1. (3+2+2+3 pont)

a) A $\{\cos(nx)\,|\,n=0,1,2,\dots\}$ fuggvenyek halmaza egy ortogonalis bazisat adja $L^2([0,\pi])$-nek. Adj meg ebben a terben egy ortonormalt bazist!

b) Legyen a 2π szerint periodikus f fuggveny erteke $f(x)=2$, ha $x\in[0,1]$, maskulonben $f=0$. Ha $f(x)=\frac{1}{\sqrt{2\pi}}\sum_{n\in\mathbb{Z}}\hat{f}_n e^{inx}$, akkor mennyi $\hat{f}(-5)$?

c) $\phi_t(t,x)=\phi_{xx}(t,x)$, $\phi(0,x)=\exp(i(5x))+\sin(2x)$. Mennyi $\phi(t,x)$?

d) $\phi_{tt}(t,x)=\phi_{xx}(t,x)$, $\phi(0,x)=\sin(3x)$, $\phi_t(0,x)=\sin(4x)$. Mennyi $\phi(t,x)$?

(a) $\int_0^\pi |\cos(0x)|^2 dx = \int_0^\pi 1\,dx = \pi \longrightarrow \frac{1}{\sqrt{\pi}}$ normája 1.

$\int_0^\pi |\cos(nx)|^2 dx = \int_0^\pi \frac{1}{2}\,dx = \frac{\pi}{2} \longrightarrow \sqrt{\frac{2}{\pi}}\cos(nx)$, $n=1,2,\dots$ normája 1.

Tehát az ortonormált bázis: $\frac{1}{\sqrt{\pi}}$, $\sqrt{\frac{2}{\pi}}\cos x$, $\sqrt{\frac{2}{\pi}}\cos(2x)$, ...

(b) $\hat{f}(-5) = \frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi} e^{-i(-5x)} f(x)\,dx = \frac{1}{\sqrt{2\pi}}\int_0^1 e^{5ix}\cdot 2\,dx = \sqrt{\frac{2}{\pi}}\left[\frac{e^{5ix}}{5i}\right]_0^1$

$= \sqrt{\frac{2}{\pi}}\cdot\left(-\frac{i}{5}\right)\left[e^{5i}-1\right]$

(c) Mivel $\frac{\partial^2}{\partial x^2} e^{5ix} = -25 e^{5ix}$ és $\frac{\partial^2}{\partial x^2}\sin(2x) = -4\sin(2x)$,

$\varphi(t,x) = e^{-25\cdot t} e^{5ix} + e^{-4\cdot t}\sin(2x)$

(d) Ha $\varphi(t,x) = c_1(t)\cdot\sin(3x)$, akkor $\frac{d^2 c_1(t)}{dt^2} = -9c_1(t)$, $c_1(0)=1$, $c_1'(0)=0$

tehát $c_1(t)=\cos(3t) \longrightarrow \varphi(t,x) = \cos(3t)\sin(3t)$

Ha $\varphi(t,x) = c_2(t)\cdot\sin(4x)$, akkor $\frac{d^2 c_2(t)}{dt^2} = -16\,c_2(t)$, $c_2(0)=0$, $c_2'(0)=1$

tehát $c_2(t) = \frac{1}{4}\sin(4t) \longrightarrow \varphi(t,x) = \frac{1}{4}\sin(4t)\sin(4x)$

Vagyis

$\varphi(t,x) = \cos(3t)\sin(3t) + \frac{1}{4}\sin(4t)\cdot\sin(4x)$

2. (2+1+(2+4+1) pont)

a) Legyen $y'' + 7y' + 8y = e^{i\omega t}$, $y = A(\omega)e^{i\omega t}$. Mennyi $A(\omega)$?

b) Legyen $f(x) = 2$, ha $x \in [-1, 1]$, maskulonben $f = 0$. Ha $f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(p) e^{ipx}\, dp$, akkor mennyi $\hat{f}(-0.5)$?

c) Keresd meg a kovetkezo DE megoldasat a ($G(t) = 0$, ha $t < 0$) feltetel mellett!
$9G'' + 27G = \delta$.
Add meg a kovetkezo mennyisegeket!
$G(0^+) - G(0^-) =$ $\qquad$ $G'(0^+) - G'(0^-) =$

Mennyi $G(t)$?

d) Ird fel az $9y'' + 27y = f(t)$ DE megoldasat, ha ($y(t) = f(t) = 0$, ha $t \ll 0$).

