Zh1, Diff.Egy., 2015.03.18. NEPTUN: Gyak.Vez.:

Név: Aláírás:

1. (3+4+3 pont)
a)

$$y' = f(x, y) = -y + 2yx;$$

Mennyi y'' ? Ird fel y masodrendu Talor polinomjat az $x = 0$ pont korul, ha $y(0) = 3$!

$$y'' = \left(\frac{\partial}{\partial x} + (-y+2yx)\cdot\frac{\partial}{\partial y}\right)(-y+2yx) =$$

$$= (2y) + (-y+2yx)\cdot(-1+2x)$$

$y(0)=3$

$y'(0) = -3+2\cdot 3\cdot 0 = -3$

$y''(0) = 2\cdot 3 + (-3+2\cdot 3\cdot 0)\cdot(-1+2\cdot 0) = 9$

$$y(x) \approx 3 - 3x + \frac{9}{2}x^2$$

b) Alkalmazd az Euler, illetve a Heun modszert a kovetkezo DE-re $\Delta x = 0.1$ lepeskozzel az $y(2) = 3$ kezdeti feltetel mellett!

$$y' = -y + 2yx;$$

Mit josol a ket modszer $y(2.1)$-re?
Euler:

$$y(2.1) \approx 3 + (-3 + 2\cdot 3\cdot 2)\cdot 0.1 = 3.9$$

Heun:

$$y(2.1) \approx 3 + \frac{1}{2}\left[(-3+2\cdot 3\cdot 2) + (-3.9 + 2\cdot 3.9\cdot 2.1)\right]\cdot 0.1$$

c) Legyen $f(x) = 1/x^2$, $x_0 = 3$. Ird fel f-nek a linearis $f(x_0 + \Delta x) \approx T_1(x_0 + \Delta x)$ kozeliteset, ha $\Delta x = 0.1$!
Mennyi $\max_{z\in[x_0,x_0+\Delta x]}|f''(z)|$? Adj nemtrivialis felso korlatot a kozelites $|\text{hiba}(\Delta x)| = |f(x_0 + \Delta x) - T_1(x_0 + \Delta x)|$ hibajara!

$f(3) = \frac{1}{9}$ $\quad f' = -2\cdot x^{-3}$ $\quad f'' = 6x^{-4}$

$f'(3) = -\frac{2}{27}$ $\quad \frac{1}{(3+0.1)^2} \approx \frac{1}{3^2} - \frac{2}{27}\cdot 0.1$

$$\max_{z\in[3,3.1]} |6z^{-4}| = \frac{6}{3^4} = \frac{2}{27}$$

$$|\text{hiba}(0.1)| \leq \frac{1}{2}\, 0.1^2 \cdot \frac{2}{27}$$

2. (5+2+3 pont)

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} -4y_1 \\ 4y_1 - 3y_2 \end{pmatrix} = A \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \qquad \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$$

Keresd meg A sajatertekeit es sajatvektorait!

$$A = \begin{pmatrix} -4 & 0 \\ 4 & -3 \end{pmatrix}$$

$$0 = \det(A - \lambda E) = \begin{vmatrix} -4-\lambda & 0 \\ 4 & -3-\lambda \end{vmatrix} = (-4-\lambda)(-3-\lambda) - 0 \cdot 4 \longrightarrow \lambda_1 = -4, \quad \lambda_2 = -3$$

sajátvektor: $(A - \lambda E)\bar{v} = 0$

$\lambda_1 = -4$

$$\begin{pmatrix} -4-(-4) & 0 \\ 4 & -3-(-4) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \longrightarrow y = -4x$$

$$\bar{V}_1 = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$$

$\lambda_2 = -3$

$$\begin{pmatrix} -4-(-3) & 0 \\ 4 & -3-(-3) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 0 \\ 4 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \longrightarrow x = 0$$

$$\bar{V}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Ird fel a DE altalanos megoldasat!

$$\bar{Y}_{ált} = C_1 e^{-4t} \begin{pmatrix} 1 \\ -4 \end{pmatrix} + C_2 e^{-3t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Szamold ki a DE partikularis megoldasat!

