PZh1, Diff.Egy., 2017.05.10. NEPTUN: :

Név: Aláírás:

(2+2+(2+4) pont)

1a. $y' = e^{t^2-5}$, $y(3) = 5$. Fejezd ki $y(7)$-et a hatarozott integralas segitsegevel!

$$y(7) = 5 + \int_3^7 e^{t^2-5}\,dt$$

1b. Legyen $f(x) = \sqrt[2]{x}$. Ird fel f linearis approximaciojat az $x_0 = 9$ pont korul! Adj minel pontosabb felso korlatot a linearis approximacio hibajara, vagyis $|f(9+\Delta x) - f(9) - f'(9)\Delta x|$-re, ha $\Delta x \in [0, 0.1]$!

$$f(9) = 3,\quad f'(x) = \frac{1}{2}x^{-1/2},\quad f'(9) = \frac{1}{6},\quad f''(x) = -\frac{1}{4}x^{-3/2},\quad f''(9) = -\frac{1}{4\cdot 27} = -\frac{1}{108}$$

$$f(9+\Delta x) \approx 3 + \frac{1}{6}\Delta x,$$

$$hiba(\Delta x) \leq \frac{1}{2}\Delta x^2 \cdot \max_{\Delta x \in [0,0.1]} |f''(9+\Delta x)| = \frac{1}{2}\Delta x^2 \cdot \frac{1}{108} = \frac{1}{216}\Delta x^2$$

1c.

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} (y_1-2)y_2 \\ (y_1-3)(y_1-2). \end{pmatrix}$$

Keresd meg a DE fixpontjat!

$$\left.\begin{matrix} (y_1-2)y_2 = 0 \\ (y_1-3)(y_1-2) = 0 \end{matrix}\right\} \quad P_1 = \begin{pmatrix} 3 \\ 0 \end{pmatrix},\quad P_2 = \begin{pmatrix} 2 \\ z \end{pmatrix}$$

Ird fel a fixpont koruli linearizalt kozelito DE-t!

$$Jac = \begin{bmatrix} \partial_{y_1}[(y_1-2)y_2] & \partial_{y_2}[(y_1-2)y_2] \\ \partial_{y_1}[(y_1-3)(y_1-2)] & \partial_{y_2}[(y_1-3)(y_1-2)] \end{bmatrix} = \begin{pmatrix} y_2 & y_1-2 \\ 2y_1-5 & 0 \end{pmatrix}$$

$$Jac(P_1) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\quad Jac(P_2) = \begin{pmatrix} z & 0 \\ -1 & 0 \end{pmatrix}$$

$$\frac{d}{dt}\begin{pmatrix} y_1-3 \\ y_2-0 \end{pmatrix} = \frac{d}{dt}\begin{pmatrix} \Delta y_1 \\ \Delta y_2 \end{pmatrix} \approx \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \Delta y_1 \\ \Delta y_2 \end{pmatrix}$$

$$\frac{d}{dt}\begin{pmatrix} y_1-2 \\ y_2-z \end{pmatrix} = \frac{d}{dt}\begin{pmatrix} \Delta y_1 \\ \Delta y_2 \end{pmatrix} \approx \begin{pmatrix} z & 0 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \Delta y_1 \\ \Delta y_2 \end{pmatrix}$$

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} (y_1-2)y_2 \\ (y_1-3)(y_2-2) \end{pmatrix}$$

$$P_1 = \begin{pmatrix} 2 \\ 2 \end{pmatrix},\quad P_2 = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$$

$$Jac = \begin{bmatrix} \partial_{y_1}[(y_1-2)y_2] & \partial_{y_2}[(y_1-2)y_2] \\ \partial_{y_1}[(y_1-3)(y_2-2)] & \partial_{y_2}[(y_1-3)(y_2-2)] \end{bmatrix}$$

$$= \begin{pmatrix} y_2 & y_1-2 \\ y_2-2 & y_1-3 \end{pmatrix}$$

$$Jac(P_1) = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix},\quad Jac(P_2) = \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix}$$

$$\frac{d}{dt}\begin{pmatrix} y_1-2 \\ y_2-2 \end{pmatrix} = \frac{d}{dt}\begin{pmatrix} \Delta y_1 \\ \Delta y_2 \end{pmatrix} \approx \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \Delta y_1 \\ \Delta y_2 \end{pmatrix}$$

$$\frac{d}{dt}\begin{pmatrix} y_1-3 \\ y_2-0 \end{pmatrix} = \frac{d}{dt}\begin{pmatrix} \Delta y_1 \\ \Delta y_2 \end{pmatrix} \approx \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix}\begin{pmatrix} \Delta y_1 \\ \Delta y_2 \end{pmatrix}$$

3a. (1+2+1+2 pont)

$y' = (y^2 - 9)y. = f(y)$

Keresd meg a DE fixpontjait!

$$y_1 = -3, \quad y_2 = 0, \quad y_3 = 3$$

Ird fel a fixpontok koruli linearizalt kozelito DE-t!

