1, Diff.Eq. 2016.03.22. NEPTUN:

Name: Signature:

1. (4+(3+3) pont)
a)

$$y' = f(x, y) = x^2 + y - 2;$$

How much is y'' ? Write down the second order Taylor polynomial of $y(x)$ around $x = 0$, if $y(0) = 3$!

$$y'' = \left(\frac{\partial}{\partial x} + [x^2+y-2]\frac{\partial}{\partial y}\right)[x^2+y-2] = 2x + [x^2+y-2]\cdot 1 \longrightarrow y''(0) = 2\cdot 0 + [0^2+3-2]\cdot 1 = 1$$

$$y'(0) = 0^2 + 3 - 2 = 1$$

$$y(x) \approx 3 + 1\cdot x + \frac{1}{2!}x^2$$

b) Apply the Euler and the Heun methods on the following DEs with stepsize $\Delta x = 0.01$!

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} y_1 - y_2 \\ 3y_2^2 + x. \end{pmatrix}, \qquad \bar{y}(2) = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

What are the predictions for $\bar{y}(2.01)$?
Euler:

$$\bar{k}_1 = \bar{y}'(0) = \begin{pmatrix} 2-3 \\ 3\cdot 3^2+2 \end{pmatrix} = \begin{pmatrix} -1 \\ 29 \end{pmatrix}$$

$$\bar{y}(2.01) \approx \bar{y}(2) + 0.01\cdot\bar{k}_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix} + 0.01\begin{pmatrix} -1 \\ 29 \end{pmatrix} = \begin{pmatrix} 1.99 \\ 3.29 \end{pmatrix}$$

Heun:

$$\bar{k}_2 = \begin{pmatrix} 1.99 - 3.29 \\ 3\cdot 3.29^2 + 2.01 \end{pmatrix}$$

$$\bar{y}(2.01) \approx \bar{y}(2) + 0.01\left(\frac{\bar{k}_1+\bar{k}_2}{2}\right) = \begin{pmatrix} 2 \\ 3 \end{pmatrix} + \frac{0.01}{2}\left[\begin{pmatrix} -1 \\ 29 \end{pmatrix} + \begin{pmatrix} 1.99-3.29 \\ 3\cdot 3.29^2+2.01 \end{pmatrix}\right]$$

3a. (1+1+1+2 pont)

$y' = (-y^2 + 4)$.

Find the fixed points of the DE!

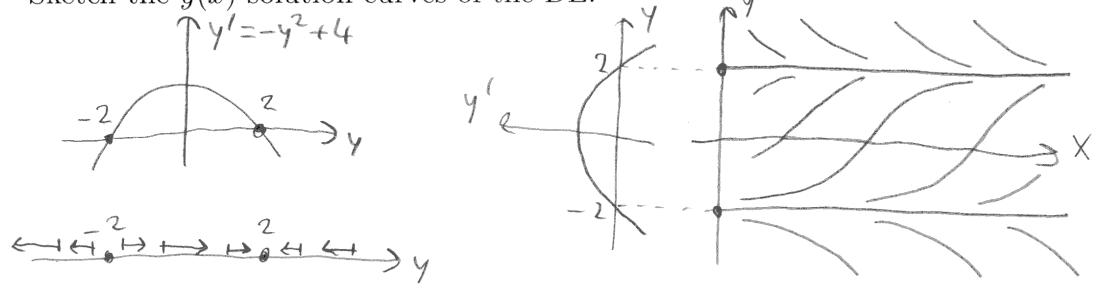
$$-y^2+4=0 \longrightarrow y_1 = 2, \quad y_2 = -2$$

If $y(0) = 0$, how much is

$\lim_{x\to\infty} y(x) =$ 2

$\lim_{x\to-\infty} y(x) = ?$ -2

Sketch the $y(x)$ solution curves of the DE!



3b. (2+3 pont)

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} -y_1 + 2 \\ 2y_2(-y_1 - 3y_2). \end{pmatrix}$$

Find the fixed points of the DE!

$$\left.\begin{aligned} -y_1+2&=0 \\ 2y_2(-y_1-3y_2)&=0 \end{aligned}\right\} \longrightarrow y_1 = 2, \quad \left[\begin{aligned} &y_2 = 0 \\ &\text{or} \\ &-2-3y_2 = 0 \rightarrow y_2 = -2/3 \end{aligned}\right.$$

$$P_1 = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \qquad P_2 = \begin{pmatrix} 2 \\ -2/3 \end{pmatrix}$$

Write down the linearized DEs around the fixed points!

$$\text{Jac} = \begin{pmatrix} \frac{\partial}{\partial y_1}(-y_1+2) & \frac{\partial}{\partial y_2}(-y_1+2) \\ \frac{\partial}{\partial y_1}[2y_2(-y_1-3y_2)] & \frac{\partial}{\partial y_2}[2y_2(-y_1-3y_2)] \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ -2y_2 & -2y_1-12y_2 \end{pmatrix}$$

$$\text{Jac}(P_1) = \begin{pmatrix} -1 & 0 \\ 0 & -4 \end{pmatrix} \qquad \text{Jac}(P_2) = \begin{pmatrix} -1 & 0 \\ 4/3 & 4 \end{pmatrix}$$

$$\frac{d}{dx}\overline{\Delta y_1} = \begin{pmatrix} -1 & 0 \\ 0 & -4 \end{pmatrix} \overline{\Delta y_1} \qquad \frac{d}{dx}\overline{\Delta y_2} = \begin{pmatrix} -1 & 0 \\ 4/3 & 4 \end{pmatrix} \overline{\Delta y_2}$$

$$\overline{\Delta y_1} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} - \begin{pmatrix} 2 \\ 0 \end{pmatrix} \qquad \overline{\Delta y_2} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} - \begin{pmatrix} 2 \\ -2/3 \end{pmatrix}$$

