A PZh2, Diff.Egy., 2015.05.12. NEPTUN: Gyak.Vez.:

Név: Aláírás:

1.(1+2+2+3+2 pont)
Legyen

$$A = \begin{pmatrix} 2i & -2i \\ 7 & 3i \end{pmatrix}.$$

Mennyi A^* ?

$(A^*)_{ij} = \overline{A_{ji}}$ $\qquad A^* = \begin{pmatrix} -2i & 7 \\ 2i & -3i \end{pmatrix}$

Legyen $f_1 = (\sin(-30°), i\cos(-30°))^T$, $f_2 = (\cos(-30°), z)^T$ egy ortonormalt bazis. Mennyi z ?

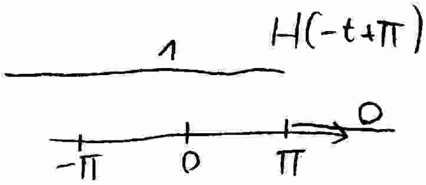
$\sin(-30°) = -\frac{1}{2}$
$\cos(-30°) = \frac{\sqrt{3}}{2}$

$0 = (f_1, f_2) = \overline{-\frac{1}{2}} \cdot \frac{\sqrt{3}}{2} + \overline{i\frac{\sqrt{3}}{2}} \cdot z = -\frac{\sqrt{3}}{4} - i\frac{\sqrt{3}}{2} z = 0$

$z = \frac{i}{2}$

A $v = (1,2)^T$ vektor kifejezheto az f-ek linearis $\alpha f_1 + \beta f_2$ kombinaciojakent! Mennyi α ?

$\alpha = (f_1, v) = \overline{-\frac{1}{2}} \cdot 1 + \overline{\frac{i\sqrt{3}}{2}} \cdot 2 = -\frac{1}{2} - \sqrt{3} \cdot i$



Legyen $f(x) = -H(-x+\pi) = \sum_{n\in\mathbb{Z}} \hat{f}_n \frac{e^{inx}}{\sqrt{2\pi}}$, ha $x \in (-\pi,\pi)$ Mennyi $\hat{f}_{-2}$?

$H = 1$ a $(-\pi, \pi)$ intervallumon, $\hat{f}_{-2} = 0$

vagy

$\hat{f}_{-2} = \left(\frac{e^{-2ix}}{\sqrt{2\pi}}, -1\right) = \int_{-\pi}^{\pi} \overline{\frac{e^{-2ix}}{\sqrt{2\pi}}} (-1) dx = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} -e^{2ix} dx = \frac{-1}{\sqrt{2\pi}} \left[\frac{e^{2ix}}{2i}\right]_{-\pi}^{\pi}$

$= \frac{-1}{\sqrt{2\pi}} \cdot \frac{1}{2i} \left(\underset{1}{\underset{\parallel}{e^{2i\pi}}} - \underset{1}{\underset{\parallel}{e^{-2i\pi}}} \right) = 0$

Legyen $f(x) = -H(-x+\pi) = \sum_{n\in\mathbb{Z}} \hat{f}_n \frac{e^{inx}}{\sqrt{2\pi}}$, ha $x \in (-\pi,\pi)$ Mennyi $\hat{f}_0$?

$\hat{f}_0 = \left(\frac{\overset{e^{i \cdot 0 \cdot x}}{1}}{\sqrt{2\pi}}, -1 \right) = \int_{-\pi}^{\pi} \overline{\frac{1}{\sqrt{2\pi}}} (-1) dx = \frac{-1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} 1 \, dx = -\sqrt{2\pi}$

1

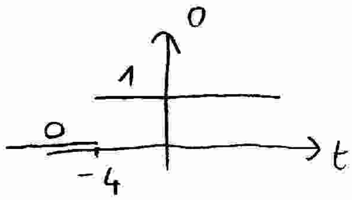
2. (2+1+3+3+1 pont)
Szamitsd ki a Laplace tr. definicioja alapjan a kovetkezoket:
a) $F(s) = \mathcal{L}(f(t)) = \mathcal{L}(\sin(3t-1))$.
$F(s)=$ $\int_0^\infty e^{-st}\,\frac{e^{i(3t-1)} - e^{-i(3t-1)}}{2i}\,dt =$

$$= \frac{1}{2i}\left\{\left[\frac{e^{-(s-3i)t}\cdot e^{-i}}{-(s-3i)}\right]_0^\infty - \left[\frac{e^{-(s+3i)t}\cdot e^{i}}{-(s+3i)}\right]_0^\infty\right\} =$$

$$= \frac{1}{2i}\left\{e^{-i}\cdot\frac{1}{(s-3i)} - e^{i}\frac{1}{s+3i}\right\}$$

Esetunkben milyen s eseten letezik a Laplace transzformaciot definialo impropius integral?

