Diff. Egy. Proba Zh2. tt

1. (a) Legyen $(\partial^2_{xx} - \cancel{\partial^2_{xx}} - 3\partial^2_{xt})\, e^{i(kx-\omega t)} = 0$. Milyen algebrai egyenletet teljesit k es ω?
   Legyen $y'' - 3y' + y = e^{5it}$. Ird fel az egyenlet egy megoldasat!

   (b) Legyen $\partial^2_{xt} e^{i(kx+\omega t)} = 0$. Milyen algebrai egyenletet teljesit k es ω? Keresd meg az egyenlet megoldasait!

a) ① $(\partial^2_{xx} - \partial^2_{tt} - 3\partial^2_{xt})\, e^{i(kx-\omega t)} = 0 = \left[(ik)^2 - (-i\omega)^2 - 3(ik)(-i\omega)\right] e^{i(kx-\omega t)}$

$$-k^2 + \omega^2 - 3k\omega = 0$$

② $y'' - 3y' + y = e^{5it}$, $\quad y = A(5)\cdot e^{5it}$, $\quad \left[(5i)^2 - 3(5i) + 1\right] A(5) e^{5it} = 1$

$$A(5) = \frac{1}{-5^2 - 3\cdot 5i + 1}, \qquad A(\omega) = \frac{1}{-\omega^2 - 3i\omega + 1}$$

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b) $\partial^2_{xt}\, e^{i(kx+\omega t)} = 0 = (ik)(i\omega)\, e^{i(kx+\omega t)}$

$$-k\omega = 0 \longrightarrow \{(k=0,\ \omega\in\mathbb{R})\} \cup \{(k\in\mathbb{R},\ \omega=0)\}$$

2. (a)

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 4y_1 \\ 4y_1 + 4y_2 \end{pmatrix} = A \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \qquad \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$$

Mennyi e^{tA} ? Ird fel a DE partikularis megoldasat!

(b)

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 5y_1 - 4y_2 \\ 4y_1 + 5y_2 \end{pmatrix} = A \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \qquad \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$$

Keresd meg A sajatertekeit es sajatvektorait! Ird fel a DE altalanos megoldasat! Mennyi e^{tA} ? Ird fel a DE partikularis megoldasat e^{tA} segitsegevel!

a) $A = \begin{pmatrix} 4 & 0 \\ 4 & 4 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 4 & 0 \end{pmatrix}$

$$\begin{pmatrix} 4t & 0 \\ 0 & 4t \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 4t & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 4t & 0 \end{pmatrix}\begin{pmatrix} 4t & 0 \\ 0 & 4t \end{pmatrix}$$

$$\exp(tA) = \exp\left[\begin{pmatrix} 4t & 0 \\ 0 & 4t \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 4t & 0 \end{pmatrix}\right] = \exp\begin{pmatrix} 4t & 0 \\ 0 & 4t \end{pmatrix} \exp\begin{pmatrix} 0 & 0 \\ 4t & 0 \end{pmatrix}$$

$$= \begin{pmatrix} e^{4t} & 0 \\ 0 & e^{4t} \end{pmatrix}\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 4t & 0 \end{pmatrix} + \frac{1}{2}\underbrace{\begin{pmatrix} 0 & 0 \\ 4t & 0 \end{pmatrix}^2}_{0} + \dots\right] = \begin{pmatrix} e^{4t} & 0 \\ 0 & e^{4t} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 4t & 1 \end{pmatrix}$$

$$= \begin{pmatrix} e^{4t} & 0 \\ 4te^{4t} & e^{4t} \end{pmatrix}. \qquad \overline{y}(t) = e^{tA}\overline{y}(0) \longrightarrow \overline{y}(t) = \begin{pmatrix} e^{4t} & 0 \\ 4te^{4t} & e^{4t} \end{pmatrix}\begin{pmatrix} 5 \\ 3 \end{pmatrix}$$

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b) $A = \begin{pmatrix} 5 & -4 \\ 4 & 5 \end{pmatrix}$. $\quad 0 = \det\begin{pmatrix} 5-\lambda & -4 \\ 4 & 5-\lambda \end{pmatrix} = (5-\lambda)^2 - (-4)\cdot 4 = \lambda^2 - 10\lambda + 41$

$$\lambda_1 = 5+4i, \quad \begin{pmatrix} 5-(5+4i) & -4 \\ 4 & 5-(5+4i) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{matrix} -4ix - 4y = 0 \\ y = -ix \end{matrix} \rightarrow \overline{V}_1 = \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

