1a. (5 pont) $y' = \cancel{e^x - e^2}$ $y' = e^y - e^2$

Keresd meg a DE fixpontjait!

Ird fel a fixpontok koruli linearizalt kozelito DE-t!

Ha $y(0) = 1.5$, mennyi
$\lim_{x\to\infty} y(x) =$ $\lim_{x\to-\infty} y(x) =$

Vazold a DE megoldasgorbeit!

1b. (5 pont) Legyen

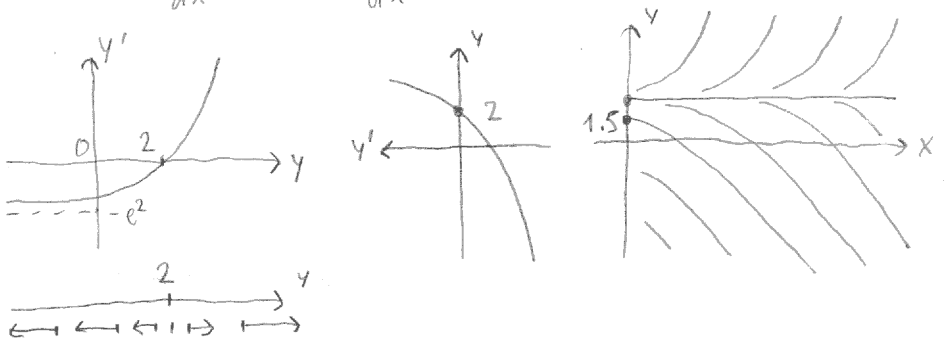
$$\partial_t\phi(t,x) = 3\partial_x^2\phi(t,x), \quad \phi(t,x+\pi) = \phi(t,x), \quad \phi(0,x) = f(x),$$

ahol $f(x) = 3$, ha $x \in [0,1]$, amugy 0 a $[0,\pi]$ intervalum tobbi reszen. Fejezd ki $\phi(t,x)$-t Fourier sor segitsegevel!

a) $y' = e^y - e^2 = f(y)$ $\qquad \dfrac{df(y)}{dy} = e^y$

Fixpont: $e^y - e^2 = 0 \Rightarrow y_1 = 2$

Lin DE: $\dfrac{d}{dx}(y-2) = \dfrac{d}{dx}\Delta y = e^2 \cdot \Delta y$



Ha $y(0) = 1.5$

$\lim_{x\to\infty} y(x) = -\infty$

$\lim_{x\to-\infty} y(x) = 2$

b) $$f(x) = \sum_{n\in\mathbb{Z}} (e_n, f)\cdot\frac{e^{2inx}}{\sqrt{\pi}} = \sum_{n\in\mathbb{Z}}\left(\int_0^\pi \frac{\overline{e^{2inx}}}{\sqrt{\pi}}\cdot f(x)\,dx\right)\cdot\frac{e^{2inx}}{\sqrt{\pi}}$$

$$= \sum_{n\in\mathbb{Z}}\left(\int_0^1 \frac{e^{-2inx}}{\sqrt{\pi}}\,dx\right)\cdot\frac{e^{2inx}}{\sqrt{\pi}} = \sum_{n\in\mathbb{Z}}\left[\frac{e^{-2inx}}{-2in\cdot\sqrt{\pi}}\right]_0^1\cdot\frac{e^{2inx}}{\sqrt{\pi}}$$

$$= \sum_{n\in\mathbb{Z}}\frac{1}{\pi}\cdot\frac{e^{-2in}-1}{-2in}\cdot e^{2inx}.$$

Mivel $3\cdot\partial_x^2\left(\dfrac{e^{2inx}}{\sqrt{\pi}}\right) = -12n^2\dfrac{e^{2inx}}{\sqrt{\pi}}$, így

$$\varphi(t,x) = \sum_{n\in\mathbb{Z}}\frac{1}{\pi}\,\frac{e^{-2in}-1}{-2in}\cdot e^{-12n^2t}\cdot e^{2inx}$$

2. (2+3+5 pont)

a) Legyen $f(x) = 1/\sqrt{x-1}$. Ird fel f linearis approximaciojat az $x_0 = 2$ pont korul! Adj minel pontosabb felso korlatot a linearis approximacio hibajara, vagyis $|f(2+\Delta x) - f(2) - f'(2)\Delta x|$-re, ha $\Delta x \in [0, 0.1]$!

b) Alkalmazd az Euler, illetve a Heun modszert a kovetkezo DE-re $\Delta x = 0.01$ lepeskozzel az $y(1) = 3$ kezdeti feltetel mellett!

$$y' = (x+y)(x-y).$$

Mit josol a ket modszer ~~$y(3.01)$~~-re? $y(1.01)$-re?