$\mathrm{Re}\ s > 0$



$F(s) = \mathcal{L}(f(t)) = \mathcal{L}(\underbrace{H(t+4)}_{=1,\ \text{ha}\ t>0}e^{5t})$ (Itt H a Heaviside fuggveny.)
$F(s)=$

$$F(s) = \int_0^\infty e^{-st}H(t+4)e^{5t}\,dt = \int_0^\infty e^{-st}\cdot e^{5t}\,dt = \left[\frac{e^{-(s-5)t}}{-(s-5)}\right]_0^\infty$$

$$= \frac{0}{-(s-5)} - \frac{1}{-(s-5)} = \frac{1}{s-5}$$

b) Szamold ki az $f(t) = t^2$ es a $g(t) = t$ fuggvenyek $h = f * g$ konvoluciojat!

$$h(t) = \int_0^t f(t-\tau)\,g(\tau)\,d\tau = \int_0^t (t-\tau)^2\cdot\tau\,d\tau$$

$$= \int_0^t t^2\tau - 2t\tau^2 + \tau^3\,d\tau = t^2\left[\frac{\tau^2}{2}\right]_0^t - 2t\left[\frac{\tau^3}{3}\right]_0^t + \left[\frac{\tau^4}{4}\right]_0^t$$

$$= \frac{t^4}{2} - 2t\frac{t^3}{3} + \frac{t^4}{4} = t^4\left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right) = \frac{t^4}{12}$$

Mennyi $\mathcal{L}(f(t))\mathcal{L}(g(t)) - \mathcal{L}(h(t))$? $= 0$

3. (5 × 2 pont)
$y'' + 4y = 5t^4,\ y(0) = 2,\ y'(0) = 3$. Mennyi $Y(s)$? ($\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}$)
$Y(s) =$

$$\left(s^2 Y(s) - 2s - 3\right) + 4Y(s) = 5 \cdot \frac{4!}{s^5}$$

$$Y(s) = \frac{1}{s^2+4}\left(2s+3+\frac{120}{s^5}\right)$$

Milyen $Y(s)$ parcialis tort felbontasanak a strukturaja? Mennyi $y(t)$?

$$Y(s) = \frac{A}{s+2i} + \frac{B}{s-2i} + \frac{C}{s^5} + \frac{D}{s^4} + \frac{E}{s^3} + \frac{F}{s^2} + \frac{G}{s}$$

$$= \frac{A}{s+2i} + \frac{B}{s-2i} + \frac{24}{s^5}\cdot\frac{C}{24} + \frac{6}{s^4}\cdot\frac{D}{6} + \frac{2}{s^3}\cdot\frac{E}{2} + \frac{F}{s^2} + \frac{G}{s}$$

$\leftarrow 4!$ $\leftarrow 3!$

$$y(t) = A e^{-2it} + B e^{2it} + \frac{C}{24}t^4 + \frac{D}{6}t^3 + \frac{E}{2}t^2 + Ft + G$$

vagy $\tilde{A}\cos(2t) + \tilde{B}\sin(2t)$

Mi a megoldasa a $y'' + 4y = 0,\ y(0) = 6,\ y'(0) = 7$ DE-nek?

$\lambda^2 + 4 = 0,\ \lambda_{1,2} = \pm 2i$

$$y_{ált} = C_1\cos(2t) + C_2\sin(2t)$$
$$y'_{ált} = -2C_1\sin(2t) + 2C_2\cos(2t)$$

$y(0) = 6 \rightarrow C_1 \cdot 1 + C_2 \cdot 0 = 6$

$y'(0) = 7 \rightarrow -2C_1 \cdot 0 + 2C_2 \cdot 1 = 7$

$$y_{part} = 6\cdot\cos(2t) + \frac{7}{2}\sin(2t)$$

Oldd meg a $G'' + 4G = \delta(t)$ egyenletet, ahol $G(t) = 0$, ha $t < 0$!

$t \approx 0$: $G'' \approx \delta(t) \longrightarrow G(0^+) - G(0^-) = 0,\ G'(0^+) - G'(0^-) = 1$

$t > 0$: $G'' + 4G = 0 \longrightarrow G(t) = \frac{1}{2}\sin(2t)$

$$G(t) = \begin{cases} 0, & \text{ha } t < 0 \\ \frac{1}{2}\sin(2t), & \text{ha } t > 0 \end{cases}$$

Ird fel $G(t)$ segitsegevel az $y'' + 4y = f(t)$ egyenlet megoldasat, ha $y(t) = f(t) = 0$ amikor $t < 0$!

$$y(t) = \int_{-\infty}^{\infty} G(t-\tau) f(\tau)\, d\tau = \int_{-\infty}^{t} \sin(2(t-\tau)) f(\tau)\, d\tau$$

$$= \int_{0}^{t} \sin(2(t-\tau)) f(\tau)\, d\tau$$

4.(2+2+4+2 pont)
Legyen $y'' + 2y' + 3y = e^{-7it}$! Ird fel az egyenlet egy megoldasat!