$$\lambda_2 = 5-4i, \quad \longrightarrow \dots \qquad \dots \rightarrow \overline{V}_2 = \begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$\overline{y}_{gen}(t) = C_1 e^{\lambda_1 t}\overline{V}_1 + C_2 e^{\lambda_2 t}\overline{V}_2 = C_1 e^{(5+4i)t}\begin{pmatrix} 1 \\ -i \end{pmatrix} + C_2 e^{(5-4i)t}\begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$A = \begin{pmatrix} | & | \\ V_1 & V_2 \\ | & | \end{pmatrix}\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\begin{pmatrix} | & | \\ V_1 & V_2 \\ | & | \end{pmatrix}^{-1}, \quad e^{tA} = \begin{pmatrix} | & | \\ V_1 & V_2 \\ | & | \end{pmatrix}\begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix}\begin{pmatrix} | & | \\ V_1 & V_2 \\ | & | \end{pmatrix}^{-1}$$

$$\overline{y}_{part}(t) = \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}\begin{pmatrix} e^{(5+4i)t} & 0 \\ 0 & e^{(5+4i)t} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}^{-1}\begin{pmatrix} 5 \\ 3 \end{pmatrix}$$

$$e^{(5+4i)t} = e^{5t}(\cos(4t) + i\cdot\sin(4t))$$

3. (a) $y_t(t,x) = y_{xx}(t,x)$, $y(0,x) = \cos(7x)$. Mennyi $y(t,x)$?
(b) $y_{tt}(t,x) = y_{xx}(t,x)$, $y(0,x) = \cos(7x)$, $y'(0,x) = \sin(5x)$. Mennyi $y(t,x)$?

a) ① $\partial^2_{xx}\cos(7x) = -7^2\cdot\cos(7x) \longrightarrow y(t,x) = e^{-7^2\cdot t}\cdot\cos(7x)$

② $y(t,x) = a(t)\cdot\cos(7x) \longrightarrow a'(t)\cos(7x) = a(t)\cdot(-7^2)\cos(7x)$

$$a'(t) = -7^2 a(t),\ a(0)=1 \longrightarrow a(t) = e^{-7^2\cdot t}$$

$$y(t,x) = e^{-7^2\cdot t}\cos(7x)$$

③ $$\cos(7x) = \sum_{n\in\mathbb{Z}} \hat{f}_n \frac{e^{inx}}{\sqrt{2\pi}} = \frac{e^{i7x}+e^{-i7x}}{2},\quad \hat{f}_7 = \sqrt{\frac{\pi}{2}},\ \hat{f}_{-7} = \sqrt{\frac{\pi}{2}}$$

$$= \sqrt{\frac{\pi}{2}}\cdot\frac{e^{i(-7)x}}{\sqrt{2\pi}} + \sqrt{\frac{\pi}{2}}\cdot\frac{e^{i\cdot 7\cdot x}}{\sqrt{2\pi}}$$

$$y(t,x) = \sum_{n\in\mathbb{Z}} \hat{f}_n\cdot e^{-n^2\cdot t}\frac{e^{inx}}{\sqrt{2\pi}} = \sqrt{\frac{\pi}{2}}\cdot e^{-(-7)^2 t}\cdot\frac{e^{i(-7)x}}{\sqrt{2\pi}} + \sqrt{\frac{\pi}{2}}\, e^{-7^2\cdot t}\cdot\frac{e^{i7t}}{\sqrt{2\pi}}$$

$$\partial^2_{xx}\left[\frac{e^{inx}}{\sqrt{2\pi}}\right] = -n^2\cdot\left[\frac{e^{inx}}{\sqrt{2\pi}}\right]$$

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b) $y(t,x) = a(t)\cdot\cos(7x) + b(t)\cdot\sin(5x)$

$y_{tt} = y_{xx}$: $a''(t)\cdot\cos(7x) + b''(t)\cdot\sin(5x) =$

$$= a(t)\cdot(-7^2)\cos(7x) + b(t)\cdot(-5^2)\sin(5x)$$

$$\longrightarrow a''(t) = -7^2 a(t),\quad b''(t) = -5^2 (t)$$

$y(0,x) = \cos(7x) \longrightarrow a(0) = 1,\ b(0) = 0$

$y_t(0,x) = \sin(5x) = a'(0)\cos(7x) + b'(0)\sin(5x) \longrightarrow$

$$\longrightarrow a'(0) = 0,\ b'(0) = 1$$

$$\underbrace{a'' = -7^2 a,\ a(0)=1,\ a'(0)=0}_{\displaystyle a(t) = \cos(7t)},\quad \underbrace{b'' = -5^2 b,\ b(0)=0,\ b'(0)=1}_{\displaystyle b(t) = \frac{1}{5}\sin(5t)}$$

$$y(t,x) = \cos(7t)\cos(7x) + \frac{1}{5}\sin(5t)\cdot\sin(5x)$$

4. (a) Legyen

$$\phi(0,x)=\sum_{n\in\mathbb{Z}} e^{-|n|}\frac{e^{inx}}{\sqrt{2\pi}}, \qquad \phi_t(0,x)=\sum_{n\in\mathbb{Z}} e^{-2|n|}\frac{1}{n^8}\frac{e^{inx}}{\sqrt{2\pi}},$$

$$\phi(t,x)=\sum_{n\in\mathbb{Z}} c_n(t)\frac{e^{inx}}{\sqrt{2\pi}}, \qquad \partial_{tt}\phi(t,x)=\frac{1}{4}\partial_{xx}^2\phi(t,x).$$

Ird fel a $c_n(t)$ fuggvenyekre vonatkozo kozonseges DE-ket (kezdeti feltetellel egyutt)!

