27 +1ev

PZh1, Diff.Egy., 2018.05.16. NEPTUN: :

Név: 2+3+4+1 pont Aláírás:

1a. Mi az $y'(t) = 4 - 2\delta(t)$, $y(3) = 1$ DE megoldasa?

$$y' = 4 - 2\delta(t) \longrightarrow y(t) = 4t - 2H(t) + C, \quad \text{ahol } H(t) = \begin{cases} 1, & \text{ha } t > 0 \\ 0, & \text{ha } t < 0 \end{cases}$$

$$y(3) = 1 \longrightarrow 1 = 4 \cdot 3 - 2 \cdot 1 + C \longrightarrow C = -9$$

$$y(t) = 4t - 2H(t) - 9 = \begin{cases} 4t - 11, & \text{ha } t > 0 \\ 4t - 9, & \text{ha } t < 0 \end{cases}$$

1b. Mi az $y''(t) = 4 - 2\delta(t)$, $y(3) = 1$, $y'(3) = 1$ DE megoldasa?

$$y'' = 4 - 2\delta(t) \longrightarrow y(t) = 4 \cdot \tfrac{1}{2} t^2 - 2K(t) + C_1 t + C_2 \quad \text{ahol } K(t) = \begin{cases} t, & \text{ha } t > 0 \\ 0, & \text{ha } t < 0 \end{cases}$$

$$y'(t) = 4t - 2H(t) + C_1$$

$$\left.\begin{array}{l} y(3) = 1 = 2 \cdot 3^2 - 2 \cdot 3 + C_1 \cdot 3 + C_2 \longrightarrow C_2 = +16 \\ y'(3) = 1 = 4 \cdot 3 - 2 \cdot 1 + C_1 \longrightarrow C_1 = -9 \end{array}\right\} \longrightarrow y(t) = 2t^2 - 2K(t) - 9t + 16$$

1c1. Mi az $y''(t) = 2y(t) + \delta(t)$ DE retardalt fundamentalis megoldasa?

$t < 0: \; y(t) = 0$

$t > 0: \; y'' = 2y \longrightarrow y(t) = C_1 \cdot e^{\sqrt{2}t} + C_2 e^{-\sqrt{2}t}$

$$t \approx 0: \; y(0^+) - y(0^-) = 0, \; y'(0^+) - y'(0^-) = 1 \longrightarrow \left.\begin{array}{l} C_1 + C_2 = 0 \\ \sqrt{2}C_1 - \sqrt{2}C_2 = 1 \end{array}\right\} \longrightarrow C_1 = \frac{1}{2\sqrt{2}}, \; C_2 = -\frac{1}{2\sqrt{2}}$$

$$G(t) = y(t) = \begin{cases} 0, & \text{ha } t < 0 \\ \dfrac{1}{2\sqrt{2}} e^{\sqrt{2}t} - \dfrac{1}{2\sqrt{2}} e^{-\sqrt{2}t}, & \text{ha } t > 0 \end{cases}$$

1c2. Mi az $y''(t) = 2y(t) + f(t)$, $y(t) = f(t) = 0$, ha $t << 0$ DE megoldasa?

$$y(t) = \int_{-\infty}^{\infty} G(t-s) f(s)\, ds = \int_{-\infty}^{t} \frac{1}{2\sqrt{2}} \left(e^{\sqrt{2}(t-s)} - e^{-\sqrt{2}(t-s)} \right) f(s)\, ds$$

2. (4+2+4 pont) Legyen

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 2y_1 - 3y_2 \\ 3y_1 + 2y_2 \end{pmatrix} = A \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \qquad \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{illetve} \quad \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Keresd meg A sajatertekeit es sajatvektorait!

$$0 = \det(A - \lambda E) = \begin{vmatrix} 2-\lambda & -3 \\ 3 & 2-\lambda \end{vmatrix} = \lambda^2 - 4\lambda + 13 \longrightarrow \lambda_{1,2} = 2 \pm 3i$$

$\lambda_1 = 2+3i$

$$\begin{pmatrix} 2-(2+3i) & -3 \\ 3 & 2-(2+3i) \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$-3iu - 3v = 0, \; v = -iu$

$$\vec{v}_1 = \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

$\lambda_2 = 2-3i$

$$\begin{pmatrix} 2-(2-3i) & -3 \\ 3 & 2-(2-3i) \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$3iu - 3v = 0, \; v = iu$

$$\vec{v}_2 = \begin{pmatrix} 1 \\ i \end{pmatrix}$$

$\lambda_1, \vec{v}_1$ komplex konjugáltja $\lambda_2, \vec{v}_2$

Ird fel a DE altalanos megoldasat!

$$\overline{y}(t) = c_1 e^{(2+3i)t} \begin{pmatrix} 1 \\ -i \end{pmatrix} + c_2 e^{(2-3i)t} \begin{pmatrix} 1 \\ i \end{pmatrix}$$

Szamold ki a DE partikularis megoldasait!

