Pot.Zh1, Diff.Egy., 2014.05.06. NEPTUN: Gyak.Vez.:

Név: Aláírás:

1. (5 × 2 pont)

1. Legyen $u = (2i, 4i)^T$, $v = (5 - 6i, 7)^T$. Mennyi az (u, v) belso szorzat?

$$= \overline{2i} \cdot (5-6i) + \overline{4i} \cdot 7 = (-2i)\cdot(5-6i) - 4i \cdot 7$$
$$= -10i - 12 - 28i = -38i - 12$$

2. Legyen $f_1 = (\cos 135°, \sin 135°)^T$, $f_2 = (\cos 225°, \sin 225°)^T$ egy ortonormalt bazis. Fejezd ki $v = (7, 8)^T$ vektort az f-ek linearis $\alpha f_1 + \beta f_2$ kombinaciojakent! Mennyi α es β ?

$$\alpha = (f_1, v) = \left(\begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \begin{pmatrix} 7 \\ 8 \end{pmatrix} \right) = -\frac{1}{\sqrt{2}} \cdot 7 + \frac{1}{\sqrt{2}} \cdot 8 = \frac{1}{\sqrt{2}}$$

$$\beta = (f_2, v) = \left(\begin{pmatrix} -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}, \begin{pmatrix} 7 \\ 8 \end{pmatrix} \right) = -\frac{1}{\sqrt{2}} \cdot 7 - \frac{1}{\sqrt{2}} \cdot 8 = -\frac{15}{\sqrt{2}}$$

3. $y' = t^2 e^t$, $y(3) = 5$. Fejezd ki $y(77)$-et a hatarozott integralas segitsegevel!

$$y(77) = 5 + \int_3^{77} t^2 e^t \, dt$$

4. Adott $f(x + \Delta x)$, $f(x)$, $f(x - \Delta x)$. Ird fel $f''(x)$ numerikus approximaciojat!

$$f''(x) \approx \frac{1}{\Delta x^2} \left(f(x+\Delta x) - 2f(x) + f(x-\Delta x) \right)$$

5. Legyen $f(x) = x = \sum_{n \in \mathbf{Z}} \hat{f}_n \frac{e^{inx}}{\sqrt{2\pi}}$, ha $x \in (-\pi, \pi)$ Mennyi $\hat{f}_{-5}$?

$$\hat{f}_{-5} = \left(e^{-5ix}, f(x) \right) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \overline{e^{-5ix}} \cdot x \, dx = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} e^{5ix} x \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{5ix} \cdot x}{5i} - \frac{e^{5ix}}{(5i)^2} \right]_{-\pi}^{\pi} =$$

mivel $e^{\pm 5i\pi} = -1$

$$= \frac{1}{\sqrt{2\pi}} \left[\left(\frac{-1 \cdot \pi}{5i} - \frac{-1}{-25} \right) - \left(\frac{-1 \cdot (-\pi)}{5i} - \frac{-1}{-25} \right) \right] = \frac{i}{5} \sqrt{\frac{2}{\pi}}$$

2. (5+2+3 pont)

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 2y_1 - 3y_2 \\ 3y_1 + 2y_2 \end{pmatrix} = A \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \quad \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

Keresd meg A sajatertekeit es sajatvektorait!

$$A = \begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix}, \quad \det(A - \lambda E) = \begin{vmatrix} 2-\lambda & -3 \\ 3 & 2-\lambda \end{vmatrix} = 0 = (2-\lambda)^2 - (-3)\cdot 3$$

$$\lambda_1 = 2+3i \quad \lambda_2 = 2-3i$$

$$\begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = (2+3i)\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -ix \end{pmatrix}$$

$$\lambda_1 = 2+3i, \quad V_1 = \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

$$\begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = (2-3i)\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ ix \end{pmatrix}$$

$$\lambda_2 = 2-3i, \quad V_2 = \begin{pmatrix} 1 \\ i \end{pmatrix}$$

Ird fel a DE altalanos megoldasat!

