Lev.Vizsga, Diff.Egy., 2017.05.19. NEPTUN: :

Név: Aláírás:

1. (2+3+5 pont)

a) Legyen $f(x) = 1/x^3$. Ird fel f linearis approximaciojat az $x_0 = 2$ pont korul! Adj minel pontosabb felso korlatot a linearis approximacio hibajara, vagyis $|f(2+\Delta x) - f(2) - f'(2)\Delta x|$-re, ha $\Delta x \in [0, 0.1]$!

$$f(x) = x^{-3},\ f'(x) = -3x^{-4},\ f''(x) = 12x^{-5},\ f(2) = \frac{1}{8},\ f'(2) = -\frac{3}{16},\ f''(2) = \frac{12}{32} = \frac{3}{8}$$

$$f(2+\Delta x) = \frac{1}{8} - \frac{3}{16}\Delta x + hiba(\Delta x)$$

$$|hiba(\Delta x)| \le \frac{1}{2}\Delta x^2 \max_{x\in[2,2+\Delta x]} |12x^{-5}| = \frac{1}{2}\Delta x^2 \cdot \frac{3}{8} = \frac{3}{16}\Delta x^2$$

b) Alkalmazd az Euler, illetve a Heun modszert a kovetkezo DE-re $\Delta x = 0.1$ lepeskozzel az $y(3) = 1$ kezdeti feltetel mellett!

$$y' = (x+y)^2 + 1.$$

Mit josol a ket modszer $y(3.1)$-re?

Euler:

$$y(3.1) \approx 1 + \left[(3+1)^2 + 1\right] \cdot 0.1 = 2.7$$

Heun:

$$y(3.1) \approx 1 + \frac{1}{2}\left\{ 17 + \left[(3.1 + 2.7)^2 + 1\right]\right\} \cdot 0.1$$

c) Keress numerikus egyenleteket a kovetkezo DE kozelito megoldasara:

$$xu''(x) + u'(x) + u(x) = 3, \quad u(0) = u(1) = 0.$$

Approximaljuk az u fuggvenyt a kovetkezo vektorral: $\vec{u}_i = u(i\Delta x)$, $i = 1, \ldots, 4$, $\Delta x = 1/5$.

- Kozelitsd $u''(x)$-t az $u(x \pm \Delta x), u(x)$ ertekek segitsegevel!

$$u''(x) = \frac{1}{\Delta x^2}\left(u(x+\Delta x) - 2u(x) + u(x-\Delta x)\right)$$

- Ird fel az ennek megfelelo veges differencias kozeliteset a DE-nek mint egy inhom.lin. egyenletet a $\vec{u}$ vektorra!

$$\begin{bmatrix} 1\Delta x & & & \\ & 2\Delta x & & \\ & & 3\Delta x & \\ & & & 4\Delta x \end{bmatrix} \cdot \frac{1}{\Delta x^2} \begin{bmatrix} -2 & 1 & & \\ 1 & -2 & 1 & \\ & 1 & -2 & 1 \\ & & 1 & -2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} + \frac{1}{\Delta x} \begin{bmatrix} -1 & 1 & & \\ & -1 & 1 & \\ & & -1 & 1 \\ & & & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \end{bmatrix}$$

3a. (5 pont)
$y' = (y-1)(2-y)(y-3).$ $= -y^3+6y^2-11y+6 = f(y)$
Keresd meg a DE fixpontjait!

$y_1 = 1 \quad , \quad y_2 = 2 \quad , \quad y_3 = 3$

$\frac{df(y)}{dy} = f'(y) = -3y^2+12y-11$

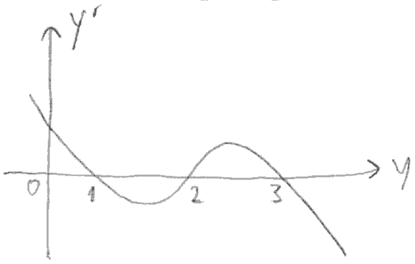
Ird fel a fixpontok koruli linearizalt kozelito DE-t!

