Zh1, Diff.Egy., 2017.03.22. NEPTUN: :

Név: Aláírás:

(2+2+(2+4) pont)

1a. $y' = \sin(t^2)$, $y(3) = 5$. Fejezd ki $y(7)$-et a hatarozott integralas segitsegevel!

$$y(7) = 5 + \int_3^7 \sin(t^2)\, dt$$

1b. Legyen $f(x) = \sqrt[3]{x}$. Ird fel f linearis approximaciojat az $x_0 = 8$ pont korul! Adj minel pontosabb felso korlatot a linearis approximacio hibajara, vagyis $|f(8 + \Delta x) - f(8) - f'(8)\Delta x|$-re, ha $\Delta x \in [0, 0.1]$!

$f(x) = x^{1/3}$ $\quad f(8) = 2$ $\quad f(8+\Delta x) \approx 2 + \frac{1}{12}\Delta x$

$f'(x) = \frac{1}{3}x^{-2/3}$ $\quad f'(8) = \frac{1}{3\cdot 2^2} = \frac{1}{12}$

$f''(x) = -\frac{2}{9}x^{-5/3}$ $\quad f''(8) = -\frac{1}{9\cdot 2^4} = -\frac{1}{144}$

$$\text{hiba}|\Delta x| \leq \frac{1}{2}\Delta x^2 \max_{z\in[8,8+\Delta x]} |f''(z)| = \frac{1}{2}\cdot\frac{1}{144}\cdot\Delta x^2 = \frac{1}{288}\Delta x^2$$

1c.

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} (y_2 - 2)y_1 \\ (y_2 - 3)(y_1 - 4). \end{pmatrix}$$

Keresd meg a DE fixpontjat!

$$\left.\begin{aligned} (y_2-2)y_1 &= 0 \\ (y_2-3)(y_1-4) &= 0 \end{aligned}\right\} \quad \begin{aligned} &y_1 = 0 \\ &y_2 - 3 = 0 \to y_2 = 3 \end{aligned} \quad \text{vagy} \quad \begin{aligned} &y_2 - 2 = 0 \to y_2 = 2 \\ &y_1 - 4 = 0 \to y_1 = 4 \end{aligned}$$

Fixpontok $\quad \bar{z}_A = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$ és $\bar{z}_B = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$

Ird fel a fixpont koruli linearizalt kozelito DE-t!

$$\text{Jac} = \begin{pmatrix} \partial_{y_1}[(y_2-2)y_1] & \partial_{y_2}[(y_2-2)y_1] \\ \partial_{y_1}[(y_2-3)(y_1-4)] & \partial_{y_2}[(y_2-3)(y_1-4)] \end{pmatrix} = \begin{pmatrix} y_2-2 & y_1 \\ y_2-3 & y_1-4 \end{pmatrix}$$

$$\text{Jac}(\bar{z}_A) = \begin{pmatrix} +1 & 0 \\ 0 & -4 \end{pmatrix} \qquad \text{Jac}(\bar{z}_B) = \begin{pmatrix} 0 & 4 \\ -1 & 0 \end{pmatrix}$$

Lin. DE:

$$\frac{d}{dt}\begin{pmatrix} y_1-0 \\ y_2-3 \end{pmatrix} = \frac{d}{dt}\begin{pmatrix} \Delta y_1 \\ \Delta y_2 \end{pmatrix} = \begin{pmatrix} +1 & 0 \\ 0 & -4 \end{pmatrix}\begin{pmatrix} \Delta y_1 \\ \Delta y_2 \end{pmatrix}$$

$$\frac{d}{dt}\begin{pmatrix} y_1-4 \\ y_2-2 \end{pmatrix} = \frac{d}{dt}\begin{pmatrix} \Delta y_1 \\ \Delta y_2 \end{pmatrix} = \begin{pmatrix} 0 & 4 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \Delta y_1 \\ \Delta y_2 \end{pmatrix}$$

3a. (1+2+1+2 pont)
$y' = (y^2-1)(2-y).$ $= -y^3+2y^2+y-2$ $\qquad \dfrac{d(-y^3+2y^2+y-2)}{dy} = -3y^2+4y+1 = \mathrm{Jac}(y)$

Keresd meg a DE fixpontjait!