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = c_1 e^{(2+3i)t}\begin{pmatrix} 1 \\ -i \end{pmatrix} + c_2 e^{(2-3i)t}\begin{pmatrix} 1 \\ i \end{pmatrix}$$

Szamold ki a DE partikularis megoldasat!

$$\begin{pmatrix} 1 \\ 3 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ -i \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ i \end{pmatrix} = \begin{pmatrix} c_1 + c_2 \\ i(-c_1 + c_2) \end{pmatrix}$$

$$\left.\begin{aligned} c_1 + c_2 &= 1 \\ -c_1 + c_2 &= 3/i = -3i \end{aligned}\right\} \rightarrow c_2 = \frac{1-3i}{2}, \quad c_1 = \frac{1+3i}{2}$$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \frac{1-3i}{2} e^{(2+3i)t}\begin{pmatrix} 1 \\ -i \end{pmatrix} + \frac{1+3i}{2} e^{(2-3i)t}\begin{pmatrix} 1 \\ i \end{pmatrix}$$

3. (5×2 pont)
Ird at az $y'' + y' + y = 0$ DE-t az uj

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} y \\ y' \end{pmatrix}$$

valtozok segitsegevel egy elsorendu $\frac{d}{dt}\bar{y} = A\bar{y}$ DE rendszerre! Mennyi A ?

$$\frac{d}{dt}\begin{pmatrix} y \\ y' \end{pmatrix} = \begin{pmatrix} y' \\ y'' \end{pmatrix} = \begin{pmatrix} y' \\ -y'-y \end{pmatrix} \Longleftrightarrow \frac{d}{dt}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

Melyek A sajatertekeik?

$$0 = \begin{vmatrix} 0-\lambda & 1 \\ -1 & -1-\lambda \end{vmatrix} = (0-\lambda)(-1-\lambda)+1 \Longrightarrow \lambda_{1,2} = \frac{-1 \pm \sqrt{1-4}}{2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$$

Ird fel az eredeti masodrendu DE altalanos megoldasat!

$$y = C_1 e^{\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)t} + C_2 e^{\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)t} = e^{-\frac{1}{2}t}\left(\tilde{C}_1 \cos\frac{\sqrt{3}}{2}t + \tilde{C}_2 \sin\frac{\sqrt{3}}{2}t\right)$$

Ird at a kovetkezo DE rendszert elsorendu DE rendszerre!

$$\frac{d^2}{dx^2}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} y_1^2 \\ y_2' - y_1' \end{pmatrix}$$

$$\frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \\ v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ y_1^2 \\ v_2 - v_1 \end{pmatrix}$$

Ird fel $e^{2-i\pi/6}$ algebrai alakjat!

$$e^{2-i\pi/6} = e^2\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right)$$

$$= e^2\left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right)$$

4.a (1+1+1+2 pont)

$y' = \cancel{\frac{1}{1+x^2}} - 1/2.$ $\quad y' = \frac{1}{1+y^2} - \frac{1}{2}$

Keresd meg a DE fixpôntjait!

$$y' = 0 = \frac{1}{1+y^2} - \frac{1}{2} \implies y_1 = 1, \quad y_2 = -1$$

Ird fel a fixpontok koruli linearizalt kozelito DE-t!

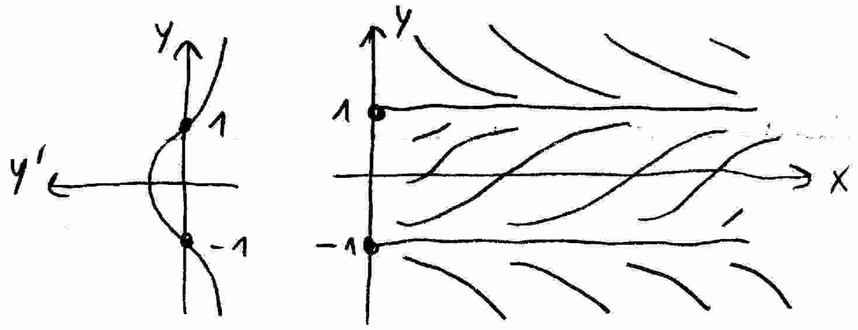
$$\frac{d}{dy}\left(\frac{1}{1+y^2} - \frac{1}{2}\right) = -\frac{2y}{(1+y^2)^2} = f'(y)$$

