Lev. Diff Egy., 2019.05.18. NEPTUN: :

Név:

Aláírás:

((3+2+1)+(3+1)) pont

1A) $y'' - 9y = (e^t - 4)^2$, $y(0) = 2$, $y'(0) = 3$. Mennyi $Y(s)$?

$= e^{2t} - 8e^t + 16$

$$(s^2 Y(s) - 2\cdot s - 3) - 9Y(s) = \frac{1}{s-2} - 8\cdot\frac{1}{s-1} + \frac{16}{s}$$

$$Y(s) = \frac{1}{s^2-9}\left[2s+3+\frac{1}{s-2} - 8\frac{1}{s-1} + \frac{16}{s}\right]$$

Hogy nez ki $Y(s)$ parcialis tort felbontasa? $s^2-9=(s+3)(s-3)$

$$Y(s) = \frac{A}{s+3} + \frac{B}{s-3} + \frac{C}{s-2} + \frac{D}{s-1} + \frac{E}{s}$$

Mennyi $y(t)$?

$$y(t) = Ae^{-3t} + Be^{3t} + Ce^{2t} + De^{t} + E$$

1B1) Mi az $G''(t) = 9G(t) + \delta(t)$ DE retardalt fundamentalis megoldasa?

$t<0$: $G(t) = 0$

$t \approx 0$: $G(0^+) - G(0^-) = 0$, $G'(0^+) - G'(0^-) = 1$

$t>0$: $G''(t) = 9G(t) \Rightarrow G(t) = Ae^{3t} + Be^{-3t}$ $\Longrightarrow G(t) = \frac{1}{6}e^{3t} - \frac{1}{6}e^{-3t}$

$$G(t) = \begin{cases} 0, & \text{ha } t<0 \\ \frac{1}{6}(e^{3t} - e^{-3t}), & \text{ha } t>0 \end{cases}$$

1B2) Mi az $y''(t) = 9y(t) + f(t)$, $y(t) = f(t) = 0$, ha $t << 0$ DE megoldasa?

$$y(t) = \int_{-\infty}^{\infty} G(t-\tau) f(\tau)\, d\tau = \int_{-\infty}^{t} \frac{1}{6}\left(e^{3(t-\tau)} - e^{-3(t-\tau)}\right) f(\tau)\, d\tau$$

4A) (5 pont)
$y' = -y^2 + 4.$
Keresd meg a DE fixpontjait!

$-y^2+4=0 \qquad y_1=-2, \qquad y_2=2$

1pont

Ird fel a fixpontok koruli linearizalt kozelito DE-t!

$$\frac{d(-y^2+4)}{dy} = -2y,$$

y_1: $\frac{d}{dt}(y-(-2)) = \frac{d}{dt}\Delta y = -2\cdot(-2)\Delta y = 4\Delta y$

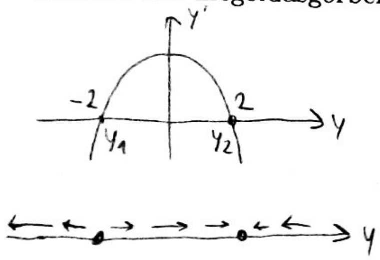
y_2: $\frac{d}{dt}(y-2) = \frac{d}{dt}\Delta y = -2\cdot 2\Delta y = -4\Delta y$

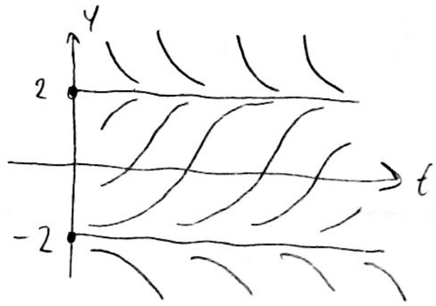
1pont

Ha $y(0) = -0.5$, mennyi
$\lim_{x\to\infty} y(x) =$ 2 $\quad \lim_{x\to-\infty} y(x) =$ -2

1 pont

Vazold a DE megoldasgorbeit!