Euler:

Heun:

c) Keress numerikus egyenleteket a kovetkezo DE kozelito megoldasara:

$$x^2u''(x) + u'(x) + xu(x) = 3x - 1, \quad u(0) = u(1) = 0.$$

Approximaljuk az u fuggvenyt a kovetkezo vektorral: $\vec{u}_i = u(i\Delta x)$, $i = 1, \ldots, 4$, $\Delta x = 1/5$.

- Kozelitsd $u''(x)$-t az $u(x \pm \Delta x), u(x)$ ertekek segitsegevel!
- Ird fel az ennek megfelelo veges differencias kozeliteset a DE-nek mint egy inhom.lin. egyenletet a $\vec{u}$ vektorra!

a) $f(x) = (x-1)^{-1/2}$, $f'(x) = -\frac{1}{2}(x-1)^{-3/2}$, $f''(x) = \frac{3}{4}(x-1)^{-5/2}$

$f(2) = 1$, $f'(2) = -\frac{1}{2}$, $f''(2) = \frac{3}{4}$.

$f(2+\Delta x) = 1 - \frac{1}{2}\Delta x + hiba(\Delta x)$, ahol $|hiba(\Delta x)| \le \frac{1}{2}\Delta x^2 \cdot \max\limits_{\Delta x \in [0,0.1]} |f''(2+\Delta x)| =$

$= \frac{1}{2}\Delta x^2 \cdot \frac{3}{4} = \frac{3}{8}\Delta x^2$

b) Euler: $y(1.01) \approx 3 + (1+3)(1-3) \cdot 0.01 = 2.92$

Heun: $y(1.01) \approx 3 + \frac{1}{2}\left\{(1+3)(1-3) + (1.01+2.92)\cdot(1.01-2.92)\right\} \cdot 0.01$

c) $u''(x) \approx \frac{1}{\Delta x^2}\left(u(x+\Delta x) - 2u(x) + u(x-\Delta x)\right)$

$$\begin{bmatrix} (1\cdot\Delta x)^2 & & & \\ & (2\cdot\Delta x)^2 & & \\ & & (3\cdot\Delta x)^2 & \\ & & & (4\cdot\Delta x)^2 \end{bmatrix} \cdot \frac{1}{\Delta x^2} \begin{bmatrix} -2 & & & \\ 1 & -2 & 1 & \\ & 1 & -2 & 1 \\ & & 1 & 2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} +$$

$$+ \frac{1}{\Delta x} \begin{bmatrix} -1 & 1 & & \\ & -1 & 1 & \\ & & -1 & 1 \\ & & & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} + \begin{bmatrix} 1\Delta x & & & \\ & 2\Delta x & & \\ & & 3\Delta x & \\ & & & 4\Delta x \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} 3\cdot 1\Delta x - 1 \\ 3\cdot 2\Delta x - 1 \\ 3\cdot 3\Delta x - 1 \\ 3\cdot 4\Delta x - 1 \end{bmatrix}$$

3. (5+2+3 pont)

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 2y_1 + 3y_2 \\ -3y_1 + 2y_2 \end{pmatrix} = A \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \qquad \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

Keresd meg A sajatertekeit es sajatvektorait!
Ird fel a DE altalanos megoldasat!
Szamold ki a DE partikularis megoldasat!

$$A = \begin{pmatrix} 2 & 3 \\ -3 & 2 \end{pmatrix}$$

sajátértékek: $0 = \det(A - \lambda E) = \begin{vmatrix} 2-\lambda & 3 \\ -3 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 + 9 = \lambda^2 - 4\lambda + 13$

$\Longrightarrow \lambda_1 = 2+3i, \ \lambda_2 = \bar{\lambda}_1 = 2-3i$

sajátvektorok: $(A - \lambda E)\vec{v} = \vec{0}$

$\lambda_1 = 2+3i$ $\quad \begin{pmatrix} 2-(2+3i) & 3 \\ -3 & 2-(2+3i) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3i & 3 \\ -3 & -3i \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow ix = y$

$\vec{v}_1 = \begin{pmatrix} 1 \\ i \end{pmatrix}$ $\quad$ Ellenőrzés: $\begin{pmatrix} 2 & 3 \\ -3 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ i \end{pmatrix} = \begin{pmatrix} 2+3i \\ -3+2i \end{pmatrix} = (2+3i) \begin{pmatrix} 1 \\ i \end{pmatrix}$