$$f'(1) = -\frac{1}{2}, \quad \frac{d}{dt}(y-1) = \frac{d}{dt}\Delta y \approx -\frac{1}{2}\Delta y \;\Big|\; f'(-1) = \frac{1}{2}, \quad \frac{d}{dt}(y-(-1)) = \frac{d}{dt}\Delta y \approx \frac{1}{2}\Delta y$$

Ha $y(0) = 0$, mennyi

$\lim_{x\to\infty} y(x) =$ 1 $\qquad$ $\lim_{x\to-\infty} y(x) =$ -1

Vazold a DE megoldasgorbeit!



4b. (2+3 pont) Mennyi

$$\exp\left[t\begin{pmatrix}-5 & 3\\ 0 & -5\end{pmatrix}\right]$$

$$= e^{t\begin{pmatrix}-5 & 0\\ 0 & -5\end{pmatrix} + t\begin{pmatrix}0 & 3\\ 0 & 0\end{pmatrix}} = e^{t\begin{pmatrix}-5 & 0\\ 0 & -5\end{pmatrix}} \cdot e^{t\begin{pmatrix}0 & 3\\ 0 & 0\end{pmatrix}}$$

nulla

$$= \begin{pmatrix}e^{-5t} & 0\\ 0 & e^{-5t}\end{pmatrix} \cdot \left[\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \begin{pmatrix}0 & 3t\\ 0 & 0\end{pmatrix} + \frac{1}{2!}\begin{pmatrix}0 & 3t\\ 0 & 0\end{pmatrix}^2 + \dots\right]$$

$$= \begin{pmatrix}e^{-5t} & 0\\ 0 & e^{-5t}\end{pmatrix}\begin{pmatrix}1 & 3t\\ 0 & 1\end{pmatrix} = \begin{pmatrix}e^{-5t} & 3te^{-5t}\\ 0 & e^{-5t}\end{pmatrix}$$

Pot.Zh2, Diff.Egy., 2014.05.15. NEPTUN: Gyak.Vez.:

Név: Aláírás:

1. (3+2+2+3 pont)
Ird fel a $y'' = \lambda y$ DE megoldasait az $y'(0) = 0$, $y'(L) = 0$ feltetelek mellett y-ra es λ-ra! (Hasznald azt a feltevest, hogy λ egy valos, nempozitiv szam!)

$$\lambda = -\omega^2, \quad y'' = -\omega^2 y \longrightarrow y = C_1 \cdot \cos(\omega x) + C_2 \sin(\omega x)$$

$$y' = -C_1 \omega \sin(\omega x) + C_2 \omega \cos(\omega x)$$

$k \in \mathbb{Z}$

$$y'(0) = 0 \longrightarrow C_2 = 0, \quad y'(L) = 0 \longrightarrow -C_1 \omega \sin(\omega L) = 0 \longrightarrow \omega = k \cdot \frac{\pi}{L}$$

Tehát

$$k = 0, 1, 2 \ldots \quad y_k = \cos\left(k \frac{\pi}{L} x\right), \quad \lambda_k = -\left(\frac{k\pi}{L}\right)^2$$

Legyen $f(x) = 3$, ha $x \in [-\pi, 0]$, maskulonben $f = 0$. Ha $f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(p) e^{ipx}\, dp$, akkor mennyi $\hat{f}(3)$?

