

Név:

Aláírás:

1. (4+(3+3) pont)

a)

$$y' = f(x, y) = x^2 + y - 2;$$

Mennyi  $y''$ ? Írd fel  $y$  másodrendű Taylor polinomját az  $x = 0$  pont körül, ha  $y(0) = 3$ !

$$y'' = \left( \frac{\partial}{\partial x} + [x^2 + y - 2] \frac{\partial}{\partial y} \right) [x^2 + y - 2] = 2x + [x^2 + y - 2] \cdot 1 \quad \rightarrow \quad y''(0) = 2 \cdot 0 + [0^2 + 3 - 2] \cdot 1 = 1$$

$$y'(0) = 0^2 + 3 - 2 = 1$$

$$y(x) \approx 3 + 1 \cdot x + \frac{1}{2!} x^2$$

b) Alkalmazd az Euler, illetve a Heun módszert a következő DE-re  $\Delta x = 0.01$  lépésközzel! az  $\bar{y}(2) =$  kezdeti feltétel mellett!

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} y_1 - y_2 \\ 3y_2^2 + x \end{pmatrix}, \quad \bar{y}(2) = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

Mit jósol a két módszer  $\bar{y}(2.01)$ -re?

Euler:  $\bar{k}_1 = \bar{y}'(0) = \begin{pmatrix} 2 - 3 \\ 3 \cdot 3^2 + 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 29 \end{pmatrix}$

$$\bar{y}(2.01) \approx \bar{y}(2) + 0.01 \cdot \bar{k}_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix} + 0.01 \begin{pmatrix} -1 \\ 29 \end{pmatrix} = \begin{pmatrix} 1.99 \\ 3.29 \end{pmatrix}$$

Heun:

$$\bar{k}_2 = \begin{pmatrix} 1.99 - 3.29 \\ 3 \cdot 3.29^2 + 2.01 \end{pmatrix}$$

$$\bar{y}(2.01) \approx \bar{y}(2) + 0.01 \left( \frac{\bar{k}_1 + \bar{k}_2}{2} \right) = \begin{pmatrix} 2 \\ 3 \end{pmatrix} + \frac{0.01}{2} \left[ \begin{pmatrix} -1 \\ 29 \end{pmatrix} + \begin{pmatrix} 1.99 - 3.29 \\ 3 \cdot 3.29^2 + 2.01 \end{pmatrix} \right]$$

3a. (1+1+1+2 pont)

$$y' = (-y^2 + 4).$$

Keresd meg a DE fixpontjait!

$$-y^2 + 4 = 0 \rightarrow y_1 = 2, \quad y_2 = -2$$

Ird fel a fixpontok koruli linearizalt kozelito DE-t!

$$\frac{d(-y^2+4)}{dy} = -2y \quad \left| \quad \frac{d}{dx}(y-2) = \frac{d}{dx} \Delta y = -2 \cdot 2 \cdot \Delta y \quad \right| \quad \frac{d}{dx}(y-(-2)) = \frac{d}{dx} \Delta y = -2 \cdot (-2) \Delta y$$

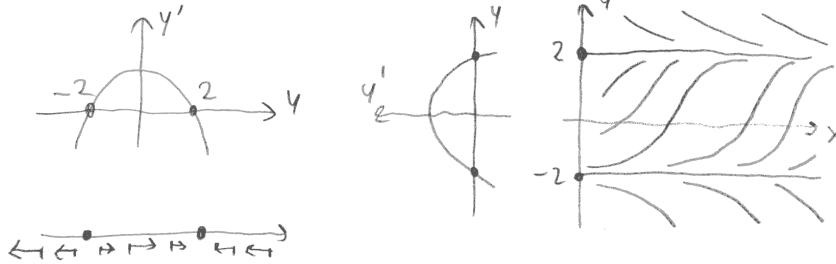
$$y_1 = 2 \qquad \qquad \qquad y_2 = -2$$

Ha  $y(0) = 0$ , mennyi

$$\lim_{x \rightarrow \infty} y(x) = 2$$

$$\lim_{x \rightarrow -\infty} y(x) = -2$$

Vazold a DE megoldasgorbeit!



3b. (2+3 pont)

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} -y_1 + 2 \\ 2y_2(-y_1 - 3y_2) \end{pmatrix}$$

Keresd meg a DE fixpontjait!

$$\begin{cases} -y_1 + 2 = 0 \\ 2y_2(-y_1 - 3y_2) = 0 \end{cases} \rightarrow \begin{cases} y_1 = 2 \\ y_2 = 0 \end{cases} \quad P_1 = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \quad P_2 = \begin{pmatrix} 2 \\ -2/3 \end{pmatrix}$$

vagy  
 $-2 - 3y_2 = 0 \rightarrow y_2 = -2/3$

Ird fel a fixpont koruli linearizalt kozelito DE-t!

$$Jac = \begin{pmatrix} \frac{\partial}{\partial y_1}(-y_1+2) & \frac{\partial}{\partial y_2}(-y_1+2) \\ \frac{\partial}{\partial y_1}[2y_2(-y_1-3y_2)] & \frac{\partial}{\partial y_2}[2y_2(-y_1-3y_2)] \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ -2y_2 & -2y_1 - 12y_2 \end{pmatrix}$$

$$Jac(P_1) = \begin{pmatrix} -1 & 0 \\ 0 & -4 \end{pmatrix}$$

$$Jac(P_2) = \begin{pmatrix} -1 & 0 \\ +4/3 & 4 \end{pmatrix}$$

$$\frac{d}{dx} \overline{\Delta y_1} = \begin{pmatrix} -1 & 0 \\ 0 & -4 \end{pmatrix} \overline{\Delta y_1}$$

$$\overline{\Delta y_1} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} - \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

$$\frac{d}{dx} \overline{\Delta y_2} = \begin{pmatrix} -1 & 0 \\ +4/3 & 4 \end{pmatrix} \overline{\Delta y_2}$$

$$\overline{\Delta y_2} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} - \begin{pmatrix} 2 \\ -2/3 \end{pmatrix}$$