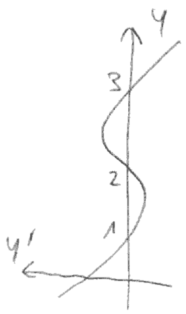
$\frac{d}{dt}(y-1) = \frac{d}{dt}\Delta y = \underbrace{f'(1)}_{-3\cdot 1^2+12\cdot 1-11}\Delta y = -2\Delta y$ | $\frac{d}{dt}(y-2) = \frac{d}{dt}\Delta y = \underbrace{1}_{-3\cdot 2^2+12\cdot 2-11}\cdot\Delta y$ | $\frac{d}{dt}(y-3) = \frac{d}{dt}\Delta y = -2\Delta y$

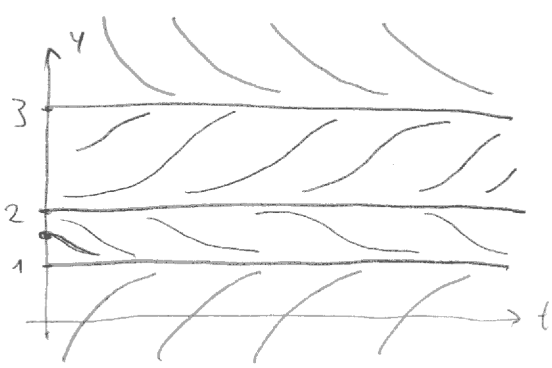
Ha $y(0) = 1.5$, mennyi
$\lim_{x\to\infty} y(x) =$ 1

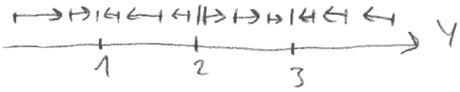
$\lim_{x\to-\infty} y(x) =$ 2

Vazold a DE megoldasgorbeit!









3b. (5 pont) Legyen

$$\partial_t\phi(t,x) = \partial_x^2\phi(t,x), \quad \phi(t,x+2\pi) = \phi(t,x), \quad \phi(0,x) = f(x),$$

ahol $f(x) = 5$, ha $x \in [2,3]$, amugy 0 a $[0,2\pi]$ intervalum tobbi reszen. Fejezd ki $\phi(t,x)$-t Fourier sor segitsegevel! ?

$$f(x) = \sum_{n\in\mathbb{Z}} (e_n, f)\cdot\frac{e^{inx}}{\sqrt{2\pi}} = \sum_{n\in\mathbb{Z}}\left(\int_0^{2\pi}\frac{e^{-inx}}{\sqrt{2\pi}}f(x)\,dx\right)\cdot\frac{e^{inx}}{\sqrt{2\pi}} =$$

$$= \sum_{n\in\mathbb{Z}}\left(\frac{1}{\sqrt{2\pi}}\int_2^3 e^{-inx}\cdot 5\,dx\right)\cdot\frac{e^{inx}}{\sqrt{2\pi}} = \sum_{n\in\mathbb{Z}} 5\cdot\frac{e^{-in\cdot 3}-e^{-in\cdot 2}}{-in}\cdot\frac{1}{2\pi}e^{inx}$$

Mivel $\partial_x^2\left(\frac{e^{inx}}{\sqrt{2\pi}}\right) = -n^2\left(\frac{e^{inx}}{\sqrt{2\pi}}\right)$, így

$$\varphi(t,x) = \sum_{n\in\mathbb{Z}} 5\cdot\frac{e^{-in\cdot 3}-e^{-in\cdot 2}}{-in}\cdot\frac{1}{2\pi}\cdot e^{inx}\cdot e^{-n^2 t}$$

2. (5+2+3 pont)

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 2y_1 \\ 3y_1 + 3y_2 \end{pmatrix} = A \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}; \qquad \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

Keresd meg A sajatertekeit es sajatvektorait!

$$A = \begin{pmatrix} 2 & 0 \\ 3 & 3 \end{pmatrix}$$

$$\det(A - \lambda E) = 0 = \begin{vmatrix} 2-\lambda & 0 \\ 3 & 3-\lambda \end{vmatrix} \Rightarrow \lambda_1 = 2, \quad \lambda_2 = 3$$

$$\overline{V}_1: \quad \begin{pmatrix} 2-2 & 0 \\ 3 & 3-2 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow v = -3u, \quad \overline{V}_1 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$$

$$\overline{V}_2: \quad \begin{pmatrix} 2-3 & 0 \\ 3 & 3-3 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 3 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow -1 \cdot u = 0, \quad \overline{V}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Ird fel a DE altalanos megoldasat!