2pont

4B). (5 pont)

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} (y_1-4)(y_2+3) \\ (5-y_1). \end{pmatrix}$$

Keresd meg a DE fixpontjat!

$$\left.\begin{matrix} (y_1-4)(y_2+3)=0 \\ 5-y_1=0 \longrightarrow y_1=5 \end{matrix}\right\} \longrightarrow (5-4)(y_2+3)=0 \longrightarrow y_2=-3$$

2 pont

$$P_{fix} = \begin{pmatrix} 5 \\ -3 \end{pmatrix}$$

Ird fel a fixpont koruli linearizalt kozelito DE-t!

$$Jac = \begin{pmatrix} \partial_{y_1}([y_1-4][y_2+3]) & \partial_{y_2}([y_1-4][y_2+3]) \\ \partial_{y_1}(5-y_1) & \partial_{y_2}(5-y_1) \end{pmatrix}$$

1pont

$$= \begin{pmatrix} y_2+3 & y_1-4 \\ -1 & 0 \end{pmatrix}$$

$$Jac(P_{fix}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\frac{d}{dt}\begin{pmatrix} y_1-5 \\ y_2-(-3) \end{pmatrix} = \frac{d}{dt}\begin{pmatrix} \Delta y_1 \\ \Delta y_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \Delta y_1 \\ \Delta y_2 \end{pmatrix}$$

2pont

2. (3+2+2+3 pont) Legyen

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 3y_2 \\ 2y_1 \end{pmatrix} = A \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

Keresd meg A sajatertekeit es sajatvektorait!

$$0 = \det(A - \lambda E) = \begin{bmatrix} -\lambda & 3 \\ 2 & -\lambda \end{bmatrix} = \lambda^2 - 6, \quad \lambda_1 = \sqrt{6}, \quad \lambda_2 = -\sqrt{6}$$

$$\begin{bmatrix} -\sqrt{6} & 3 \\ 2 & -\sqrt{6} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} \sqrt{6} & 3 \\ 2 & \sqrt{6} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-\sqrt{6}u + 3v = 0 \qquad \sqrt{6}u + 3v = 0$$

$$v = \frac{\sqrt{6}}{3}u = \sqrt{\frac{2}{3}}u \qquad v = -\frac{\sqrt{6}}{3}u = -\sqrt{\frac{2}{3}}u$$

$$\vec{v}_1 = \begin{pmatrix} 1 \\ \sqrt{\frac{2}{3}} \end{pmatrix} \qquad \vec{v}_2 = \begin{pmatrix} 1 \\ -\sqrt{\frac{2}{3}} \end{pmatrix}$$

Ird fel a DE altalanos megoldasat!

$$\vec{y}_{ált}(t) = C_1 e^{\sqrt{6}t} \begin{pmatrix} 1 \\ \sqrt{2/3} \end{pmatrix} + C_2 e^{-\sqrt{6}t} \begin{pmatrix} 1 \\ -\sqrt{2/3} \end{pmatrix}$$

Legyen

$$\begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} 13 \\ 77 \end{pmatrix}$$

Szamold ki a DE partikularis megoldasait!

$$\begin{pmatrix} 13 \\ 77 \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ \sqrt{2/3} \end{pmatrix} + C_2 \begin{pmatrix} 1 \\ -\sqrt{2/3} \end{pmatrix}$$

$$\left.\begin{aligned} C_1 + C_2 &= 13 \\ \sqrt{2/3}\,C_1 - \sqrt{2/3}\,C_2 &= 77 \\ C_1 - C_2 &= 77 \cdot \sqrt{\tfrac{3}{2}} \end{aligned}\right\} \quad \begin{aligned} C_1 &= \frac{1}{2}\left(13 + 77 \cdot \sqrt{\tfrac{3}{2}}\right) \\ C_2 &= \frac{1}{2}\left(13 - 77 \cdot \sqrt{\tfrac{3}{2}}\right) \end{aligned}$$

$$\vec{y}_{part} = \frac{1}{2}\left(13 + 77 \cdot \sqrt{\tfrac{3}{2}}\right) e^{\sqrt{6}t} + \frac{1}{2}\left(13 - 77 \cdot \sqrt{\tfrac{3}{2}}\right) e^{-\sqrt{6}t}$$