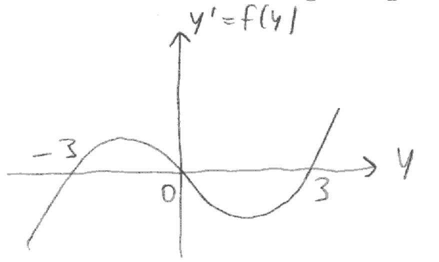
$$\frac{d}{dy}\left[(y^2-9)\cdot y\right] = 3y^2 - 9 = f'(y)$$

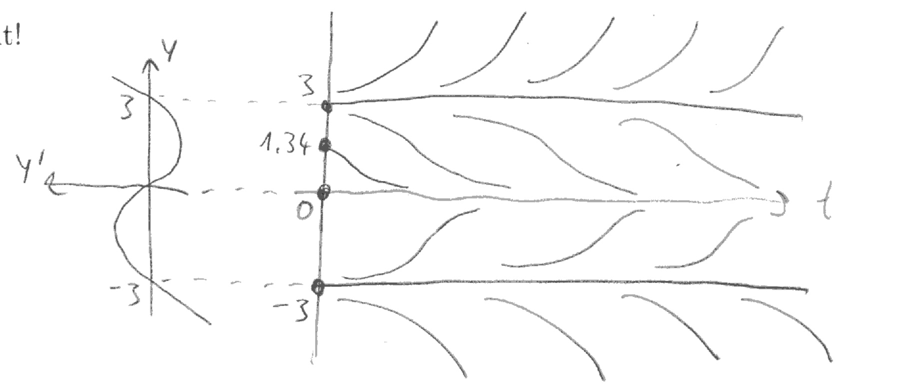
$f'(-3) = 18$	$f'(0) = -9$	$f'(3) = 18$
$\frac{d}{dt}(y-(-3)) = \frac{d}{dt}\Delta y \approx 18\Delta y$	$\frac{d}{dt}(y-0) = \frac{d}{dt}\Delta y = -9\cdot\Delta y$	$\frac{d}{dt}(y-3) = \frac{d}{dt}\Delta y = 18\Delta y$

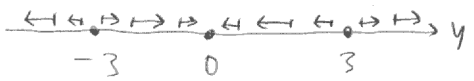
Ha $y(0) = 1.34$, mennyi

$\lim_{x\to\infty} y(x) = 0$ $\qquad\qquad$ $\lim_{x\to-\infty} y(x) = 3$

Vazold a DE megoldasgorbeit!







3b. (4 pont) Mennyi

$$\exp\left[t\begin{pmatrix}5 & 6 & 0\\ 0 & 5 & 0\\ 0 & 0 & 7\end{pmatrix}\right] =$$

$e^{A+B} = e^A \cdot e^B$, ha $AB = BA$

$$\exp\underbrace{\begin{bmatrix}5t & 0 & 0\\ 0 & 5t & 0\\ 0 & 0 & 7t\end{bmatrix}}_{A} \cdot \exp\underbrace{\begin{bmatrix}0 & 6t & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}}_{B} =$$

$$= \begin{bmatrix}e^{5t} & 0 & 0\\ 0 & e^{5t} & 0\\ 0 & 0 & e^{7t}\end{bmatrix} \cdot \left(\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} + \begin{bmatrix}0 & 6t & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix} + \frac{1}{2!}\overbrace{\begin{bmatrix}0 & 6t & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}^2}^{=0} + \ldots\right)$$

$$= \begin{bmatrix}e^{5t} & 0 & 0\\ 0 & e^{5t} & 0\\ 0 & 0 & e^{7t}\end{bmatrix}\begin{bmatrix}1 & 6t & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} = \begin{bmatrix}e^{5t} & 6te^{5t} & 0\\ 0 & e^{5t} & 0\\ 0 & 0 & e^{7t}\end{bmatrix}$$