(b) Legyen

$$\phi(0,x)=\sum_{n\in\mathbb{Z}} e^{-|n|}\frac{e^{inx}}{\sqrt{2\pi}}, \qquad \phi(t,x)=\sum_{n\in\mathbb{Z}} d_n(t)\frac{e^{inx}}{\sqrt{2\pi}}, \qquad \partial_t\phi(t,x)=\frac{1}{2}\partial_{xx}^2\phi(t,x).$$

Ird fel a $d_n(t)$ fuggvenyekre vonatkozo kozonseges DE-ket (kezdeti feltetellel egyutt)!

a) $\varphi''_{tt}=\frac{1}{4}\varphi''_{xx} \longrightarrow \sum_{n\in\mathbb{Z}} c''_n(t)\frac{e^{inx}}{\sqrt{2\pi}}=\frac{1}{4}\cdot\sum_{n\in\mathbb{Z}} c_n(t)(-n^2)\frac{e^{inx}}{\sqrt{2\pi}}$

$\longrightarrow c''_n(t)=\frac{1}{4}\cdot(-n^2)\,c_n(t)$

$\varphi(0,x)=\ldots \longrightarrow c_n(0)=e^{-|n|}, \quad \varphi'_t(0,x)=\ldots \longrightarrow c'_n(t)=\frac{e^{-2|n|}}{n^8}$

$c''_n=-\frac{n^2}{4}c_n \longrightarrow c_n=A\cos(\frac{n}{2}t)+B\sin(\frac{n}{2}t)$

$c'_n=-\frac{n}{2}A\sin(\frac{n}{2}t)+\frac{n}{2}B\cos(\frac{n}{2}t)$

$c_n(0)=e^{-|n|} \longrightarrow A=e^{-|n|}$

$c'_n(0)=\frac{e^{-2|n|}}{n^8} \longrightarrow B=\frac{e^{-2|n|}/n^8}{n/2}$

$c_n(t)=e^{-|n|}\cdot\cos(\frac{n}{2}t)+\frac{e^{-2|n|}/n^8}{n/2}\cdot\sin(\frac{n}{2}t)$

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b) $\varphi'_t=\frac{1}{2}\varphi''_{xx} \longrightarrow \sum_{n\in\mathbb{Z}} d'_n(t)\frac{e^{inx}}{\sqrt{2\pi}}=\frac{1}{2}\cdot\sum_{n\in\mathbb{Z}} d_n(t)(-n^2)\frac{e^{inx}}{\sqrt{2\pi}}$

$\longrightarrow d'_n(t)=-\frac{n^2}{2}d_n(t)$

$\varphi(0,x)=\ldots \longrightarrow d_n(0)=e^{-|n|}$

$d_n(t)=e^{-|n|}\cdot e^{-\frac{n^2}{2}\cdot t}$

5. (a) • Legyen

$$A=\begin{pmatrix}2i & 1-2i\\ -i-1 & -3\end{pmatrix}.$$

Mennyi A^* ?

• Mennyi a $v=(2+3i,4-i)^T$ es a $w=(4,i)^T$ vektorok belso szorzata?

• Legyen $f_1=(\sin(30°, i\cos(30°)^T$, $f_2=(\cos(30°, z)^T$ egy ortonormalt bazis. Mennyi z ?

• A $v=(5,6)^T$ vektor kifejezheto az f-ek linearis $\alpha f_1+\beta f_2$ kombinaciojakent! Mennyi α ?

(b) • Legyen $f(x)=\sin(4x)=\sum_{n\in\mathbb{Z}}\hat{f}_n\frac{e^{inx}}{\sqrt{2\pi}}$, ha $x\in(-\pi,\pi)$ Mennyi $\hat{f}_2$ es $\hat{f}_4$?

• Legyen $f(x)=H(t-1)H(2-t)=\sum_{n\in\mathbb{Z}}\hat{f}_n\frac{e^{inx}}{\sqrt{2\pi}}$, ha $x\in(-\pi,\pi)$ Mennyi $\hat{f}_2$ es $\hat{f}_3$?

