Zh2, Diff.Egy., 2017.05.10. NEPTUN: Név:

Aláírás:

1. (2+1+3+3+1 pont)

Szamitsd ki a Laplace tr. definicioja alapjan a kovetkezoket:

a) $F(s) = \mathcal{L}(f(t)) = \mathcal{L}(\sin(-2t+7))$.

$F(s)=$

$$\int_0^\infty e^{-st}\,\frac{e^{i(-2t+7)}-e^{-i(-2t+7)}}{2i}\,dt = \frac{e^{7i}}{2i}\int_0^\infty e^{(-s-2i)t}\,dt - \frac{e^{-7i}}{2i}\int_0^\infty e^{(-s+2i)t}\,dt$$

$$= \frac{e^{7i}}{2i}\left.\frac{e^{(-s-2i)t}}{-s-2i}\right|_0^\infty - \frac{e^{-7i}}{2i}\left.\frac{e^{(-s+2i)t}}{-s+2i}\right|_0^\infty = \frac{e^{7i}}{2i}\,\frac{1}{s+2i} - \frac{e^{-7i}}{2i}\,\frac{1}{s-2i}$$

Esetunkben milyen s eseten letezik a Laplace transzformaciot definialo impropius integral?

$$Re(s) > 0$$

$F(s) = \mathcal{L}(f(t)) = \mathcal{L}(\underbrace{H(2t+3)}_{=1,\ \text{ha}\ t>0}e^{-5t-1})$ (Itt H a Heaviside fuggveny.)

$F(s)=$

$$\int_0^\infty e^{-st}\cdot e^{-5t-1}\,dt = e^{-1}\cdot\int_0^\infty e^{(-s-5)t}\,dt = e^{-1}\left.\frac{e^{(-s-5)t}}{-s-5}\right|_0^\infty$$

$$= \frac{1}{e}\cdot\frac{1}{s+5}$$

b) Szamold ki az $f(t) = 4$ es a $g(t) = 5$ fuggvenyek $h = f * g$ konvoluciojat!

$$h(t) = \int_0^t f(t-\tau)g(\tau)\,d\tau = \int_0^t 5\cdot 4\,dt = 20t$$

Mennyi $\mathcal{L}(f(t))\mathcal{L}(g(t)) - \mathcal{L}(h(t))$? $= 0$

4. (2+3+5 pont)
A) Legyen $\left(2\partial_{xx}^2 + 3\partial_{tt}^2 - 9\partial_{xt}^2\right) e^{i(kx-\omega t)} = 0$. Milyen algebrai egyenletet teljesit k es ω?

$$2(ik)^2 + 3(i\omega)^2 - 9(ik)(i\omega) = 0$$

$$-2k^2 - 3\omega^2 + 9k\omega = 0$$

B) Legyen

$$\partial_t \phi(t,x) = \partial_x^2 \phi(t,x), \quad \phi(t, x+2\pi) = \phi(t,x), \quad \phi(0,x) = f(x),$$

ahol $f(x) = 5$, ha $x \in [0,3]$, amugy 0 a $[0,2\pi]$ intervalum tobbi reszen.
1. Ird fel egy ortonormalt bazist $L^2([0,2\pi], dx)$-nek!

$$e_n(x) = \frac{e^{inx}}{\sqrt{2\pi}}, \quad n \in \mathbb{Z}$$

2. Mennyi $\phi(t,x)$?

$$f(x) = \sum_{n\in\mathbb{Z}} \hat{f}_n \frac{e^{inx}}{\sqrt{2\pi}}, \quad \text{ahol: } \hat{f}_n = (e_n, f) =$$

$$= \int_0^{2\pi} \overline{\frac{e^{inx}}{\sqrt{2\pi}}} \cdot f(x)\,dx = \frac{5}{\sqrt{2\pi}} \int_0^3 e^{-inx}\,dx =$$

$$= \frac{5}{\sqrt{2\pi}} \cdot \frac{e^{-inx}}{-in}\Big|_0^3 = \frac{5}{\sqrt{2\pi}} \cdot \frac{i}{n}\left(e^{-3in} - 1\right)$$

Mivel $\partial_x^2 e_n = -n^2 e_n$, így

$$\varphi(x,t) = \sum_{n\in\mathbb{Z}} \left(\frac{5}{\sqrt{2\pi}} \cdot \frac{i}{n}\left(e^{-3in} - 1\right)\right) \cdot e^{-n^2 t} \cdot \frac{e^{inx}}{\sqrt{2\pi}}$$

2. (4+(3+1+2) pont)

A) Veges differenciak.