Mennyi e^{tA} ?

$$e^{tA} = e^{tSDS^{-1}} = S e^{tD} S^{-1} =$$

$$= \begin{bmatrix} 1 & 1 \\ \sqrt{2/3} & -\sqrt{2/3} \end{bmatrix} \begin{bmatrix} e^{\sqrt{6} \cdot t} & 0 \\ 0 & e^{-\sqrt{6} \cdot t} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ \sqrt{2/3} & -\sqrt{2/3} \end{bmatrix}^{-1}$$

((2+1)+(3+1)+3 pont)

3a. Legyen $f(t) = t$, $g(t) = (4-t)$. Mennyi $h(t) = (f * g)(t)$?

$$h(t) = \int_0^t f(t-\tau)g(\tau)\,d\tau = \int_0^t (t-\tau)(4-\tau)\,d\tau = \int_0^t 4t - (4+t)\tau + \tau^2\,d\tau$$

$$= 4t^2 - (4+t)\frac{t^2}{2} + \frac{t^3}{3} = -\frac{1}{6}t^3 + 2t^2$$

Mennyi $\mathcal{L}(h) - \mathcal{L}(f)\mathcal{L}(g)$?

$= 0$ (automatikusan). Valóban: $-\frac{1}{6}\frac{3!}{s^4} + 2\cdot\frac{2!}{s^3} = \frac{1}{s^2}\cdot\left(\frac{4}{s} - \frac{1}{s^2}\right)$

3b. Legyen

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} -2y_1 + 3y_2 \\ -2y_2 \end{pmatrix} = A\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{bmatrix} -2 & 3 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

Mennyi e^{tA} ?

$$e^{tA} = \exp\begin{bmatrix} -2t & 3t \\ 0 & -2t \end{bmatrix} = \exp\left(\begin{bmatrix} -2t & 0 \\ 0 & -2t \end{bmatrix} + \begin{bmatrix} 0 & 3t \\ 0 & 0 \end{bmatrix}\right) =$$

$$= \exp\begin{bmatrix} -2t & 0 \\ 0 & -2t \end{bmatrix}\cdot \exp\begin{bmatrix} 0 & 3t \\ 0 & 0 \end{bmatrix} =$$

$$= \begin{bmatrix} e^{-2t} & 0 \\ 0 & e^{-2t} \end{bmatrix}\cdot\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 3t \\ 0 & 0 \end{bmatrix} + \frac{1}{2!}\underbrace{\begin{bmatrix} 0 & 3t \\ 0 & 0 \end{bmatrix}^2}_{=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}} + \dots\right) =$$

$$= \begin{bmatrix} e^{-2t} & 3t\,e^{-2t} \\ 0 & e^{-2t} \end{bmatrix}$$

Mi az elozo DE partikularis megoldasa az $(y_1(0), y_2(0))^T = (4,5)$ kezdeti feltetel mellett?

$$\vec{y}(t) = \begin{bmatrix} e^{-2t} & 3t\,e^{-2t} \\ 0 & e^{-2t} \end{bmatrix}\begin{bmatrix} 4 \\ 5 \end{bmatrix} = \begin{bmatrix} e^{-2t}\cdot(4+15t) \\ 5\,e^{-2t} \end{bmatrix}$$

3c. Legyen $f \in L^2([-\pi,\pi], dx)$ egyenlo nullaval, ha $x < 1$, amugy meg legyen az erteke ketto. Mennyi $\hat{f}_5$, ha $f(x) = (1/\sqrt{2\pi})\sum_{n\in\mathbb{Z}} \hat{f}_n e^{inx}$?

$$\hat{f}_5 = \frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi} e^{-i5x} f(x)\,dx = \frac{1}{\sqrt{2\pi}}\int_1^{\pi} e^{-i5x}\cdot 2\,dx = \frac{1}{\sqrt{2\pi}}\left[\frac{e^{-i5x}}{-5i}\right]_1^{\pi} =$$

$$= \frac{1}{\sqrt{2\pi}}\left(\underbrace{e^{-i5\pi}}_{=-1} - e^{-i\cdot 5}\right)\cdot\frac{1}{-5i}$$