• Legyen $f(x)=1$, ha $0\le x\le\pi$, maskulonben nulla. Ha $f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(p)e^{ipx}\,dx$, akkor mennyi $\hat{f}(1)$?

a) ① $(A^*)_{ij}=\overline{A}_{ji}$, $A^*=\overline{A^T}$. $\begin{pmatrix}2i & 1-2i\\ -i-1 & -3\end{pmatrix}^*=\begin{pmatrix}\overline{2i} & \overline{-i-1}\\ \overline{1-2i} & \overline{-3}\end{pmatrix}=\begin{pmatrix}-2i & i-1\\ 1+2i & -3\end{pmatrix}$

② $(v,w)=\sum_i\overline{v_i}w_i$. $\left(\begin{pmatrix}2+3i\\4-i\end{pmatrix},\begin{pmatrix}4\\i\end{pmatrix}\right)=\overline{(2+3i)}\cdot4+\overline{(4-i)}\cdot i=$

$=(8+12i)+(4i+1)=9+16i$

③ $0=(f_1,f)=\overline{\sin 30°}\cdot\cos30°+\overline{i\cdot\cos30°}\cdot z=\frac{1}{2}\cdot\frac{\sqrt{3}}{2}+(-i\frac{\sqrt{3}}{2}\cdot z)$

$\longrightarrow z=-\frac{1}{2}i$

④ $v=\alpha f_1+\beta f_2\longrightarrow(f_1,v)=(f_1,\alpha f_1+\beta f_2)=\alpha\overbrace{(f_1,f_1)}^{1}+\beta\overbrace{(f_1,f_2)}^{0}=\alpha$

$\alpha=\left(\begin{pmatrix}1/2\\ i\frac{\sqrt{3}}{2}\end{pmatrix},\begin{pmatrix}5\\6\end{pmatrix}\right)=\overline{\frac{1}{2}}\cdot5+\overline{i\frac{\sqrt{3}}{2}}\cdot6=\frac{5}{2}-3\sqrt{3}\,i$

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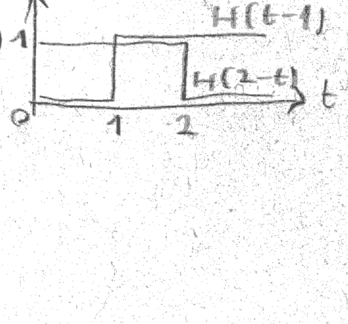
① $\hat{f}_2=\frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi}e^{-2ix}\sin4x\,dx=\frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi}e^{-2ix}\,\frac{e^{4ix}-e^{-4ix}}{2i}\,dx$

$=\frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi}\frac{1}{2i}\left[e^{2ix}-e^{-6ix}\right]dx=\frac{1}{\sqrt{2\pi}}\cdot\frac{1}{2i}\left(\left[\frac{e^{2ix}}{2i}\right]_{-\pi}^{\pi}-\left[\frac{e^{-6ix}}{-6i}\right]_{-\pi}^{\pi}\right)$

$=0$,

$\hat{f}_4=\frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi}\frac{1}{2i}\left[e^{0\cdot ix}-e^{-8ix}\right]dx=\frac{1}{\sqrt{2\pi}}\cdot\frac{1}{2i}\cdot2\pi=\frac{\sqrt{\pi}}{i\cdot\sqrt{2}}$

②



$\hat{f}_2=\frac{1}{\sqrt{2\pi}}\int_{1}^{2}e^{-2ix}\cdot1\,dx=\frac{1}{\sqrt{2\pi}}\left[\frac{e^{-2ix}}{-2i}\right]_1^2$

$=\frac{1}{\sqrt{2\pi}}\cdot\frac{1}{-2i}\left[e^{-4i}-e^{-2i}\right]$

6. (a)
- Oldd meg! $y'' - 25y = 0$, $y(0) = 0$, $y'(0) = 1$.
- Oldd meg! $G'' - 25G = \delta$, es $G(t) = 0$ negativ t-kre.
- Ird fel G segitsegevel az $y'' - 25y = f(t)$ egyenlet megoldasat, ha $y(t) = f(t) = 0$ negativ t-kre!

(b)
- Oldd meg! $y'' + 25y = 0$, $y(0) = 0$, $y'(0) = 1$.
- Oldd meg! $G'' + 25G = \delta$, es $G(t) = 0$ negativ t-kre.
- Ird fel G segitsegevel az $y'' + 25y = f(t)$ egyenlet megoldasat, ha $y(t) = f(t) = 0$ negativ t-kre!

a) 1. $y'' - 25y = 0$, $y = e^{\lambda t}$, $\lambda^2 - 25 = 0$, $\lambda_1 = 5$, $\lambda_2 = -5$

$y = Ae^{5t} + Be^{-5t}$, $y' = 5Ae^{5t} - 5Be^{-5t}$

$y(0) = 0 \to A + B = 0$, $y'(0) = 1 \to 5A - 5B = 0$ $\}$ $\to y = \frac{1}{10}e^{5t} - \frac{1}{10}e^{-5t}$