A $\overline{y}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$: $\quad c_1 \begin{pmatrix} 1 \\ -i \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ i \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$c_1 + c_2 = 1 \longrightarrow c_1 = c_2 = \frac{1}{2}$$
$$-ic_1 + ic_2 = 0 \longrightarrow c_1 = c_2$$

$$\overline{y}^A_{part}(t) = \frac{1}{2} e^{(2+3i)t} \begin{pmatrix} 1 \\ -i \end{pmatrix} + \frac{1}{2} e^{(2-3i)t} \begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{1}{2} \begin{pmatrix} e^{(2+3i)t} + e^{(2-3i)t} \\ -ie^{(2+3i)t} + ie^{(2+3i)t} \end{pmatrix}$$

B $\overline{y}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$: $\quad c_1 \begin{pmatrix} 1 \\ -i \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ i \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$\left.\begin{aligned} c_1 + c_2 &= 0 \\ -ic_1 + ic_2 &= 1 \end{aligned}\right\} \begin{aligned} c_1 &= \frac{i}{2} \\ c_2 &= -\frac{i}{2} \end{aligned}$$

$$\overline{y}^B_{part}(t) = \frac{i}{2} e^{(2+3i)t} \begin{pmatrix} 1 \\ -i \end{pmatrix} - \frac{i}{2} e^{(2-3i)t} \begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{1}{2} \begin{pmatrix} ie^{(2+3i)t} - ie^{(2-3i)t} \\ e^{(2+3i)t} + e^{(2-3i)t} \end{pmatrix}$$

3 + 1 pont

3a1. Mennyi e^{tA} ?

$$e^{tA} = \begin{pmatrix} \bar{y}^{A}_{part} & \bar{y}^{B}_{part} \end{pmatrix} = \frac{1}{2}\begin{bmatrix} e^{(2+3i)t} + e^{(2-3i)t} & ie^{(2+3i)t} - ie^{(2-3i)t} \\ -ie^{(2+3i)t} + ie^{(2-3i)t} & e^{(2+3i)t} + e^{(2-3i)t} \end{bmatrix}$$

Vagy

$$e^{tA} = Se^{tD}S^{-1} = \left(\underset{\uparrow\, \vec{v}_1}{\boxed{\begin{matrix} 1 \\ -i \end{matrix}}} \ \underset{\uparrow\, \vec{v}_2}{\boxed{\begin{matrix} 1 \\ i \end{matrix}}} \right) \begin{pmatrix} \underset{\uparrow\, \lambda_1}{e^{(2+3i)t}} & 0 \\ 0 & \underset{\uparrow\, \lambda_2}{e^{(2-3i)t}} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}^{-1}$$

3a2. Mi az $\frac{d}{dt}\bar{y}(t) = A\bar{y}(t) + \bar{f}(t),\ \bar{y}(t) = \bar{f}(t) = 0$, ha $t << 0$, DE megoldasa?

$$\bar{y}(t) = \int_{-\infty}^{t} e^{(t-s)A}\,\bar{f}(s)\,ds$$

3b. Legyen 4 + 2 pont

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 2y_1 \\ \cancel{2y_2}\ 2y_1 + 2y_2 \end{pmatrix} = A\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

Mennyi e^{tA} ?

$$e^{tA} = e^{t\cdot\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} + t\cdot\begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}} = e^{\begin{pmatrix} 2t & 0 \\ 0 & 2t \end{pmatrix}} \cdot e^{\begin{pmatrix} 0 & 0 \\ 2t & 0 \end{pmatrix}} =$$

$$= \begin{pmatrix} e^{2t} & 0 \\ 0 & e^{2t} \end{pmatrix} \cdot \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 2t & 0 \end{pmatrix} + \frac{1}{2!}\underbrace{\begin{pmatrix} 0 & 0 \\ 2t & 0 \end{pmatrix}^2}_{= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}} + \ldots \right] =$$

$$= \begin{pmatrix} e^{2t} & 0 \\ 0 & e^{2t} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 2t & 1 \end{pmatrix} = \begin{pmatrix} e^{2t} & 0 \\ e^{2t}\cdot 2t & e^{2t} \end{pmatrix}$$