(a) $$\left[(i\omega)^2 + 7i\omega + 8\right] A(\omega) e^{i\omega t} = e^{i\omega t} \longrightarrow A(\omega) = \frac{1}{-\omega^2 + 7i\omega + 8}$$

(b) $$\hat{f}(-0.5) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i(-0.5)x} f(x)\, dx = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{ix/2} \cdot 2\, dx = \frac{2}{\sqrt{2\pi}} \left[\frac{e^{ix/2}}{1/2 \cdot i} \right]_{-1}^{1}$$
$$= \frac{2\sqrt{2}}{\pi \cdot i} \left[e^{i/2} - e^{-i/2} \right]$$

(c) $$G(0^+) - G(0^-) = 0, \quad G'(0^+) - G'(0^-) = \frac{1}{9} \qquad (*)$$

① $G(t) = 0$, ha $t < 0$

② $9G''(t) + 27G(t) = 0$, ha $t > 0$

tehát ekkor $G(t) = C_1 \cos(\sqrt{3}t) + C_2 \sin(\sqrt{3}t)$

a $(*)$ feltétel miatt
$$G(t) = \frac{1}{9\sqrt{3}} \sin(\sqrt{3}t)$$

Tehát
$$G(t) = \begin{cases} 0, & \text{ha } t < 0 \\ \frac{1}{9\sqrt{3}} \sin(\sqrt{3}t), & \text{ha } t > 0 \end{cases}$$

(d) $$y(t) = \int_{-\infty}^{\infty} G(t-\tau) f(\tau)\, d\tau = \int_{-\infty}^{t} \frac{1}{9\sqrt{3}} \sin(\sqrt{3}(t-\tau)) f(\tau)\, d\tau$$

3. (2+2+(3+2+1) pont)

a) Szamold ki az $f = 1 - H(t-1)$ fuggveny Laplace transzformaltjat a definicio alapjan!

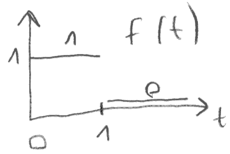
b) Szamold ki az alabbi f, g fuggvenypar $f * g$ konvoluciojat! $f(t) = 2, \quad g(t) = \sin 5t$

c) Legyen $y'' - 25y = (1-t)^2, \quad y(0) = 2,\ y'(0) = 3$. Mennyi $\mathcal{L}(y(t)) = Y(s)$? ($\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}$.)

d) Ird fel azt, hogy hogyan nez ki $Y(s)$ parcialis tort felbontasa! (Az egyutthatokat nem kell kiszamolni!)

e) Mennyi $y(t)$?

(a)



$f(t) = \begin{cases} 1, & \text{ha } 0 \le t < 1 \\ 0, & \text{ha } 1 < t \end{cases}$

$$F(s) = \int_0^\infty e^{-st} f(t)\,dt = \int_0^1 e^{-st}\,dt = \left[\frac{e^{-st}}{-s}\right]_0^1 = \frac{1}{s}\left(1 - e^{-s}\right)$$

(b)

$$(f*g)(t) = \int_0^t \underbrace{2}_{f(t-\tau)=2} \cdot \sin(5\tau)\,d\tau = -\frac{2}{5}\left[\cos(5\tau)\right]_0^t = -\frac{2}{5}\left(\cos(5t) - 1\right)$$