2. $t < 0 \to G(t) = 0$

$t \approx 0 \to G'' - 25G \approx G'' \approx \delta \to G(0^+) - G(0^-) = 0$

$G'(0^+) - G'(0^-) = 1$

$t > 0 \to G'' - 25G = 0$, $G(0^+) = 0$, $G'(0^+) = 1$

$$G(t) = \begin{cases} 0, & t < 0 \\ \frac{1}{10}e^{5t} - \frac{1}{10}e^{-5t}, & t > 0 \end{cases}$$

3. $$y(t) = \int_{-\infty}^{\infty} G(t-\tau) f(\tau)\, d\tau = \int_{-\infty}^{t} \left(\frac{1}{10}e^{5(t-\tau)} - \frac{1}{10}e^{-5(t-\tau)}\right) f(\tau)\, d\tau$$

$$= \int_0^t \left(\frac{1}{10}e^{5(t-\tau)} - \frac{1}{10}e^{-5(t-\tau)}\right) f(\tau)\, d\tau$$

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b) 1. $y'' + 25y = 0$, $\lambda^2 + 25 = 0$, $\lambda_1 = 5i$, $\lambda_2 = -5i$

$y = A\cos(5t) + B\sin(5t)$, $y(t) = \frac{1}{5}\sin(5t)$

2. $t > 0 \to G'' + 25G = 0$

$$G(t) = \begin{cases} 0 & t < 0 \\ \frac{1}{5}\sin(5t) & t > 0 \end{cases}$$

3. $$y(t) = \int_{-\infty}^{t} \frac{1}{5}\sin(5[t-\tau])\, f(\tau)\, d\tau$$

$$= \int_0^t \frac{1}{5}\sin(5[t-\tau])\, f(\tau)\, d\tau$$

7. (a)
- Oldd meg! $y' - 25y = 0$, $y(0) = 1$.
- Oldd meg! $G' - 25G = \delta$, es $G(t) = 0$ negativ t-kre.
- Ird fel G segitsegevel az $y' - 25y = f(t)$ egyenlet megoldasat, ha $y(t) = f(t) = 0$ negativ t-kre!

a) 1 $y' - 25y = 0 \longrightarrow y = e^{25 \cdot t} \cdot C$, $y(0) = 1 \longrightarrow y = e^{25 \cdot t}$

2 $t < 0 \longrightarrow G(t) = 0$

$t \approx 0 \longrightarrow G' - 25G \approx G' \approx \delta \longrightarrow G(0^+) - G(0^-) = 1$

$t > 0 \longrightarrow G' - 25G = 0$, $G(0^+) = 1 \longrightarrow G = e^{25 \cdot t}$

$$G(t) = \begin{cases} 0, & t < 0 \\ e^{25 \cdot t}, & t > 0 \end{cases}$$

3 $$y(t) = \int_{-\infty}^{\infty} G(t-\tau) \cdot f(\tau)\, d\tau = \int_{-\infty}^{t} e^{25(t-\tau)} f(\tau)\, d\tau$$

$$= \int_{0}^{t} e^{25(t-\tau)} f(\tau)\, d\tau$$

8. (a)
- Szamitsd ki a Laplace tr. definicioja alapjan: $\mathcal{L}(\sin(2t))$
- Szamitsd ki a Laplace tr. definicioja alapjan: $\mathcal{L}(H(-t+4)e^{-5t})$
- Szamold ki az $f(t) = H(t-3)$ es a $g(t) = e^{5t}$ fuggvenyek $h = f * g$ konvoluciojat!
- Mennyi $\mathcal{L}(f(t))\mathcal{L}(g(t)) - \mathcal{L}(h(t))$?

(b)
- Szamitsd ki a Laplace tr. definicioja alapjan: $\mathcal{L}(tH(t-2))$
- Szamitsd ki a Laplace tr. definicioja alapjan: $\mathcal{L}(H(-t-4)e^{-5t})$
- Szamold ki az $f(t) = \sin(t)$ es a $g(t) = e^{5t}$ fuggvenyek $h = f * g$ konvoluciojat!
- Mennyi $\mathcal{L}(f(t))\mathcal{L}(g(t)) - \mathcal{L}(h(t))$?

a) ① $\mathcal{L}(\sin(2t)) = \int_0^\infty e^{-st}\frac{e^{2it}-e^{-2it}}{2i}\,dt = \frac{1}{2i}\int_0^\infty e^{-(s-2i)t} - e^{-(s+2i)t}\,dt$

$$= \frac{1}{2i}\left(\left[\frac{e^{-(s-2i)t}}{-(s-2i)}\right]_0^\infty - \left[\frac{e^{-(s+2i)t}}{s+2i}\right]_0^\infty\right) =$$

$$= \frac{1}{2i}\left(\left[0-\frac{1}{-(s-2i)}\right]-\left[0-\frac{1}{-(s+2i)}\right]\right) = \frac{2}{s^2+2^2}$$