Keress numerikus egyenleteket a kovetkezo DE kozelito megoldasara:

$$xu''(x) + u'(x) = x(1-x), \quad u(0) = u(1) = 0.$$

Approximaljuk az u fuggvenyt a kovetkezo vektorral: $\vec{u}_i = u(i\Delta x)$, $i = 1, \ldots, 4$, $\Delta x = 1/5$.

- Kozelitsd $u''(x)$-t az $u(x \pm \Delta x), u(x)$ ertekek segitsegevel!

(1)

$$U''(x) \approx \frac{1}{\Delta x^2}\left(U(x+\Delta x) - 2U(x) + U(x-\Delta x)\right)$$

(3)

- Ird fel az ennek megfelelo veges differencias kozeliteset a DE-nek mint egy inhom.lin. egyenletet a $\vec{u}$ vektorra!

$$\begin{bmatrix} 1\Delta x & & & \\ & 2\Delta x & & \\ & & 3\Delta x & \\ & & & 4\Delta x \end{bmatrix} \cdot \frac{1}{\Delta x^2} \cdot \begin{bmatrix} -2 & 1 & & \\ 1 & -2 & 1 & \\ & 1 & -2 & 1 \\ & & 1 & -2 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix} + \underbrace{\frac{1}{\Delta x} \begin{bmatrix} -1 & 1 & & \\ & -1 & 1 & \\ & & -1 & 1 \\ & & & -1 \end{bmatrix}}_{\approx \frac{d}{dx}} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix} = \begin{bmatrix} 1\Delta x(1-1\Delta x) \\ 2\Delta x(1-2\Delta x) \\ 3\Delta x(1-3\Delta x) \\ 4(\Delta x(1-4\Delta x) \end{bmatrix}$$

$\frac{d}{dx}$ egy jobb közelítése: $\frac{1}{2\cdot\Delta x} \begin{bmatrix} 0 & 1 & & \\ -1 & 0 & 1 & \\ & -1 & 0 & 1 \\ & & -1 & 0 \end{bmatrix}$

B) Veges elemek, variacios elv.

Oszd fel a $[0, 2]$ intervallumot 4 reszre a kovetkezo pontokkal: $x_i = 0.3, 0.5, 0.8$. Legyen $v(x)$ az a folytonos fuggveny, amelyik affine az alintervallumokon es az ertekei az $x = 0$, ~~0.2~~ 0.3, 0.5, 0.8, 2 pontokban a kovetkezoek: $0, v_1, v_2, v_3, 0$.

- Szamitsd ki, mennyi

$$Energy[v] = \int_0^2 (v')^2 + (1+x)v\,dx$$

kozelitoleg vagy pontosan! Kozelito szamitas eseten add meg, hogy milyen kozelitest hasznaltal!

- Ird fel az EL egyenleteket az $Energy[u]$ funkcionalra!
- Ird fel az igy kapott DE gyenge megfogalmazasat!

(3)

- Integrál közelítése: középpont módszer

$$Energy[V] \approx 0.3\cdot\left(\frac{V_1-0}{0.3}\right)^2 + 0.2\left(\frac{V_2-V_1}{0.2}\right)^2 + 0.3\left(\frac{V_3-V_2}{0.3}\right)^2 + 1.2\left(\frac{0-V_3}{1.2}\right)^2 +$$

$$+ 0.3\cdot\left[\left(1+\frac{0+0.3}{2}\right)\left(\frac{0+V_1}{2}\right)\right] + 0.2\left[\left(1+\frac{0.3+0.5}{2}\right)\cdot\left(\frac{V_1+V_2}{2}\right)\right] +$$

$$+ 0.3\cdot\left[\left(1+\frac{0.5+0.8}{2}\right)\left(\frac{V_2+V_3}{2}\right)\right] + 1.2\left[\left(1+\frac{0.8+2}{2}\right)\left(\frac{V_3+0}{2}\right)\right]$$

(1)

- $\frac{d}{dx}\frac{\partial L}{\partial v'} - \frac{\partial L}{\partial v} = 0 \qquad \frac{d}{dx}(2v') - (1+x) = \boxed{0 = 2v'' - (1+x)}$

(2)

- $\int_0^2 \varphi(x)\left[2v''(x) - (1+x)\right]dx = \int_0^2 -2\varphi'(x)v(x) - (1+x)\varphi(x)\,dx = 0$