$$\overline{y}_{ált}(t) = C_1 e^{2t} \begin{pmatrix} 1 \\ -3 \end{pmatrix} + C_2 e^{3t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Szamold ki a DE partikularis megoldasat!

$$\begin{pmatrix} 3 \\ 1 \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ -3 \end{pmatrix} + C_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\left.\begin{aligned} 3 &= C_1 \\ 1 &= -3C_1 + C_2 \end{aligned}\right\} \quad C_1 = 3, \quad C_2 = 10$$

$$\overline{y}_{part}(t) = 3 e^{2t} \begin{pmatrix} 1 \\ -3 \end{pmatrix} + 10 e^{3t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

((2+2+1)+(3+1+1) pont)
A) $y'' + 2y' + 3y = (t^2+1)^2$, $y(0) = 4$, $y'(0) = 5$. Mennyi $Y(s)$? ($\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}$) .

$$(s^2 Y(s) - 4s - 5) + 2(sY(s) - 4) + 3Y(s) = \frac{4!}{s^5} + 2\cdot\frac{2!}{s^3} + \frac{1}{s}$$

$$Y(s) = \frac{1}{s^2+2s+3}\left(\frac{4!}{s^5} + 2\cdot\frac{2!}{s^3} + \frac{1}{s} + 4s + 13\right)$$

$$s^2+2s+3=0 \longrightarrow s_{1,2} = \frac{-2\pm\sqrt{-8}}{2} = -1\pm\sqrt{2}\cdot i$$

Ird fel $Y(s)$ parcialis tort felbontasat! (Az egyutthatokat nek kell kiszamolni.)

$$Y(s) = \frac{A}{s-(-1+\sqrt{2}i)} + \frac{B}{s-(-1-\sqrt{2}i)} + \frac{C}{s^5} + \frac{D}{s^4} + \frac{E}{s^3} + \frac{F}{s^2} + \frac{G}{s}$$

Mennyi $y(t)$?

$$y(t) = Ae^{(-1+\sqrt{2}i)t} + Be^{(-1-\sqrt{2}i)t} + \frac{C}{4!}t^4 + \frac{D}{3!}t^3 + \frac{E}{2!}t^2 + Ft + G$$

B) Oldd meg a $y'' + 9y = f(t)$ DE-t: 1.Keresd meg a G retardalt Green fuggvenyt!

- $G(t) = 0$, ha $t < 0$
- $G''(t) + 9G(t) = 0$, ha $t > 0$ $\Longrightarrow$ $G(t) = C_1\cos(3t) + C_2\sin(3t)$
- ha $t \approx 0$, $G''(t) \approx \delta(t)$, tehát $G(0^+) = 0$, $G'(0^+) = 1$.

így $C_1 = 0$, $C_2 = \frac{1}{3}$

$$G(t) = \begin{cases} 0, & \text{ha } t < 0 \\ \frac{1}{3}\sin(3t), & \text{ha } t > 0 \end{cases}$$

vagy $G(t) = K_1 e^{3it} + K_2 e^{-3it}$

$$\left.\begin{array}{l} K_1 + K_2 = 0 \\ 3iK_1 - 3iK_2 = 1 \end{array}\right\} K_1 = \frac{1}{6i},\ K_2 = -\frac{1}{6i}$$

$$G(t) = \frac{1}{3}\,\frac{e^{3it} - e^{-3it}}{2i} = \frac{1}{3}\sin(3t)$$

2. G segitsegevel fejezd ki y-t, ha $y(t) = f(t) = 0$ amikor $t \ll 0$!

$$y(t) = (G * f)(t) = \int_{-\infty}^{\infty} G(t-\tau)f(\tau)\,d\tau = \int_{-\infty}^{t} \frac{1}{3}\sin(3(t-\tau))f(\tau)\,d\tau = \int_{0}^{t} \frac{1}{3}\sin(3(t-\tau))f(\tau)\,d\tau.$$

3. Hasznald G-t arra, hogy kifejezd a megoldast $t > 0$-ra, ha $y(0) = 7$!

$$y(t) = 7\cdot G(t) + \int_0^{\infty} G(t-\tau)f(\tau)\,d\tau = 7\cdot\frac{1}{3}\sin(3t) + \int_0^t \frac{1}{3}\sin(3(t-\tau))f(\tau)\,d\tau$$