② $\mathcal{L}(H(-t+4)e^{-5t}) = \int_0^\infty e^{-st}H(-t+4)e^{-5t}\,dt =$

$$= \int_0^4 e^{-(s+5)t}\,dt = \left[\frac{e^{-(s+5)t}}{-(s+5)}\right]_0^4 = \left(-\frac{e^{-(s+5)\cdot 4}}{s+5}\right)-\left(-\frac{1}{s+5}\right)$$

③ $(f*g)(t) = \int_0^t H(t-3-\tau)e^{5\tau}\,d\tau = \begin{cases}\int_{t-3}^{t} e^{5\tau}\,d\tau, & t>3\\ 0, & t<3\end{cases}$

$$= \begin{cases}\left[\frac{e^{5\tau}}{5}\right]_{t-3}^{t}, & t>3\\ 0, & t<3\end{cases} \qquad \frac{1}{5}\cdot\left(e^{5t}-e^{5(t-3)}\right)$$

④ $=0$

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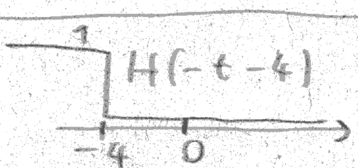
② $=0$



③ $(f*g)(t) = \int_0^t \sin(t-\tau)e^{3\tau}\,d\tau = \int_0^t \frac{e^{i(t-\tau)}-e^{-i(t-\tau)}}{2i}\cdot e^{3\tau}\,d\tau$

$$= \frac{e^{it}}{2i}\int_0^t e^{(3-i)\tau}\,d\tau - \frac{e^{-it}}{2i}\int_0^t e^{(3+i)\tau}\,d\tau$$

$$= \frac{e^{it}}{2i}\cdot\frac{1}{3-i}\left(e^{(3-i)t}-1\right) - \frac{e^{-it}}{2i}\cdot\frac{1}{3+i}\left(e^{(3+i)t}-1\right)$$

④ $=0$

9. (a) $y' - 4y = 3$, $y(0) = 2$.

- Szamold ki $y(t)$-nek az $Y(s)$ Laplace transzformaltjat!
- Szamold ki $Y(s)$ parcialis tort felbontasat!
- Mennyi $\mathcal{L}^{-1}(Y(s))$?
- Mennyi $y(t)$?

$$\left[\begin{array}{l} \mathcal{L}(y'(t)) = s\cdot\mathcal{L}(y(t)) - y(0) = sY(s) - y(0) \\ \mathcal{L}(y''(t)) = s^2\cdot\mathcal{L}(y(t)) - s\,y(0) - y'(0) = s^2 Y s - s\,y(0) - y'(0) \\ \mathcal{L}(1) = \frac{1}{s},\ \mathcal{L}(e^{at}) = \frac{1}{s-a} \end{array}\right]$$

(1) $y' - 4y = 3,\ y(0) = 2 \xrightarrow{\mathcal{L}} (sY(s) - 2) - 4Y(s) = 3\cdot\frac{1}{s}$

$$Y(s) = \frac{1}{s-4}\cdot\left(\frac{3}{s} + 2\right)$$

(2) $\frac{3}{(s-4)\cdot s} = \frac{A}{s-4} + \frac{B}{s} = \frac{As + B(s-4)}{(s-4)\cdot s} = \frac{(A+B)s - 4B}{(s-4)\cdot s}$

$A + B = 0,\ -4B = 3$, $\frac{3}{(s-4)\cdot s} = \frac{3/4}{s-4} + \frac{-3/4}{s}$

$B = -\frac{3}{4},\ A = \frac{3}{4}$

$$Y(s) = \frac{2\frac{3}{4}}{s-4} - \frac{3/4}{s}$$

(3,4) $y(t) = \mathcal{L}^{-1}(Y(s)) = 2\frac{3}{4}\cdot\mathcal{L}^{-1}\left(\frac{1}{s-4}\right) - \frac{3}{4}\mathcal{L}^{-1}\left(\frac{1}{s}\right)$

$= 2\frac{3}{4}\cdot e^{4t} - \frac{3}{4}\cdot 1$

10. (a) $y'' - 4y = 3$, $y(0) = 2$, $y'(0) = 20$

- Szamold ki $y(t)$-nek az $Y(s)$ Laplace transzformaltjat!
- Szamold ki $Y(s)$ parcialis tort felbontasanak a strukturajat!
- Mennyi $y(t)$?