$$\tilde{f}(3) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i3x} \cdot f(x)\, dx = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{0} e^{-i3x} \cdot 3\, dx =$$

$$= \frac{3}{\sqrt{2\pi}} \left[\frac{e^{-i3x}}{-i \cdot 3} \right]_{-\pi}^{0} = \frac{3}{\sqrt{2\pi}} \left(\frac{1}{-i \cdot 3} - \frac{-1}{-i \cdot 3} \right) = \sqrt{\frac{2}{\pi}} \cdot i$$

$\phi_t(t, x, y) = \phi_{xx}(t, x, y) + \phi_{yy}(t, x, y)$, $\varphi(0, x) = \exp(i(5x + 7y)) + \sin(2x)$. Mennyi $\varphi(t, x, y)$?

$$\varphi(t,x,y) = e^{-(5^2+7^2)t} \cdot e^{i(5x+7y)} + e^{-2^2 \cdot t} \sin(2x)$$

$$\text{Mivel } (\partial_{xx} + \partial_{yy}) e^{i(5x+7y)} = -(5^2+7^2) e^{i(5x+7y)}$$

$$(\partial_{xx} + \partial_{yy}) \sin(2x) = -2^2 \sin(2x)$$

$\phi_t(t, x, y) = \phi_{xx}(t, x, y)$, $\phi(0, x, y) = \sin(3x)\sin(4y)$. Mennyi $\cancel{\phi(t,x)}$? $\varphi(t,x,y)$

$$\varphi(t,x,y) = e^{-3^2 t} \sin(3x) \sin(4y)$$

$$\text{Mivel } \partial_{xx}\left(\sin(3x)\sin(4y)\right) = -3^2 \sin(3x) \sin(4y)$$

2.(2+1+1+4+2 pont)
Keresd meg a kovetkezo DE altalanos megoldasat! $y' = 3\delta(x) + 4$.

$$y(x) = \begin{cases} 4x + C, & \text{ha } x < 0 \\ 4x + C + 3, & \text{ha } x > 0 \end{cases}$$

Keresd meg a kovetkezo DE megoldasat az ($y(x) = 0$, ha $x < 0$) feltetel mellett! $y'' = \delta(x)$.

$$y(x) = \begin{cases} 0, & \text{ha } x < 0 \\ x, & \text{ha } x > 0 \end{cases}$$

Keresd meg a kovetkezo DE megoldasat a ($G(x) = 0$, ha $x < 0$) feltetel mellett!
$G'' - 16G = \delta(x)$.
Add meg a kovetkezo mennyisegeket!
$G(0^+) - G(0^-) =$ 0 $G'(0^+) - G'(0^-) =$ 1

Mennyi $G(x)$? Az $y'' - 16y = 0$, $y(0) = 0$, $y'(0) = 1$ DE megoldása: $\lambda^2 - 16 = 0 \longrightarrow \lambda_{1,2} = \pm 4$

$y = C_1 e^{4x} + C_2 e^{-4x}$, $y' = 4C_1 e^{4x} - 4C_2 e^{-4x}$

$C_1 + C_2 = 0$, $4C_1 - 4C_2 = 1 \longrightarrow C_1 = \frac{1}{8}$, $C_2 = -\frac{1}{8}$

Tehát

$$G(x) = \begin{cases} 0, & \text{ha } x < 0 \\ \frac{1}{8}\left(e^{4x} - e^{-4x}\right), & \text{ha } x > 0 \end{cases}$$

Ird fel az $y'' - 16y = f(x)$ DE megoldasat, ha ($y(x) = f(x) = 0$, ha $x \ll 0$).

$$y(x) = \int_{-\infty}^{\infty} G(x-z) f(z)\,dz =$$

$$= \int_{-\infty}^{x} \frac{1}{8}\left(e^{4(x-z)} - e^{-4(x-z)}\right) f(z)\,dz$$

3. (2+2+3+2+1 pont)

Szamold ki az $f = 3H(5-t)$ fuggveny Laplace transzformaltjat a definicio alapjan!