(c)

$$y'' - 25y = 1 - 2t + t^2, \quad y(0) = 2, \quad y'(0) = 3$$

$$\left(s^2 Y(s) - s \cdot 2 - 3\right) - 25\,Y(s) = \frac{1}{s} - 2 \cdot \frac{1}{s^2} + \frac{2}{s^3}$$

$$Y(s) = \frac{1}{s^2 - 25}\left(2s + 3 + \frac{1}{s} - \frac{2}{s^2} + \frac{2}{s^3}\right)$$

(d)

$$Y(s) = \frac{A}{s-5} + \frac{B}{s+5} + \frac{C}{s} + \frac{D}{s^2} + \frac{E}{s^3}$$

Mivel $s^2 - 25 = (s-5)(s+5)$

(e)

$$y(t) = A e^{5t} + B e^{-5t} + C + Dt + \frac{E}{2} t^2$$

Mivel $\mathcal{L}^{-1}\left(\frac{2}{s^3}\right) = t^2$

4.(1+1+2+3+3 pont)

a) Legyen

$$A = \begin{pmatrix} 2i & -3i \\ 3 & 4i \end{pmatrix}.$$

Mennyi A^* ?

b) Legyen $f_1 = (i/\sqrt{2}, i/\sqrt{2})^T$, $f_2 = (z, -1/\sqrt{2})^T$ egy ortonormalt bazis. Mennyi z ?

c) A $v = (2,3)^T$ vektor kifejezheto az f-ek linearis $\alpha f_1 + \beta f_2$ kombinaciojakent! Mennyi α ?

d) Elegitse ki $\phi(t,x)$ a $\partial_{tt}^2\phi - \partial_{tx}^2\phi + 8\partial_{xx}^2\phi = 0$ egyenletet. Ha $\phi(t,x) = e^{i(\omega t + kx)}$, akkor milyen osszefugges all fenn ω es k kozott?

e) Legyen

$$\frac{d}{dt}\bar{y} + \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}\bar{y} = \begin{pmatrix} t^2 \\ t^4 \end{pmatrix}, \quad \bar{y}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Mennyi $\bar{Y}(s)$? (Nem kell elvegezni a matrixok invertalasat!)

a) $$A^* = \begin{pmatrix} \overline{2i} & \overline{3} \\ \overline{-3i} & \overline{4i} \end{pmatrix} = \begin{pmatrix} -2i & 3 \\ 3i & -4i \end{pmatrix}$$

b) $$(f_1, f_2) = 0 = \overline{\frac{i}{\sqrt{2}}} \cdot z + \overline{\frac{i}{\sqrt{2}}} \cdot \left(-\frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}\left(-iz + \frac{i}{\sqrt{2}}\right) \rightarrow z = +\frac{1}{\sqrt{2}}$$

c) $$\alpha = (f_1, v) = \left(\begin{pmatrix} i/\sqrt{2} \\ i/\sqrt{2} \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \end{pmatrix}\right) = \overline{\frac{i}{\sqrt{2}}} \cdot 2 + \overline{\frac{i}{\sqrt{2}}} \cdot 3 = -\frac{5i}{\sqrt{2}}$$

d) $$(i\omega)^2 - (i\omega)(ik) + 8(ik)^2 = 0$$

$$-\omega^2 + \omega k - 8k^2 = 0$$

e) $$\begin{pmatrix} sY_1(s) - 0 \\ sY_2(s) - 0 \end{pmatrix} + \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} Y_1(s) \\ Y_2(s) \end{pmatrix} = \begin{pmatrix} \frac{2!}{s^3} \\ \frac{4!}{s^5} \end{pmatrix}$$

$$\overline{Y}(s) = \begin{pmatrix} Y_1(s) \\ Y_2(s) \end{pmatrix} = \begin{pmatrix} 2+s & 3 \\ 3 & 2+s \end{pmatrix}^{-1} \begin{pmatrix} 2!/s^3 \\ 4!/s^5 \end{pmatrix}$$