2. (5+2+3 pont)

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 2y_1 - 3y_2 \\ 3y_1 + 2y_2 \end{pmatrix} = A \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \qquad \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

Keresd meg A sajatertekeit es sajatvektorait!

$$A = \begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix} \qquad \det(A - \lambda E) = \begin{vmatrix} 2-\lambda & -3 \\ 3 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 + 9 = \lambda^2 - 4\lambda + 13 = 0$$

$$\lambda_1 = 2 + 3i$$

$$\begin{pmatrix} -3i & -3 \\ 3 & -3i \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \vec{v} = \begin{pmatrix} u \\ -iu \end{pmatrix}$$

$$\vec{v}_1 = \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

$$\lambda_2 = 2 - 3i$$

$$\begin{pmatrix} 3i & -3 \\ 3 & 3i \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \vec{v} = \begin{pmatrix} u \\ iu \end{pmatrix}$$

$$\vec{v}_2 = \begin{pmatrix} 1 \\ i \end{pmatrix}$$

Ird fel a DE altalanos megoldasat!

$$\vec{y}(t) = C_1 e^{(2+3i)t} \begin{pmatrix} 1 \\ -i \end{pmatrix} + C_2 e^{(2-3i)t} \begin{pmatrix} 1 \\ i \end{pmatrix}$$

Szamold ki a DE partikularis megoldasat!

$$\vec{y}(0) = C_1 \begin{pmatrix} 1 \\ -i \end{pmatrix} + C_2 \begin{pmatrix} 1 \\ i \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$$\left.\begin{aligned} C_1 + C_2 &= 3 \\ -iC_1 + iC_2 &= 1 \end{aligned}\right\} \Rightarrow \left.\begin{aligned} C_1 + C_2 &= 3 \\ C_1 - C_2 &= i \end{aligned}\right\} \Rightarrow \begin{aligned} C_1 &= \frac{3}{2} + \frac{1}{2}i \\ C_2 &= \frac{3}{2} - \frac{1}{2}i \end{aligned}$$

$$\vec{y}_{part}(t) = \left(\frac{3}{2} + \frac{1}{2}i\right) e^{(2+3i)t} \begin{pmatrix} 1 \\ -i \end{pmatrix} + \left(\frac{3}{2} - \frac{1}{2}i\right) e^{(2-3i)t} \begin{pmatrix} 1 \\ i \end{pmatrix}$$

(5 × 2 pont)

Ird fel, hogy milyen osszefugges van A es a sajatertekeket tartalmazo diagonalis D matrixok kozott!

$$S^{-1}AS = D \qquad A = SDS^{-1}$$

$$\begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}^{-1} \begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} = \begin{pmatrix} 2+3i & 0 \\ 0 & 2-3i \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} \begin{pmatrix} 2+3i & 0 \\ 0 & 2-3i \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}^{-1} = \begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix}$$

Mennyi e^{tA}? (Elegendo az, hogy kifejezed az eredmenyt D, illetve egy S matrix es annak inverze szorzatakent!)

$$e^{tA} = S e^{tD} S^{-1} = \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} \begin{pmatrix} e^{(2+3i)t} & 0 \\ 0 & e^{(2-3i)t} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}^{-1}$$

Ird fel a partikularis megoldast e^{tA} segitsegevel!

$$\vec{y}_{part}(t) = e^{tA} \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

2b) Ird at a kovetkezo DE rendszert elsorendu idofuggetlen DE rendszerre!

$$\frac{d^2}{dt^2}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} ty_1' - y_1^2 \\ t^2 y_2' - y_1' - \cos(t) \end{pmatrix}$$

$$\frac{d}{dt}\begin{pmatrix} y_1 \\ v_1 \\ y_2 \\ v_2 \\ s \end{pmatrix} = \begin{pmatrix} v_1 \\ s v_1 - y_1^2 \\ v_2 \\ s^2 v_2 - v_1 - \cos(s) \\ 1 \end{pmatrix}$$

Ird fel $e^{2-i\pi/4}$ algebrai alakjat!

$$e^{2-i\pi/4} = e^2\left(\frac{1}{\sqrt{2}} - i\cdot\frac{1}{\sqrt{2}}\right)$$