$\lambda_2 = 2-3i$, $\vec{v}_2 = \overline{\vec{v}_1} = \overline{\begin{pmatrix} 1 \\ i \end{pmatrix}} = \begin{pmatrix} 1 \\ -i \end{pmatrix}$

Általános megoldás:

$$y(t) = C_1 e^{(2+3i)t} \cdot \begin{pmatrix} 1 \\ i \end{pmatrix} + C_2 e^{(2-3i)t} \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

Partikuláris megoldás:

$$C_1 \begin{pmatrix} 1 \\ i \end{pmatrix} + C_2 \begin{pmatrix} 1 \\ -i \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \qquad \left.\begin{matrix} C_1 + C_2 = 3 \\ iC_1 - iC_2 = 1 \\ -C_1 + C_2 = i \end{matrix}\right\} \Rightarrow \begin{matrix} C_2 = \frac{3+i}{2} \\ C_1 = \frac{3-i}{2} \end{matrix}$$

$$y_{part}(t) = \frac{3-i}{2} e^{(2+3i)t} \begin{pmatrix} 1 \\ i \end{pmatrix} + \frac{3-i}{2} e^{(2-3i)t} \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

4. ((2+2+1)+(3+1+1) pont)
A) $2y'' + 5y = (t + e^{4t})^2$, $y(0) = 3$, $y'(0) = 2$. Mennyi $Y(s)$? ($\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}$)

Ird fel $Y(s)$ parcialis tort felbontasat! (Az egyutthatokat nek kell kiszamolni.)

Mennyi $y(t)$?

B) Oldd meg a $y'' - 9y = f(t)$ DE-t:
1. Keresd meg a G retardalt Green fuggvenyt!
2. G segitsegevel fejezd ki y-t, ha $y(t) = f(t) = 0$ amikor $t \ll 0$!
3. Hasznald G-t arra, hogy kifejezd a megoldast $t > 0$-ra, ha $y(0) = 7$!

A) $2y''+5y=t^2-2te^{4t}+e^{8t}$ $\qquad \mathcal{L}(te^{at})=\int_0^\infty e^{-st}te^{at}=\left[t\cdot\frac{e^{(a-s)t}}{a-s}\right]_0^\infty-\int_0^\infty\frac{e^{(a-s)t}}{a-s}dt = \frac{1}{(a-s)^2}$

$$2(s^2Y-3s-2)+5Y(s)=\frac{2}{s^3}-2\frac{1}{(4-s)^2}+\frac{1}{s-8}$$

$$Y(s)=\frac{1}{2s^2+5}\left(6s+4+\frac{2}{s^3}-2\frac{1}{(s-4)^2}+\frac{1}{s-8}\right)$$

$$Y(s)=\frac{A}{s-\sqrt{\frac{5}{2}}\cdot i}+\frac{B}{s+\sqrt{\frac{5}{2}}\cdot i}+\frac{C}{s^3}+\frac{D}{s^2}+\frac{E}{s}+\frac{F}{(s-4)^2}+\frac{G}{(s-4)}+\frac{H}{s-8}$$

$$y(t)=Ae^{\sqrt{\frac{5}{2}}it}+Be^{-\sqrt{\frac{5}{2}}it}+\frac{C}{2}t^2+Dt+E+Fte^{4t}+Ge^{4t}+He^{8t}$$

B) 1)

$G(t)=0$, ha $t<0$

$G''(t)-9G(t)=0$, ha $t>0$ $\quad\Rightarrow G(t)=c_1e^{3t}+c_2e^{-3t}$

$G(0^+)=0,\ G'(0^+)=1$ $\quad\Rightarrow G(t)=\begin{cases}0, & \text{ha } t<0\\ \frac{1}{6}e^{3t}-\frac{1}{6}e^{-3t}, & \text{ha } t>0\end{cases}$

2)

$$y(t)=(G*F)(t)=\int_{-\infty}^{\infty}G(t-\tau)f(\tau)d\tau=$$
$$=\int_{-\infty}^{t}\frac{1}{6}\left(e^{3(t-\tau)}-e^{-3(t-\tau)}\right)f(\tau)d\tau$$

3) $t>0$:

$$y(t)=7\cdot G(t)+\int_0^\infty G(t-\tau)f(\tau)d\tau=$$
$$=7\cdot\frac{1}{6}\left(e^{3t}-e^{-3t}\right)+\int_0^t\frac{1}{6}\left(e^{3(t-\tau)}-e^{-3(t-\tau)}\right)f(\tau)d\tau.$$