"bármely" olyan $\varphi(x)$-re, amelyre teljesül $\varphi(0) = \varphi(2) = 0$

t^3-3t^2+3t-1

3. (5 × 2 pont)

$y'' - 5y' + 4y = (t-1)^3$, $y(0) = 2$, $y'(0) = 3$. Mennyi $Y(s)$? ($\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}$)

$$(s^2 Y(s) - 2s - 3) - 5(Y(s) - 2) + 4Y(s) = \frac{3!}{s^4} - 3\cdot\frac{2!}{s^3} + 3\cdot\frac{1}{s^2} - \frac{1}{s}$$

$$Y(s) = \frac{1}{s^2 - 5s + 4}\left(\frac{6}{s^4} - \frac{6}{s^3} + \frac{3}{s^2} - \frac{1}{s} + 2s + 7\right)$$

Oldd meg a $G'' - 5G' + 4G = \delta(t)$ egyenletet, ahol $G(t) = 0$, ha $t < 0$!

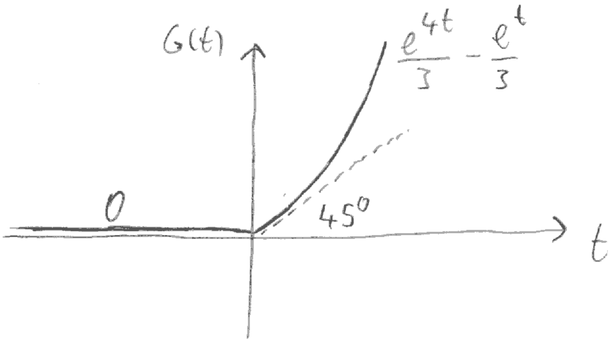
$$G(0^-) = G'(0^-) = 0, \quad G(0^+) = 0, \quad G'(0^+) = 1$$

Ha $t > 0$ $\quad G''(t) - 5G'(t) + 4G(t) = 0 \quad \Rightarrow G(t) = C_1 e^t + C_2 e^{4t}$

$$\left.\begin{array}{l} G(0^+) = 0 \Rightarrow C_1 + C_2 = 0 \\ G'(0^+) = 1 \Rightarrow C_1 + 4C_2 = 1 \end{array}\right\} \Rightarrow \begin{array}{l} C_1 = -\frac{1}{3} \\ C_2 = \frac{1}{3} \end{array}$$

$$G(t) = \begin{cases} 0, & \text{ha } t < 0 \\ -\frac{1}{3}e^t + \frac{1}{3}e^{4t}, & \text{ha } t > 0 \end{cases}$$

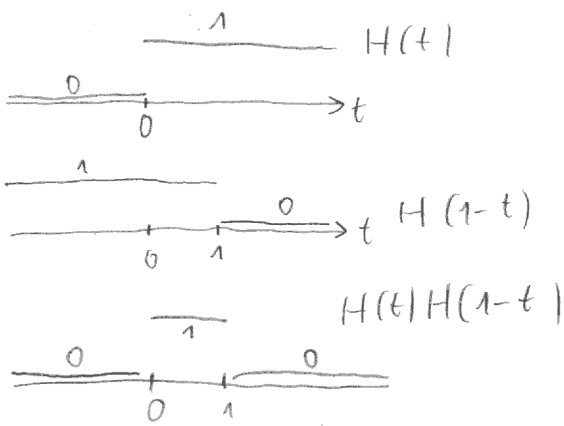
Rajzold le $G(t)$-t!



Ird fel a $y'' - 5y' + 4y = f(t)$ egyenlet megoldasat, ha $y(t) = f(t) = 0$ amikor $t < 0$!

$$y(t) = \int_{-\infty}^{\infty} G(t-\tau) f(\tau)\,d\tau = \int_{0}^{\infty} G(t-\tau) f(\tau)\,d\tau = \int_0^t \left(-\frac{1}{3}e^{t-\tau} + \frac{1}{3}e^{4(t-\tau)}\right) f(\tau)\,d\tau$$

Szamitsd ki az $\mathbb{R}$-en adott $\underbrace{H(t)H(1-t)}_{=f(t)}$ fuggveny Fourier transzformaltjat!



$$\tilde{f}(p) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ipt} f(t)\,dt =$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^1 e^{-ipt}\,dt = \frac{1}{\sqrt{2\pi}} \cdot \left.\frac{e^{-ipt}}{-ip}\right|_0^1 =$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{i}{p} \cdot \left(e^{-ip} - 1\right)$$