① a) $(f,f)=\int_0^\pi \bar f(x) f(x)\,dx = \int_0^\pi f^2(x)\,dx$ ← for real functions, so $(\sin(nx),\sin(nx)) = \int_0^\pi \sin^2(nx)\,dx$

$= \int_0^\pi \frac{1}{2}\,dx = \frac{\pi}{2}$ ← becuse $\sin^2\alpha+\cos^2\alpha=1$

Consequently $\left(\sqrt{\frac{2}{\pi}}\sin(nx), \sqrt{\frac{2}{\pi}}\sin(nx)\right)=1$,

So the orthonormed basis is $\sqrt{\frac{2}{\pi}}\sin(nx)$, $n=1,2,\dots$ 3

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b) $\hat f(-3) = \frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi} e^{-i\cdot(-3)\cdot x} f(x)\,dx = \frac{1}{\sqrt{2\pi}}\int_{-1}^{0} e^{3ix}\cdot 3\,dx = \frac{3}{\sqrt{2\pi}}\left[\frac{e^{3ix}}{3i}\right]_{-1}^{0} =$

$= \frac{-i}{\sqrt{2\pi}}\left(1-e^{-3i}\right)$ 2

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c) $\frac{\partial^2}{\partial x^2} e^{i5x} = -5^2 e^{i5x}$

$\frac{\partial^2}{\partial x^2}\sin(2x) = -2^2\sin(2x)$

so $\varphi(t,x) = e^{-25t} e^{i5x} + e^{-4t}\sin(2x)$ 3

d) $\frac{\partial^2}{\partial x^2}\sin(3x) = -3^2\sin(3x)$

$\frac{\partial^2}{\partial x^2} 5 = 0 = 0\cdot 5$

so $\varphi(t,x) = e^{-9t}\sin(3x)+5$ 2

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② a) $\left[(i\omega)^2+7(i\omega)+8\right]A(\omega)\,e^{i\omega t} = e^{i\omega t}$ 2

so $A(\omega) = \frac{1}{-\omega^2+7i\omega+8}$

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b) $\hat f(-0{,}5) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{+\frac{ix}{2}} f(x)\,dx = \frac{1}{\sqrt{2\pi}}\int_{-1}^{1} e^{\frac{ix}{2}}\cdot 2\,dx = \frac{1}{\sqrt{2\pi}}\cdot 2\cdot\left[\frac{e^{ix/2}}{i/2}\right]_{-1}^{1}$

$-(-0{,}5)x$

$= -\frac{2\sqrt{2}\,i}{\sqrt{\pi}}\left(e^{+i/2} - e^{-i/2}\right)$ 1

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c) 2

$G(t)=0$, if $t<0$

$G(0^+)-G(0^-)=0$

$G'(0^+)-G'(0^-)=\frac{1}{9}$

$9G''(t)+27G(t)=0$, if $t>0$

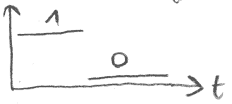
since if $t\approx 0$, $G''(t)=\frac{1}{9}\cdot\delta(t)$

$$\Rightarrow G(t)=\begin{cases}0, & \text{if } t<0\\ \frac{1}{9\sqrt{3}}\sin(\sqrt{3}t), & \text{if } t>0\end{cases}$$ 4

since $9G''+27G=0$ for $t>0$
implies $G=C_1\cos\sqrt{3}t + C_2\sin\sqrt{3}t$
and $G(0^+)=0$, $G'(0^+)=\frac{1}{9} \to C_1=0$, $C_2=\frac{1}{9\cdot\sqrt{3}}$

d) $y(t) = \int_{-\infty}^{\infty} G(t-\tau) f(\tau)\,d\tau = \int_{-\infty}^{t}\frac{1}{9\sqrt{3}}\sin(\sqrt{3}(t-\tau))\,f(\tau)\,d\tau$. 1

③ ⓐ $\mathcal{L}(f) = \int_0^\infty e^{-st} f(t)dt = \int_0^1 e^{-st} dt = \frac{e^{-st}}{-s}\Big|_0^1 = \frac{1}{s}(1-e^{-s})$

F(t): 

2

ⓑ $f * g(t) = \int_0^t 2 \cdot \sin(5\tau)\, d\tau = -\frac{2}{5}\cos(5\tau)\Big|_0^t = -\frac{2}{5}(1-\cos(5t))$ 2

ⓒ $s^2Y(s) - 2\cdot s - 3 - 25Y(s) = \frac{1}{s} - 2\cdot\frac{1}{s^2} + \frac{2}{s^3} \leftarrow \mathcal{L}(1-2t+t^2)$

3 $Y(s) = \frac{1}{s^2-25}\left(\frac{1}{s} - \frac{2}{s^2} + \frac{2}{s^3} + 2s + 3\right) = \frac{A}{s+5} + \frac{B}{s-5} + \frac{C}{s^3} + \frac{D}{s^2} + \frac{E}{s}$ 2

$y(t) = Ae^{-5t} + Be^{5t} + \frac{C}{2}t^2 + Dt + E$ 1

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④ ⓐ $A^* = \begin{pmatrix} -2i & 3 \\ 3i & -4i \end{pmatrix}$ 1

ⓑ $0 = (f_1, f_2) = \overline{\frac{i}{\sqrt{2}}} \cdot z + \overline{\frac{i}{\sqrt{2}}} \cdot \left(-\frac{1}{\sqrt{2}}\right) \longrightarrow z = \frac{1}{\sqrt{2}}$ 1

ⓒ $\alpha = (f_1, v) = \overline{\frac{i}{\sqrt{2}}} \cdot 2 + \overline{\frac{i}{\sqrt{2}}} \cdot 3 = -\frac{5}{\sqrt{2}} i$ 2

ⓓ $\left[(i\omega)^2 - (i\omega)(ik) + 8(ik)^2\right] e^{i(\omega t + kx)} = 0$ 3

so $-\omega^2 + \omega k - 8k^2 = 0$

ⓔ $\begin{pmatrix} sY_1(s) - 0 \\ sY_2(s) - 0 \end{pmatrix} + \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} Y_1(s) \\ Y_2(s) \end{pmatrix} = \begin{pmatrix} 2!/s^3 \\ 4!/s^4 \end{pmatrix}$

$\begin{pmatrix} Y_1(s) \\ Y_2(s) \end{pmatrix} = \begin{pmatrix} 2+s & 3 \\ 3 & 2+s \end{pmatrix}^{-1} \begin{pmatrix} 2/s^3 \\ 24/s^4 \end{pmatrix}$ 3