B Test 2, Diff.Eq., 2015.05.04. NEPTUN: Name:

1. (2+1+2+3+1+1)

Use the definition of the Laplace tr. for the computation of $F(s) = \mathcal{L}(f(t)) = \mathcal{L}(e^{5t-7})$.

$F(s)=$

$$\int_0^\infty e^{-st} e^{5t-7}\, dt = e^{-7} \int_0^\infty e^{-(s-5)t}\, dt =$$

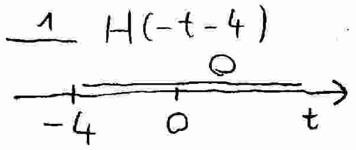
$$= e^{-7} \left[\frac{e^{-(s-5)t}}{-(s-5)} \right]_0^\infty = e^{-7} \cdot \frac{1}{s-5}$$

For what values of s does the improper integral exist?

$$Re(s) > 5$$

$F(s) = \mathcal{L}(f(t)) = \mathcal{L}(H(-t-4)e^{-5t})$ (Here H is the Heaviside function.)

$F(s)=$



As $H(-t-4) = 0$ if $t > 0$,

$$\mathcal{L}(f(t)) = 0$$

Compute the $h = f * g$ convolution of $f(t) = 4t$ and $g(t) = 3$!

$$(f*g)(t) = \int_0^t 4(t-\tau)\cdot 3\, d\tau = 12 \int_0^t t - \tau\, d\tau = 12 \left[t\tau - \frac{\tau^2}{2} \right]_0^t$$

$$= 12\left(t^2 - \frac{t^2}{2}\right) = 6t^2$$

Compute the $h = g * f$ convolution of $f(t) = 4t$ and $g(t) = 3$! $(f*g)(t) = (g*f)(t) = 6t^2$

convolution is commutative

How much is $\mathcal{L}(f(t))\mathcal{L}(g(t)) - \mathcal{L}(h(t))$?

$$= 0$$

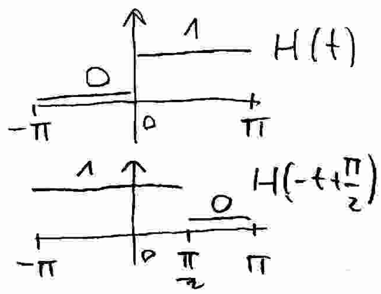
2. (2+2+3+3)
Let $f_1 = (i/\sqrt{2}, i/\sqrt{2})^T$, $f_2 = (1/\sqrt{2}, z)^T$ be an orthonormal basis of $\mathbb{C}^2$. How much is z?

$$(f_1, f_2) = \overline{\frac{i}{\sqrt{2}}} \cdot \frac{1}{\sqrt{2}} + \overline{\frac{i}{\sqrt{2}}} \cdot z = -\frac{i}{2} - \frac{i}{\sqrt{2}} z = 0$$

$$z = \frac{-1}{\sqrt{2}}$$

The vector $v = (7, 8)^T$ can be expressed as a linear combination $v = \alpha f_1 + \beta f_2$! Compute α !

$$\alpha = (f_1, v) = \overline{\frac{i}{\sqrt{2}}} \cdot 7 + \overline{\frac{i}{\sqrt{2}}} \cdot 8 = -i \cdot \frac{15}{\sqrt{2}}$$



Let $f(x) = H(t)H(-t + \pi/2) = \sum_{n \in \mathbb{Z}} \hat{f}_n \frac{e^{inx}}{\sqrt{2\pi}}$, if $x \in (-\pi, \pi)$ Compute $\hat{f}_5$!

$$\hat{f}_5 = \left(\frac{e^{i5x}}{\sqrt{2\pi}}, f(x) \right) = \int_{-\pi}^{\pi} \frac{\overline{e^{i5x}}}{\sqrt{2\pi}} \cdot f(x)\, dx$$

$$= \int_0^{\pi/2} \frac{1}{\sqrt{2\pi}} e^{-i5x}\, dx = \frac{1}{\sqrt{2\pi}} \left[\frac{e^{-i5x}}{-i \cdot 5} \right]_0^{\pi/2} = \frac{i}{\sqrt{2\pi} \cdot 5} \left(e^{-\frac{5\pi}{2} i} - 1 \right)$$

$$e^{-\frac{5\pi}{2} i} = e^{-\frac{\pi}{2} i} = -i$$

$$= \frac{1}{5\sqrt{2\pi}} (1 - i)$$

Express $\hat{f}_{-5} \frac{e^{i(-5)x}}{\sqrt{2\pi}} + \hat{f}_5 \frac{e^{i5x}}{\sqrt{2\pi}}$ with the help of trigonometric functions!

$$\hat{f}_{-5} = \frac{1}{5\sqrt{2\pi}} (1 + i)$$

$$= \frac{1}{5\sqrt{2\pi}} \cdot \frac{1}{\sqrt{2\pi}} \cdot \Big((1 - i)(\cos[5x] + i \cdot \sin[5x]) + (1 - i)(\cos(5x) - i \cdot \sin(5x)) \Big)$$

$$= \frac{1}{5\pi} \left(\cos[5x] + \sin[5x] \right)$$

$= t^2 + 2t + 1$

3. $(3+2+1+4)$

$y'' - 4y = (t+1)^2$, $y(0) = 6$, $y'(0) = 7$. How much is $Y(s)$? ($\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}$)