$y_1 = -1, \; y_2 = 1, \; y_2 = 2$

Ird fel a fixpontok koruli linearizalt kozelito DE-t!

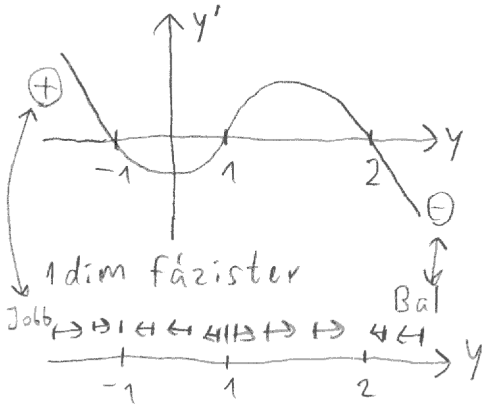
$y_1 = -1$: $\mathrm{Jac}(-1) = -6$ $\qquad \dfrac{d}{dt}(y-(-1)) = \dfrac{d}{dt}\Delta y = -6\Delta y$

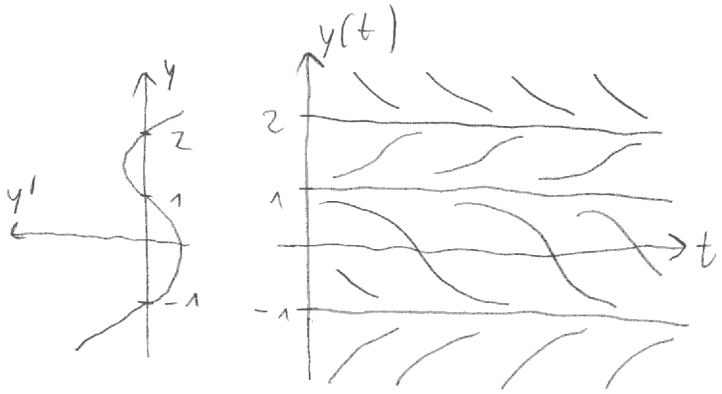
$y_2 = 1$: $\mathrm{Jac}(1) = 2$ $\qquad \dfrac{d}{dt}(y-1) = \dfrac{d}{dt}\Delta y = 2\Delta y$

$y_3 = 2$: $\mathrm{Jac}(2) = -3$ $\qquad \dfrac{d}{dt}(y-2) = \dfrac{d}{dt}\Delta y = -3\Delta y$

Ha $y(0) = 1.34$, mennyi
$\lim_{t\to\infty} y(t) = 2$ $\qquad \lim_{t\to-\infty} y(t) = 1$

Vazold a DE megoldasgorbeit!





3b. (4 pont) Mennyi

Blokk diagonális mátrix, tehát elég kiszámítani:

$$\exp\left[t\begin{pmatrix}5 & 6 & 0\\ 0 & 5 & 0\\ 0 & 0 & 7\end{pmatrix}\right] = \exp\left[t\begin{pmatrix}5 & 6\\ 0 & 5\end{pmatrix}\right] \text{ és } \exp[(t\cdot 7)]$$

$$= \exp\left[\begin{pmatrix}5t & 0 & 0\\ 0 & 5t & 0\\ 0 & 0 & 7t\end{pmatrix}\right]\cdot\exp\left[\begin{pmatrix}0 & t & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\right] =$$

$$= \begin{pmatrix}e^{5t} & 0 & 0\\ 0 & e^{5t} & 0\\ 0 & 0 & e^{7t}\end{pmatrix}\cdot\begin{pmatrix}1 & 6t & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} = \left(\begin{array}{cc|c}e^{5t} & 6te^{5t} & 0\\ 0 & e^{5t} & 0\\ \hline 0 & 0 & e^{7t}\end{array}\right)$$

2. (5+2+3 pont)