$$\bar{y}(0) = C_1 e^{-4\cdot 0} \begin{pmatrix} 1 \\ -4 \end{pmatrix} + C_2 e^{-3\cdot 0} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} C_1 \\ -4C_1 + C_2 \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$$

$$\longrightarrow c_1 = 5, \quad c_2 = 23$$

$$\bar{y} = 5 \cdot e^{-4t} \begin{pmatrix} 1 \\ -4 \end{pmatrix} + 23 \cdot e^{-3t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

2b. $(2 + (1 + 2) + 5$ pont)

a) $x_0 = 7$, $x_{n+1} = \phi(x_n) = 3x_n + 7$. Mennyi x_n ?

fixpont: $x_f = 3x_f + 7 \longrightarrow x_f = -\frac{7}{2}$

$$x_n = 3^n \cdot \left(7 - \left(-\frac{7}{2}\right)\right) + \left(-\frac{7}{2}\right)$$

b) Ird fel a kovetkezo Lagrange fuggvenyekhez tartozo Euler-Lagrange egyenlet(ek)et!

$$L = y^2(y')^2 - y^4, \qquad M = (y_1')^2 + (y_2')^2 + y_1' y_2.$$

L: $\frac{d}{dx}(y^2 \cdot 2y') - (2yy'^2 - 4y^3) = 0$

M: $\frac{d}{dx}(2y_1' + 0 + y_2) - (0 + 0 + 0) = 0 = 2y_1'' + y_2' = 0$

$\frac{d}{dx}(0 + 2y_2' + 0) - (0 + 0 + y_1') = 0 = 2y_2'' - y_1' = 0$

c) Legyen adott az $S[u] = \int_0^1 (y'(x))^2 + (1 - x)y(x)\, dx$ funkcional a $[0, 1]$-en ertelmezett es a vegpontokban eltuno fuggvenyek H teren. Legyen V a $[0, 1]$-en ertelmezett, a vegpontokban eltuno es a $[0, 1/4]$, $[1/4, 3/4]$, $[3/4, 1]$ intervallumokon affin folytonos fuggvenyek tere. Legyen ϕ_1 es ϕ_2 ennek e ternek egy bazisa, ahol $\phi_1(1/4) = \phi_2(3/4) = 1$ es $\phi_2(1/4) = \phi_1(3/4) = 0$. Legyen $u_h = c_1\phi_1 + c_2\phi_2$. Szamitsd ki az $S[u_h] = s(c_1, c_2)$ ketvaltozos fuggvenyt! (Az $(1 - x)y(x)$-os tag kiszamitasara az integralban hasznalj valamilyen kozelito modszert es add is meg a modszer nevet!)

$S[u_h] \approx$

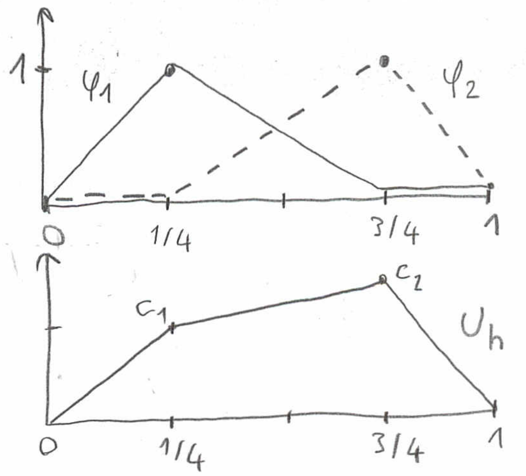
$\Delta x = \frac{3}{4} - \frac{1}{4}$

$$\approx \Bigg((4c_1)^2 \cdot \frac{1}{4} + (2(c_2 - c_1))^2 \cdot \frac{1}{2} + (-4c_2)^2 \cdot \frac{1}{4}$$

$$+ \left(\frac{1}{2}\left((1-0)\cdot 0 + \left(1 - \frac{1}{4}\right)\cdot c_1\right) \cdot \frac{1}{4} + \right.$$

$$+ \frac{1}{2}\left(\left(1 - \frac{1}{4}\right)\cdot c_1 + \left(1 - \frac{3}{4}\right)c_2\right) \cdot \frac{1}{2} +$$

$$\left. + \frac{1}{2}\left(\left(1 - \frac{3}{4}\right)\cdot c_2 + (1-1)\cdot 0\right) \cdot \frac{1}{4} \right) \Bigg)$$