2. (5+2+3 pont)

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} -y_1 - 2y_2 \\ 2y_1 - y_2 \end{pmatrix} = A \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \quad \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

Keresd meg  $A$  sajátértékeit és sajátvektorait!

$$A = \begin{pmatrix} -1 & -2 \\ 2 & -1 \end{pmatrix} \quad \det(A - \lambda E) = \begin{vmatrix} -1-\lambda & -2 \\ 2 & -1-\lambda \end{vmatrix} = (-1-\lambda)^2 - (-2) \cdot 2 = \lambda^2 + 2\lambda + 5$$

$$\left. \begin{array}{l} \lambda_1 = -1 + 2i \\ \begin{pmatrix} -1 - (-1 + 2i) & -2 \\ 2 & -1 - (-1 + 2i) \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} -2i & -2 \\ 2 & -2i \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow v = -iu \\ \bar{V}_1 = \begin{pmatrix} 1 \\ -i \end{pmatrix} \end{array} \right\} \begin{array}{l} \lambda_2 = -1 - 2i \\ \begin{pmatrix} -1 - (-1 - 2i) & -2 \\ 2 & -1 - (-1 - 2i) \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 2i & -2 \\ 2 & 2i \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow v = iu \\ \bar{V}_2 = \begin{pmatrix} 1 \\ i \end{pmatrix} \end{array}$$

Ird fel a DE általános megoldását!

$$\bar{y}_{\text{ált}} = C_1 e^{(-1+2i)x} \begin{pmatrix} 1 \\ -i \end{pmatrix} + C_2 e^{(-1-2i)x} \begin{pmatrix} 1 \\ i \end{pmatrix}$$

Számold ki a DE partikularis megoldását!

$$\bar{y}_{\text{ált}}(0) = C_1 \begin{pmatrix} 1 \\ -i \end{pmatrix} + C_2 \begin{pmatrix} 1 \\ i \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$\left. \begin{array}{l} C_1 + C_2 = 1 \\ -iC_1 + iC_2 = 3 \rightarrow C_1 - C_2 = 3i \end{array} \right\} \rightarrow C_1 = \frac{1+3i}{2} \rightarrow C_2 = \frac{1-3i}{2}$$

$$\bar{y}_{\text{part}}(x) = \frac{1+3i}{2} e^{(-1+2i)x} \begin{pmatrix} 1 \\ -i \end{pmatrix} + \frac{1-3i}{2} e^{(-1-2i)x} \begin{pmatrix} 1 \\ i \end{pmatrix}$$

4+3x2 pont  
(5x2 pont)

$$\text{mivel } \begin{pmatrix} -x & 0 \\ 0 & -x \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 6x & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 6x & 0 \end{pmatrix} \begin{pmatrix} -x & 0 \\ 0 & -x \end{pmatrix}$$

$$A = \begin{pmatrix} -1 & 0 \\ 6 & -1 \end{pmatrix}$$

Mennyi  $e^{xA}$ ?

$$e^{xA} = e^{\begin{pmatrix} -x & 0 \\ 0 & -x \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 6x & 0 \end{pmatrix}} = e^{\begin{pmatrix} -x & 0 \\ 0 & -x \end{pmatrix}} e^{\begin{pmatrix} 0 & 0 \\ 6x & 0 \end{pmatrix}}$$

$$= \begin{pmatrix} e^{-x} & 0 \\ 0 & e^{-x} \end{pmatrix} \cdot \left[ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 6x & 0 \end{pmatrix} + \frac{1}{2!} \underbrace{\begin{pmatrix} 0 & 0 \\ 6x & 0 \end{pmatrix}^2}_{= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}} + \dots \right]$$

$$= \begin{pmatrix} e^{-x} & 0 \\ 0 & e^{-x} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 6x & 1 \end{pmatrix} = \begin{pmatrix} e^{-x} & 0 \\ 6xe^{-x} & e^{-x} \end{pmatrix}$$

Ird fel a

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} -y_1 \\ 6y_1 - y_2 \end{pmatrix}, \quad \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

DE partikularis megoldásat  $e^{xA}$  segítségével!

$$\bar{y}_{\text{part}} = \begin{pmatrix} e^{-x} & 0 \\ 6xe^{-x} & e^{-x} \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

2b) Ird at a következő DE rendszert elsőrendű DE rendszerre!

$$\frac{d^2}{dx^2} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} y_1'' - y_2 \\ 2y_2' - 3y_1' \end{pmatrix}$$

$$\frac{d}{dx} \begin{pmatrix} y_1 \\ y_2 \\ v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ v_1^2 - y_2 \\ 2v_2 - 3v_1 \end{pmatrix}$$

Mennyi  $z = e^{-1-i\pi}$ ?

$$z = e^{-1} e^{-i\pi} = \frac{1}{e} (\cos(-\pi) + i \sin(-\pi)) = \frac{1}{e} \cdot (-1) = -\frac{1}{e}$$