(b) $y'' + 4y = 3$, $y(0) = 2$, $y'(0) = 20$

- Szamold ki $y(t)$-nek az $Y(s)$ Laplace transzformaltjat!
- Ird fel $Y(s)$ parcialis tort felbontasanak a strukturajat!
- Mennyi $\mathcal{L}^{-1}(Y(s))$?
- Mennyi $y(t)$?

a) 1 $\left(s^2 Y(s) - 2s - 20\right) - 4Y(s) = \frac{3}{s}$

$$Y(s) = \frac{1}{s^2-4}\left(\frac{3}{s} + 2s + 20\right)$$

2 $Y(s) = \dfrac{pol_2(s)}{(s-2)(s+2)\cdot s} = \dfrac{A}{s-2} + \dfrac{B}{s+2} + \dfrac{C}{s}$

3,4 $\mathcal{L}^{-1}(Y(s)) = y(t) = A\mathcal{L}^{-1}\left(\frac{1}{s-2}\right) + B\mathcal{L}^{-1}\left(\frac{1}{s+2}\right) + C\mathcal{L}^{-1}\left(\frac{1}{s}\right)$

$$= Ae^{2t} + Be^{-2t} + C\cdot 1$$

---

b) 1 $Y(s) = \dfrac{1}{s^2+4}\left(\dfrac{3}{s} + 2s + 20\right)$

2 $Y(s) = \dfrac{pol_2(s)}{(s-2i)(s+2i)s} = \dfrac{A}{s-2i} + \dfrac{B}{s+2i} + \dfrac{C}{s}$

3,4 $y(t) = \mathcal{L}^{-1}(Y(s)) = Ae^{2it} + Be^{-2it} + C$

$$= \tilde{A}\cos(2t) + \tilde{B}\sin(2t) + C$$

11. (a) Oldd meg: $y'' - 25y = 0$, $y(0) = 10$, $y'(0) = 20$.

(b) Oldd meg az allando varialasanak a modszerevel: $y' + 3y = 5$.

a) 1 $y''-25y=0$, $y=e^{\lambda t}$, $y'=\lambda e^{\lambda t}$, $y''=\lambda^2 e^{\lambda t}$

$\lambda^2 e^{\lambda t}-25e^{\lambda t}=0=e^{\lambda t}(\lambda^2-25)$

$(\lambda^2-25)=0 \longrightarrow \lambda_1=5$, $\lambda_2=-5$

$y=C_1\cdot e^{5t}+C_2\cdot e^{-5t}$

$y'=5C_1\cdot e^{5t}-5C_2\cdot e^{-5t}$

$y(0)=10 \longrightarrow C_1+C_2=10$ $\longrightarrow$ $C_1=7$

$y'(0)=20 \longrightarrow 5C_1-5C_2=20$ $\quad C_2=3$

$y_{part}=7e^{5t}+3\cdot e^{-5t}$

2 Oldd meg: $y''+25y=0$, $y(0)=10$, $y'(0)=20$

$(\lambda^2+25)=0 \longrightarrow \lambda_1=5i$, $\lambda_2=-5i$

A $y=C_1 e^{5i\cdot t}+C_2 e^{-5i\cdot t}$

B $y=\tilde{C}_1\cos(5t)+\tilde{C}_2\sin(5t)$

$y'=-5\tilde{C}_1\sin(5t)+5\tilde{C}_2\cos(5t)$

$y(0)=10 \longrightarrow \tilde{C}_1=10$

$y'(0)=20 \longrightarrow \tilde{C}_2=4$

$y_{part}(t)=10\cos(5t)+4\sin(4t)$

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b) $\boxed{\begin{array}{l} y'+3y=0 \\ y=C\cdot e^{-3t}\end{array}}$ $\quad y=C(t)\cdot e^{-3t}$

$y'=C'(t)e^{-3t}+C(t)\cdot(-3)e^{-3t}$

$(C'\cdot e^{-3t}-3Ce^{-3t})+3Ce^{-3t}=5$ $\quad \longleftarrow y'+3y=5$

$C'e^{-3t}=5$

$C'=5e^{3t} \longrightarrow C=\int 5e^{3t}dt=\frac{5}{3}e^{3t}+k$

$y=C\cdot e^{-3t}=\left(\frac{5}{3}e^{3t}+k\right)e^{-3t}=\frac{5}{3}+ke^{-3t}$

12. (a) Legyen

$$\frac{d}{dt}\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} + \begin{pmatrix} 0 & 3 \\ -3 & 0 \end{pmatrix}\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} 1+t \\ t+t^2 \end{pmatrix}, \qquad \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}.$$

Mennyi

$$\begin{pmatrix} Y_1(s) \\ Y_2(s) \end{pmatrix} ?$$

(Nem kell kiszamolni a matix inverzet!)