Mi az elozo DE partikularis megoldasa az $(y_1(0), y_2(0))^T = (4, 5)$ kezdeti feltetel mellett?

$$\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = e^{tA}\begin{pmatrix} 4 \\ 5 \end{pmatrix} = \begin{pmatrix} e^{2t} & 0 \\ 2te^{2t} & e^{2t} \end{pmatrix}\begin{pmatrix} 4 \\ 5 \end{pmatrix}$$

4a. (4 pont)

$y' = e^{-3y} - 4. = f(y)$

Keresd meg a DE fixpontjait!

$$0 = e^{-3y} - 4 \longrightarrow y_1 = -\frac{\ln 4}{3} \approx -0.46$$

$$\frac{d}{dy}\left(e^{-3y} - 4\right) = -3e^{-3y} = \frac{df}{dy} = f'(y)$$

Ird fel a fixpontok koruli linearizalt kozelito DE-t!

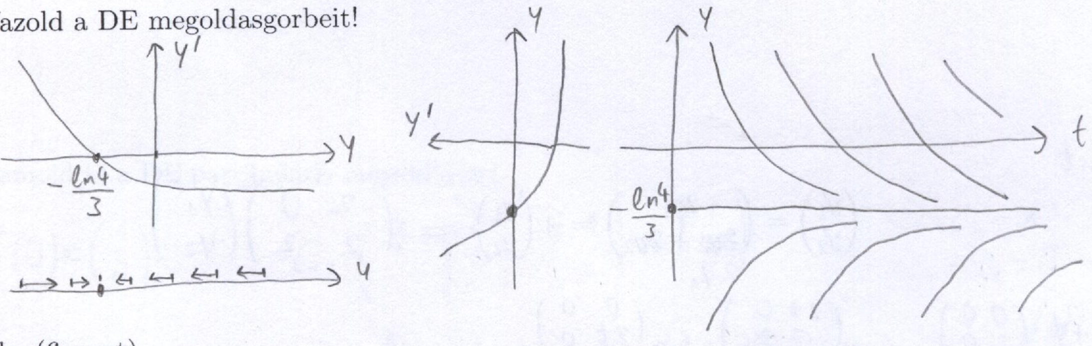
$$f'\left(-\frac{\ln 4}{3}\right) = -3e^{-3 \cdot \frac{\ln 4}{-3}} = -3 \cdot 4 = -12$$

$$\frac{d}{dt}\left(y - \left(-\frac{\ln 4}{3}\right)\right) = \frac{d}{dt}\Delta y = -12\,\Delta y$$

Ha $y(0) = 2$, mennyi

$\lim_{x\to\infty} y(x) = -\frac{\ln 4}{3}$ $\lim_{x\to-\infty} y(x) = \infty$

Vazold a DE megoldasgorbeit!



4b. (6 pont)

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} (y_2 - 2)(y_1^2 + 1) \\ (y_1 - 4)(y_2 + 5). \end{pmatrix}$$

Keresd meg a DE fixpontjat!

$$(y_2 - 2)(y_1^2 + 1) = 0 \longrightarrow y_2 = 2$$

$$(y_1 - 4)\cdot(2 + 5) = 0 \longrightarrow y_1 = 4$$

fixpont: $P = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$

$$\mathrm{Jac} = \begin{pmatrix} \partial_{y_1}\left[(y_2-2)(y_1^2+1)\right] & \partial_{y_2}\left[(y_2-2)(y_1^2+1)\right] \\ \partial_{y_1}\left[(y_1-4)(y_2+5)\right] & \partial_{y_2}\left[(y_1-4)(y_2+5)\right] \end{pmatrix} = \begin{pmatrix} (y_2-2)\cdot 2y_1 & y_1^2+1 \\ y_2+5 & y_1-4 \end{pmatrix}$$

Ird fel a fixpont koruli linearizalt kozelito DE-t!

$$\mathrm{Jac}(P) = \begin{pmatrix} 0 & 17 \\ 7 & 0 \end{pmatrix}$$

$$\frac{d}{dt}\begin{pmatrix} y_1 - 4 \\ y_2 - 2 \end{pmatrix} = \frac{d}{dt}\begin{pmatrix} \Delta y_1 \\ \Delta y_2 \end{pmatrix} = \begin{pmatrix} 0 & 17 \\ 7 & 0 \end{pmatrix}\begin{pmatrix} \Delta y_1 \\ \Delta y_2 \end{pmatrix}$$