$$\mathcal{L}(f) = \int_0^\infty e^{-st} \cdot 3H(\underbrace{5-t}_{=1,\text{ ha } t<5})\,dt = \int_0^5 3\cdot e^{-st}\,dt = \left[3\,\frac{e^{-st}}{-s}\right]_0^5$$

$$= -\frac{3}{s}\left(e^{-5s}-1\right)$$

(a Laplace-tr.-es értelemben)

Szamold ki az alabbi fuggvenypar $f*g$ konvoluciojat! $\quad f(t) = t, \quad g(t) = 5t$

$$(f*g)(t) = \int_0^t f(t-\tau)\,g(\tau)\,d\tau = \int_0^t (t-\tau)\cdot 5\tau\,d\tau = \int_0^t 5t\tau - 5\tau^2\,d\tau$$

$$= 5t\frac{t^2}{2} - 5\frac{t^3}{3} = \frac{5}{6}t^3$$

Legyen $y'' - 16y = (t-1)$, $\quad y(0) = 3$, $y'(0) = 4$. Mennyi $\mathcal{L}(y(t)) = Y(s)$? ($\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}$.)

$$\left(s^2Y(s) - 3s - 4\right) - 16\,Y(s) = \frac{1}{s^2} - \frac{1}{s}$$

$$Y(s) = \frac{1}{s^2-16}\left(\frac{1}{s^2} - \frac{1}{s} + 3s + 4\right)$$

Ird fel azt, hogy hogyan nez ki $Y(s)$ parcialis tort felbontasa! (Az egyutthatokat nem kell kiszamolni!)

$$s^2-16 = (s+4)(s-4)$$

$$Y(s) = \frac{A}{s+4} + \frac{B}{s-4} + \frac{C}{s^2} + \frac{D}{s}$$

Mennyi $y(t)$? ($\mathcal{L}(e^{at}) = \frac{1}{s-a}$.)

$$y(t) = A\,e^{-4t} + B\,e^{4t} + Ct + D$$

4. (3+2+3+2 pont)
a)

$$y'(x) = f(x,y) = 5 - x - y;$$

Mennyi y'' ? Ird fel y masodrendu Talor polinomjat az $x = 0$ pont korul, ha $y(0) = 3$!

$$y'' = \left(\frac{\partial}{\partial x} + f \cdot \frac{\partial}{\partial y}\right) f = \left(\frac{\partial}{\partial x} + (5-x-y)\frac{\partial}{\partial y}\right)(5-x-y) =$$

$$= -1 + (5-x-y)(-1) = -6 + x + y$$

$$y(0) = 3, \quad y'(0) = 5-0-3 = 2, \quad y''(0) = -6+0+3 = -3$$

$$y(x) \approx 3 + 2x - \frac{3}{2!} x^2$$

b) Alkalmazd az Euler, illetve a Heun modszert a kovetkezo DE-re $\Delta x = 0.01$ lepeskozzel!

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} y_1 \\ x + y_2 + 1. \end{pmatrix}, \qquad \bar{y}(2) = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

Mit josol a ket modszer $\bar{y}(2.01)$-re?
Euler:

$$y(2.01) \approx \begin{pmatrix} 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 2 \\ 2+3+1 \end{pmatrix} \cdot 0.01 = \begin{pmatrix} 2.02 \\ 3.06 \end{pmatrix}$$

Heun:

$$y(2.01) \approx \begin{pmatrix} 2 \\ 3 \end{pmatrix} + \frac{1}{2} \cdot 0.01 \cdot \left(\begin{pmatrix} 2 \\ 2+3+1 \end{pmatrix} + \begin{pmatrix} 2.02 \\ 2.01 + 3.06 + 1 \end{pmatrix} \right)$$

c) $x_0 = 0$, $x_{n+1} = 7x_n + 4$, Mennyi x_n?

fixpont: $7x_f + 4 = x_f \longrightarrow x_f = -\frac{2}{3}$

$$x_n = 7^n \cdot \left(0 - \left(-\frac{2}{3}\right)\right) + \left(-\frac{2}{3}\right) = 7^n \cdot \frac{2}{3} - \frac{2}{3}$$