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 2y_1 \\ 3y_1 + 4y_2 \end{pmatrix} = A \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \qquad \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

Keresd meg A sajatertekeit es sajatvektorait!

karakterisztikus egyenlet: $\det(A-\lambda E)=0 = \begin{vmatrix} 2-\lambda & 0 \\ 3 & 4-\lambda \end{vmatrix} = (2-\lambda)(4-\lambda) - 0\cdot 3$

tehát $\lambda_1 = 2$, $\lambda_2 = 4$. (Ez automatikus, mivel A triangulásis)

$(A-\lambda E)\overline{v} = 0$

$\lambda_1 = 2:$ $\begin{pmatrix} 2-2 & 0 \\ 3 & 4-2 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$3u + 2v = 0 \longrightarrow v = -\frac{3}{2}u$

$\overline{v}_1 = \begin{pmatrix} 1 \\ -3/2 \end{pmatrix}$

$\lambda_2 = 4:$ $\begin{pmatrix} 2-4 & 0 \\ 3 & 4-4 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ 3 & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$-2u = 0$, $3u = 0 \longrightarrow u = 0$

$\overline{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Ird fel a DE altalanos megoldasat!

$$\overline{y}_{ált}(t) = C_1 e^{2t}\begin{pmatrix} 1 \\ -3/2 \end{pmatrix} + C_2 e^{4t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Szamold ki a DE partikularis megoldasat!

$$\overline{y}(0) = \begin{pmatrix} 3 \\ 1 \end{pmatrix} = C_1 \cdot e^{2\cdot 0}\begin{pmatrix} 1 \\ -3/2 \end{pmatrix} + C_2 \cdot e^{4\cdot 0}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= C_1\begin{pmatrix} 1 \\ -3/2 \end{pmatrix} + C_2\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\longrightarrow C_1 = 3, \quad C_2 = \frac{11}{2}$$

$$\overline{y}_{part}(t) = 3e^{2t}\begin{pmatrix} 1 \\ -3/2 \end{pmatrix} + \frac{11}{2}e^{4t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

5 × 2 pont)

rd fel, hogy milyen osszefugges van A es a sajatertekeket tartalmazo diagonalis D matrixok kozott!

$$D = S^{-1} A S$$

$$\begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -3/2 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 2 & 0 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -3/2 & 1 \end{pmatrix}, \quad \text{vagy}$$

$$A = SDS^{-1} = \begin{pmatrix} 1 & 0 \\ -3/2 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -3/2 & 1 \end{pmatrix}^{-1}$$

Mennyi e^{tA}? (Elegendo az, hogy kifejezed az eredmenyt D, illetve egy S matrix es annak inverze szorzatakent!)

$$e^{tA} = e^{tSDS^{-1}} = S e^{tD} S^{-1} =$$

$$= \begin{pmatrix} 1 & 0 \\ -3/2 & 1 \end{pmatrix} \begin{pmatrix} e^{2t} & 0 \\ 0 & e^{4t} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -3/2 & 1 \end{pmatrix}^{-1}$$

rd fel a partikularis megoldast e^{tA} segitsegevel!

$$\bar{y}_{part}(t) = e^{tA} \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

2b) Ird at a kovetkezo DE rendszert elsorendu idofuggetlen DE rendszerre!

$$\frac{d^2}{dt^2} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} ty_1' - y_2^2 \\ t^2 y_2' - y_1' - \sin(t) \end{pmatrix}$$

$$\frac{d}{dt} \begin{pmatrix} y_1 \\ y_2 \\ v_1 \\ v_2 \\ s \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ s v_1 - y_2^2 \\ s^2 v_2 - v_1 - \sin(s) \\ 1 \end{pmatrix}$$

rd fel $e^{2-i\pi/3}$ algebrai alakjat!

$$e^{2-i\pi/3} = e^2 \left[\cos\left(-\frac{\pi}{3}\right) + i \sin\left(-\frac{\pi}{3}\right) \right] = e^2 \cdot \left(+ \frac{1}{2} - \frac{\sqrt{3}}{2} i \right)$$