$Y(s) =$

$$\left(s^2 Y(s) - 6s - 7\right) - 4Y(s) = \frac{2}{s^3} + 2 \cdot \frac{1}{s^2} + \frac{1}{s}$$

$$Y(s) = \frac{1}{s^2 - 4}\left(\frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s} + 6s + 7\right)$$

$$s^2 - 4 = (s+2)(s-2)$$

Write down the partial fraction decomposition of $Y(s)$! (Do not compute the coefficients!)

$$Y(s) = \frac{A}{s+2} + \frac{B}{s-2} + \frac{C}{s^3} + \frac{D}{s^2} + \frac{E}{s}$$

How much is $y(t)$?

$$y(t) = A e^{-2t} + B e^{2t} + \frac{C}{2} t^2 + Dt + E$$

Let

$$\frac{d}{dt}\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} + \begin{pmatrix} 0 & 3 \\ -3 & 0 \end{pmatrix}\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} 1+t \\ t+t^2 \end{pmatrix}, \qquad \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}.$$

Compute

$$\begin{pmatrix} Y_1(s) \\ Y_2(s) \end{pmatrix}$$

(Do not compute the inverse matrix!)

$$\begin{pmatrix} sY_1(s) - 3 \\ sY_2(s) - 4 \end{pmatrix} + \begin{pmatrix} 0 & 3 \\ -3 & 0 \end{pmatrix}\begin{pmatrix} Y_1(s) \\ Y_2(s) \end{pmatrix} = \begin{pmatrix} \frac{1}{s} + \frac{1}{s^2} \\ \frac{1}{s^2} + \frac{2}{s^3} \end{pmatrix}$$

$$\begin{pmatrix} Y_1(s) \\ Y_2(s) \end{pmatrix} = \begin{pmatrix} 0+s & 3 \\ -3 & 0+s \end{pmatrix}^{-1}\begin{pmatrix} \frac{1}{s} + \frac{1}{s^2} + 3 \\ \frac{1}{s^2} + \frac{2}{s^3} + 4 \end{pmatrix}$$

4. (3+2+3+2)
Let

$$\phi(0,x) = \sum_{n\in\mathbb{Z}} n^{-2}\sin(n)\frac{e^{inx}}{\sqrt{2\pi}}, \quad \phi(t,x) = \sum_{n\in\mathbb{Z}} c_n(t)\frac{e^{inx}}{\sqrt{2\pi}}, \quad \partial_t\phi(t,x) = 6\partial^2_{xx}\phi(t,x).$$

What ordinary differential equations are satisfied by the functions $c_n(t)$? (Do not forget the initial conditions!)

$$\partial_t \sum c_n(t)\frac{e^{inx}}{\sqrt{2\pi}} = \sum c_n'(t)\frac{e^{inx}}{\sqrt{2\pi}}$$

$$\partial_{xx} \sum c_n(t)\frac{e^{inx}}{\sqrt{2\pi}} = \sum c_n(t)\,(+in)^2\frac{e^{inx}}{\sqrt{2\pi}} = \sum c_n(t)$$

$$c_n'(t) = 6\cdot(-n^2)\,c_n\,, \quad c_n(0) = n^{-2}\sin(n)$$

Compute $c_5(6)$!

$$c_5(6) = 5^{-2}\cdot\sin(5)\cdot e^{-6\cdot 5^2\cdot t} = \frac{\sin(5)\,e^{-150t}}{25}$$

Let $(f,g) = \int_0^\pi \bar{f}(x)g(x)\,dx$.
Compute $(\sin(x), \sin(2x))$!

$$= \int_0^\pi \overline{\frac{e^{ix}-e^{-ix}}{2i}}\cdot\frac{e^{2ix}-e^{-2ix}}{2i}\,dx = -\frac{1}{4}\int_0^\pi e^{3ix}+e^{-3ix}-e^{ix}-e^{-ix}\,dx$$

$$= -\frac{1}{4}\left[\frac{e^{3ix}}{3i}+\frac{e^{-3ix}}{-3i}-\frac{e^{ix}}{i}-\frac{e^{-ix}}{-i}\right]_0^\pi = 0 \qquad \begin{aligned} e^{3i\pi} &= e^{-3i\pi} \\ &= e^{i\pi} = e^{-i\pi} = -1 \end{aligned}$$

numerators take the same values at π, (and at 0, too)

Compute $(\sin(x), \cos(x))$!

$$= \int_0^\pi \overline{\frac{e^{ix}-e^{-ix}}{2i}}\cdot\frac{e^{ix}+e^{-ix}}{2}\,dx = \int_0^\pi \frac{e^{-ix}-e^{ix}}{-2i}\cdot\frac{e^{ix}+e^{-ix}}{2}\,dx$$

$$= -\frac{1}{4i}\int_0^\pi \underset{e^{-ix}\cdot e^{ix}}{1} - 1 - e^{2ix} + e^{-2ix}\,dx = -\frac{1}{4i}\left[-\frac{e^{2ix}}{2i}+\frac{e^{-2ix}}{-2i}\right]_0^\pi$$

$$= 0 \qquad 1 = e^{2i\pi} = e^{-2i\pi}$$