2. (5+2+3 pont)

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} -y_1 - 2y_2 \\ 2y_1 - y_2 \end{pmatrix} = A \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \qquad \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

Find the eigenvalues and eigenvectors of A !

$$A = \begin{pmatrix} -1 & -2 \\ 2 & -1 \end{pmatrix} \qquad \det(A - \lambda E) = \begin{vmatrix} -1-\lambda & -2 \\ 2 & -1-\lambda \end{vmatrix} = (-1-\lambda)^2 - (-2)\cdot 2 = \lambda^2 + 2\lambda + 5$$

$\lambda_1 = -1 + 2i$

$$\begin{pmatrix} -1-(-1+2i) & -2 \\ 2 & -1-(-1+2i) \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -2i & -2 \\ 2 & -2i \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow v = -iu$$

$\vec{v}_1 = \begin{pmatrix} 1 \\ -i \end{pmatrix}$, (or any nonzero multiple of $\vec{v}_1$)

$\lambda_2 = -1 - 2i$

$$\begin{pmatrix} -1-(-1-2i) & -2 \\ 2 & -1-(-1-2i) \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2i & -2 \\ 2 & 2i \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{matrix} 2iu - 2v = 0 \\ 2u + 2iv = 0 \end{matrix}$$

so $v = iu$

$\vec{v}_2 = \begin{pmatrix} 1 \\ i \end{pmatrix}$

Remark: Since A is a real matr[ix], $\lambda_2 = \overline{\lambda_1}$, and $\vec{v}_2$ ca[n] be the complex conjugate of $\vec{v}_1$

Write down the general solution of the DE!

$$\overline{y}_{gen}(x) = c_1 e^{(-1+2i)x} \begin{pmatrix} 1 \\ -i \end{pmatrix} + c_2 e^{(-1-2i)x} \begin{pmatrix} 1 \\ i \end{pmatrix}$$

Compute the particular solution!

$$\overline{y}(0) = c_1 \begin{pmatrix} 1 \\ -i \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ i \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$\left.\begin{matrix} c_1 + c_2 = 1 \\ -ic_1 + ic_2 = 3 \longrightarrow c_1 - c_2 = 3i \end{matrix}\right\} \longrightarrow c_1 = \frac{1+3i}{2}, \quad c_2 = \frac{1-3i}{2}$$

$$\overline{y}_{part}(x) = \frac{1+3i}{2} e^{(-1+2i)x} \begin{pmatrix} 1 \\ -i \end{pmatrix} + \frac{1-3i}{2} e^{(-1-2i)x} \begin{pmatrix} 1 \\ i \end{pmatrix}$$

4 + 3·2 points
(~~5 × 2~~ pont)

Since $\begin{pmatrix} -x & 0 \\ 0 & -x \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 6x & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 6x & 0 \end{pmatrix}\begin{pmatrix} -x & 0 \\ 0 & -x \end{pmatrix}$

$$A = \begin{pmatrix} -1 & 0 \\ 6 & -1 \end{pmatrix}$$

How much is e^{xA}?

$$e^{xA} = e^{\begin{pmatrix} -x & 0 \\ 0 & -x \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 6x & 0 \end{pmatrix}} = e^{\begin{pmatrix} -x & 0 \\ 0 & -x \end{pmatrix}} e^{\begin{pmatrix} 0 & 0 \\ 6x & 0 \end{pmatrix}}$$

$$= \begin{pmatrix} e^{-x} & 0 \\ 0 & e^{-x} \end{pmatrix} \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 6x & 0 \end{pmatrix} + \frac{1}{2!} \underbrace{\begin{pmatrix} 0 & 0 \\ 6x & 0 \end{pmatrix}^2}_{= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}} + \ldots \right]$$

$$= \begin{pmatrix} e^{-x} & 0 \\ 0 & e^{-x} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 6x & 1 \end{pmatrix} = \begin{pmatrix} e^{-x} & 0 \\ 6xe^{-x} & e^{-x} \end{pmatrix}$$

Express the solution of the following DE

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} -y_1 \\ 6y_1 - y_2 \end{pmatrix}, \qquad \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

with e^{xA} !

$$\bar{y}_{part} = \begin{pmatrix} e^{-x} & 0 \\ 6x\,e^{-x} & e^{-x} \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} e^{-x} \\ (6x+3)\,e^{-x} \end{pmatrix}$$

2b) Express the following DE as a first order system!

$$\frac{d^2}{dx^2} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} y_1'^2 - y_2 \\ 2y_2' - 3y_1' \end{pmatrix}$$

Introduce $v_1 = y_1'$, $v_2 = y_2'$, then

$$\frac{d}{dx} \begin{pmatrix} y_1 \\ y_2 \\ v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ v_1^2 - y_2 \\ 2v_2 - 3v_1 \end{pmatrix}$$

How much is $z = e^{-1-i\pi}$?

$$z = e^{-1} e^{-i\pi} = \frac{1}{e}\left(\cos(-\pi) + i\sin(-\pi)\right) = \frac{1}{e}\cdot(-1) = -1/e$$