U_h': $4c_1$, $2(c_2 - c_1)$, $-4c_2$

3a. (1+1+1+2 pont)
$y' = y^4 - 16.$
Keresd meg a DE fixpontjait!

$$\frac{d(y^4-16)}{dy} = 4y^3$$

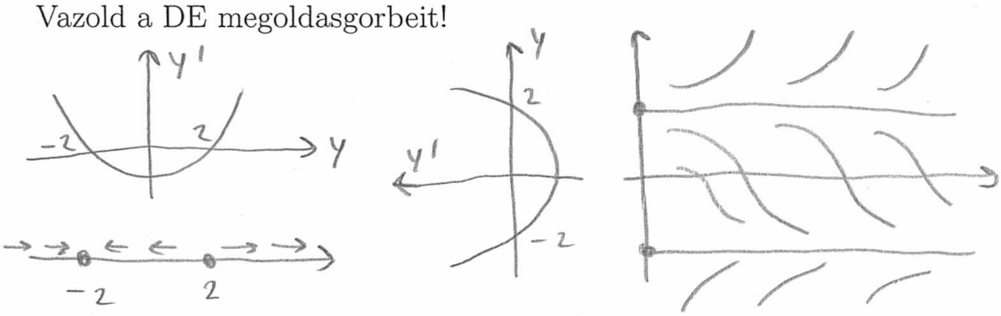
$$y^4 - 16 = 0 \longrightarrow y_1 = -2, \quad y_2 = 2$$

Ird fel a fixpontok koruli linearizalt kozelito DE-t!

$$\Delta y = y-(-2), \quad \frac{d}{dx}\Delta y = 4\cdot(-2)^3\cdot\Delta y = -32\,\Delta y \quad \Big| \quad \Delta y = y-2, \quad \frac{d}{dx}\Delta y = 4\cdot 2^3\cdot\Delta y = 32\,\Delta y$$

Ha $y(0) = 0$, mennyi
$\lim_{x\to\infty} y(x) =$ -2 $\qquad$ $\lim_{x\to-\infty} y(x) =$ 2

Vazold a DE megoldasgorbeit!



3b. (2+3 pont)

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} (y_2^2+8)(y_1-3) \\ (y_2^2-1)y_1. \end{pmatrix}$$

Keresd meg a DE fixpontjat!

$$\left.\begin{aligned} (y_2^2+8)(y_1-3) &= 0 \\ (y_2^2-1)y_1 &= 0 \end{aligned}\right\} \quad \begin{aligned} y_1 &= 3 \\ y_2 &= \pm 1 \end{aligned} \qquad P_1 = \begin{pmatrix} 3 \\ -1 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$$\mathrm{Jac} = \begin{pmatrix} \frac{\partial}{\partial y_1}\left[(y_2^2+8)(y_1-3)\right] & \frac{\partial}{\partial y_2}\left[(y_2^2+8)(y_1-3)\right] \\ \frac{\partial}{\partial y_1}\left[(y_2^2-1)y_1\right] & \frac{\partial}{\partial y_2}\left[(y_2^2-1)y_1\right] \end{pmatrix} = \begin{pmatrix} y_2^2+8 & 2y_2(y_1-3) \\ y_2^2-1 & 2y_2y_1 \end{pmatrix}$$

Ird fel a fixpont koruli linearizalt kozelito DE-t!

$$\mathrm{Jac}(P_1) = \begin{pmatrix} 9 & 0 \\ 0 & -6 \end{pmatrix}$$

$$\mathrm{Jac}(P_2) = \begin{pmatrix} 9 & 0 \\ 0 & 6 \end{pmatrix}$$

$$\frac{d}{dx}\left[\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} - \begin{pmatrix} 3 \\ -1 \end{pmatrix}\right] = \frac{d}{dx}\overline{\Delta y} = \begin{pmatrix} 9 & 0 \\ 0 & -6 \end{pmatrix}\overline{\Delta y}$$ közelítő lin. DE.

$$\frac{d}{dx}\left[\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} - \begin{pmatrix} 3 \\ 1 \end{pmatrix}\right] = \frac{d}{dx}\overline{\Delta y} = \begin{pmatrix} 9 & 0 \\ 0 & 6 \end{pmatrix}\overline{\Delta y}$$