$y = A e^{-7it}$ $\qquad A = \frac{1}{-49-14i+3}$

$A((-7i)^2 + 2\cdot(-7i)+3) = 1$

$y(t) = \frac{e^{-7it}}{-49-14i+3}$

Legyen

$$\phi(0,x) = \sum_{n\in\mathbb{Z}} e^{-|n|}\frac{e^{inx}}{\sqrt{2\pi}}, \qquad \phi_t(0,x) = \sum_{n\in\mathbb{Z}} e^{-2|n|}\frac{e^{inx}}{\sqrt{2\pi}},$$

$$\phi(t,x) = \sum_{n\in\mathbb{Z}} c_n(t)\frac{e^{inx}}{\sqrt{2\pi}}, \qquad \partial_{tt}\phi(t,x) = \frac{1}{4}\partial^2_{xx}\phi(t,x).$$

Ird fel a $c_n(t)$ fuggvenyekre vonatkozo kozonseges DE-ket (kezdeti feltetellel egyutt)!

$c_n''(t) = \frac{1}{4}(-n^2)\cdot c_n(t)$ $\qquad c_n(0) = e^{-|n|}$

$(in)^2$

$c_n'(0) = e^{-2|n|}$

Legyen

$$\phi(0,x) = \cos(2x) + 3\sin(x) = \sum_{n\in\mathbb{Z}} \hat{f}_n\frac{e^{inx}}{\sqrt{2\pi}}, \qquad \phi(t,x) = \sum_{n\in\mathbb{Z}} d_n(t)\frac{e^{inx}}{\sqrt{2\pi}}, \qquad \partial_t\phi(t,x) = \frac{1}{4}\partial^2_{xx}\phi(t,x).$$

Melyek a nem null $\hat{f}_n$-ek? Ird fel a $d_n(t)$ fuggvenyekre vonatkozo kozonseges DE-ket (kezdeti feltetellel egyutt)! Mi a DE megoldasa?

$$\boxed{\begin{aligned}\cos x &= \frac{e^{ix}+e^{-ix}}{2}\\ \sin x &= \frac{e^{ix}-e^{-ix}}{2i}\end{aligned}}$$

$$\cos(2x)+3\sin(x) = \frac{e^{i2x}+e^{-i2x}}{2} + 3\,\frac{e^{i3x}-e^{-i3x}}{2i}$$

$$= \frac{3\sqrt{2\pi}}{2i}\cdot\frac{e^{i3x}}{\sqrt{2\pi}} - \frac{3\sqrt{2\pi}}{2i}\frac{e^{-i3x}}{\sqrt{2\pi}} + \frac{\sqrt{2\pi}}{2}\frac{e^{i2x}}{\sqrt{2\pi}} + \frac{\sqrt{2\pi}}{2}\frac{e^{-i2x}}{\sqrt{2\pi}}$$

Nem nulla $\hat{f}$: $\hat{f}_3 = \frac{3\sqrt{2\pi}}{2i}$, $\hat{f}_{-3} = -\frac{3\sqrt{2\pi}}{2i}$, $\hat{f}_2 = \frac{\sqrt{2\pi}}{2}$, $\hat{f}_{-2} = \frac{\sqrt{2\pi}}{2}$

$d_n'(t) = \frac{1}{4}(-n^2)d_n(t)$, $d_n(0) = 0$, Kivéve $n = 2,-2,3,-3$, ekkor $d_n(0) = \hat{f}_n$

$\varphi(t,x) = e^{-\frac{1}{4}\cdot 2^2}\cdot\cos(2x) + 3\cdot e^{-\frac{1}{4}\cdot 1^2}\cdot\sin(x)$

Mennyi

$$\exp\left[t\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\right] \quad ?$$

$e^{t\begin{pmatrix}1&1\\0&1\end{pmatrix}} = e^{\begin{pmatrix}t&0\\0&t\end{pmatrix}+\begin{pmatrix}0&t\\0&0\end{pmatrix}}$

$= e^{\begin{pmatrix}t&0\\0&t\end{pmatrix}}\cdot e^{\begin{pmatrix}0&t\\0&0\end{pmatrix}} = \begin{pmatrix}e^t & 0\\ 0 & e^t\end{pmatrix}\left[\begin{pmatrix}1&0\\0&1\end{pmatrix} + \begin{pmatrix}0&t\\0&0\end{pmatrix} + \frac{1}{2}\underbrace{\begin{pmatrix}0&t\\0&0\end{pmatrix}^2}_{=0} + \ldots\right] =$

$= \begin{pmatrix}e^t&0\\0&e^t\end{pmatrix}\begin{pmatrix}1&t\\0&1\end{pmatrix} = \begin{pmatrix}e^t & te^t\\ 0 & e^t\end{pmatrix}$

mivel $\begin{pmatrix}t&0\\0&t\end{pmatrix}\begin{pmatrix}0&t\\0&0\end{pmatrix} = \begin{pmatrix}0&t\\0&0\end{pmatrix}\begin{pmatrix}t&0\\0&t\end{pmatrix}$