$$\begin{pmatrix} s\cdot Y_1(s) - 3 \\ s\cdot Y_2(s) - 4 \end{pmatrix} + \begin{pmatrix} 0 & 3 \\ -3 & 0 \end{pmatrix}\begin{pmatrix} Y_1(s) \\ Y_2(s) \end{pmatrix} = \begin{pmatrix} \frac{1}{s} + \frac{1}{s^2} \\ \frac{1}{s^2} + \frac{2}{s^3} \end{pmatrix} \longleftarrow \mathcal{L}(t^n) = \frac{n!}{s^{n+1}}$$

$$\begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix}\begin{pmatrix} Y_1(s) \\ Y_2(s) \end{pmatrix} + \begin{pmatrix} 0 & 3 \\ -3 & 0 \end{pmatrix}\begin{pmatrix} Y_1(s) \\ Y_2(s) \end{pmatrix} = \begin{pmatrix} \frac{1}{s} + \frac{1}{s^2} + 3 \\ \frac{1}{s^2} + \frac{2}{s^3} + 4 \end{pmatrix}$$

$$\begin{pmatrix} 0+s & 3 \\ -3 & 0+s \end{pmatrix}\begin{pmatrix} Y_1(s) \\ Y_2(s) \end{pmatrix} = \text{—\,''\,—}$$

$$\begin{pmatrix} Y_1(s) \\ Y_2(s) \end{pmatrix} = \begin{pmatrix} 0+s & 3 \\ -3 & 0+s \end{pmatrix}^{-1}\begin{pmatrix} \frac{1}{s} + \frac{1}{s^2} + 3 \\ \frac{1}{s^2} + \frac{2}{s^3} + 4 \end{pmatrix}$$

13. (a) Legyen

$$\frac{d}{dt}\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} + \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix}, \qquad f_1(t) = f_2(t) = y_1(t) = y_2(t) = 0, \text{ ha } t < 0.$$

Keresd meg a rendszer fundamentalis megoldasat, es a segitsegevel ird fel a DE megoldasat!

$$① \ \frac{d}{dt}\begin{pmatrix} G_{11} \\ G_{21} \end{pmatrix} + \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} G_{11} \\ G_{21} \end{pmatrix} = \begin{pmatrix} \delta \\ 0 \end{pmatrix}, \quad \frac{d}{dt}\begin{pmatrix} G_{12} \\ G_{22} \end{pmatrix} + \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} G_{12} \\ G_{22} \end{pmatrix} = \begin{pmatrix} 0 \\ \delta \end{pmatrix}$$

$$\frac{d}{dt}\underbrace{\begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix}}_{G} + \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} = \begin{pmatrix} \delta & 0 \\ 0 & \delta \end{pmatrix}$$

$$t < 0 \longrightarrow G\,\cancel{(t)} = 0$$

$$t \approx 0 \longrightarrow \begin{pmatrix} G_{11}(0^+) & G_{12}(0^+) \\ G_{21}(0^+) & G_{22}(0^+) \end{pmatrix} - \begin{pmatrix} G_{11}(0^-) & G_{12}(0^+) \\ G_{21}(0^-) & G_{22}(0^-) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$t > 0 \quad \frac{d}{dt} G = \begin{pmatrix} -3 & -1 \\ -1 & -3 \end{pmatrix} G$$

$$\frac{d}{dt}\begin{pmatrix} G_{11} \\ G_{21} \end{pmatrix} = -\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} G_{11} \\ G_{21} \end{pmatrix}, \begin{pmatrix} G_{11}(0) \\ G_{21}(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \frac{d}{dt}\begin{pmatrix} G_{12} \\ G_{22} \end{pmatrix} = -\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} G_{12} \\ G_{22} \end{pmatrix},$$

$$\begin{pmatrix} G_{12}(0) \\ G_{22}(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} -3 & -1 \\ -1 & -3 \end{pmatrix} \longrightarrow \begin{matrix} \lambda_1 = -4 & \overline{V}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ \lambda_2 = -2 & \overline{V}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \end{matrix}$$

$t > 0$

$$\begin{pmatrix} G_{11} \\ G_{21} \end{pmatrix} = \frac{1}{2} e^{-4t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{2} e^{-2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \begin{pmatrix} G_{12} \\ G_{22} \end{pmatrix} = \frac{1}{2} e^{-4t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{1}{2} e^{-2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$G(t) = \begin{cases} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, & t < 0 \\ \frac{1}{2} e^{-4t}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \frac{1}{2} e^{-2t}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, & t > 0 \end{cases}$$

$$\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \int_{-\infty}^{\infty} \begin{pmatrix} G_{11}(t-\tau) & G_{12}(t-\tau) \\ G_{21}(t-\tau) & G_{22}(t-\tau) \end{pmatrix}\begin{pmatrix} f_1(\tau) \\ f_2(\tau) \end{pmatrix} d\tau$$