $x_f = 3x_f + 6 \Rightarrow x_f = -3$

$x_n = 3^n(5-(-3)) + (-3) = 3^n\cdot 8 - 3$