Differential equations. Quiz.2 problems

1. Let $u = (1, i)^T$, $v = (3 - i, 2 + 4i)^T$. Compute (u, v) and (v, u) !

   Sol. $(u, v) = \bar{1}(3 - i) + \bar{i}(2 + 4i) = 7 - 3i$, $(v, u) = \overline{(u, v)} = 7 + 3i$.

2. $e_1 = \frac{1}{\sqrt{2}}(1, i)^T$, $e_2 = \frac{1}{\sqrt{2}}(i, 1)^T$, $u = (3 + i, 4 - 2i)^T$, $u = \alpha_1 e_1 + \alpha_2 e_2$. Compute $\alpha_{1,2}$!

   Sol. $e_{1,2}$ is an othonormed basis, so $\alpha_n = (e_n, u)$. In particular $\alpha_1 = \frac{1}{\sqrt{2}}(1 - 3i)$, $\alpha_2 = \frac{1}{\sqrt{2}}(5 - 5i)$.

3. If $e_1 = \frac{1}{\sqrt{2}}(i, i)^T$, $e_2 = \frac{1}{\sqrt{2}}(i, \alpha)^T$ is an orthonormed basis, then how much is α?

   Sol. $(e_1, e_2) = \frac{1}{2}((-i)i + (-i)\alpha) = 0 \implies \alpha = -i$.

4. Let
$$A = \begin{pmatrix} i & 3 - 2i \\ 4 + 2i & 8 - i \end{pmatrix}.$$
   Compute A^*!

   Sol. $(A^*)_{i,j} = \bar{A}_{ji}$, so
$$A^* = \begin{pmatrix} -i & 4 - 2i \\ 3 + 2i & 8 + i \end{pmatrix}.$$

5. $$B = \begin{pmatrix} 3 & 3 - 2i \\ 3 + 2i & 8 \end{pmatrix}.$$
   If $\lambda_{1,2}$ are the eigenvalues of B, than how much is $Im(\lambda_1 - \lambda_2)$?

   Sol. B is self-adjoint, so its eigenvalues are real numbers, consequently the answer is 0.

6. $$B = \begin{pmatrix} 3i & 3i - 2 \\ 3i + 2 & 8i \end{pmatrix}.$$
   If $\lambda_{1,2}$ are the eigenvalues of B, than how much is $Re(\lambda_1 - \lambda_2)$?

   Sol. iB is self-adjoint, so the answer is 0.

7. We claim that the set of functions $e_n(x) = \sin(nx)$, $n = 1, 2, \dots$ is an orthogonal complete basis in $\mathcal{H} = L^2([0, \pi], dx)$. Construct an orthonormed basis in $\mathcal{H}$!

   Sol. The average value of $\sin^2(x)$ and $\cos^2(x)$ is $1/2$ over an interval of length π, since $\sin^2(x) + \cos^2(x) = 1$. So the basis $\tilde{e}(x) = \sqrt{\frac{2}{\pi}} \sin(nx)$, $n = 1, 2, \dots$ has the proper normalization.

8. Let f be a function on $[0, \pi]$ and express $f(x)$ as $\sum_{n>0}^{\infty} \hat{f}_n \sin(nx)$. Compute $\hat{f}_n$! (Repeat this exercise for different orthonormed basis functions (ex. 9-11)!)

Outline of the theory of orthogonal series (Fourier transform): Let e_n be an othonormed basis. Then a vector u can be expressed as $u = \sum_n \hat{f}_n e_n = \sum_n (e_n, u) e_n$. The application of this scheme for $L^2([-\pi,\pi], dx)$ (as known as the Fourier transform) is the following:

Othonormed basis: $e_n = \frac{1}{\sqrt{2\pi}} e^{inx}, n \in \mathbb{Z}$.

Fourier transform $f \to \hat{f}$:

$$\hat{f}_n = (e_n, f) = \int_{-\pi}^{\pi} \overline{\frac{1}{\sqrt{2\pi}} e^{inx}} f(x)\, dx = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} e^{-inx} f(x)\, dx.$$

Inverse Fourier transform $\hat{f} \to f$:

$$f(x) = \left(\sum_{n\in\mathbb{Z}} \hat{f}_n e_n \right)(x) = \sum_{n\in\mathbb{Z}} \hat{f}_n \frac{1}{\sqrt{2\pi}} e^{inx}$$

If an orthogonal (but unnormalized) basis e_n was used, then we can use the following transformations

$$f(x) \xrightarrow{FT} \hat{f}_n = \frac{1}{(e_n, e_n)} (e_n, f) \xrightarrow{IFT} f = \sum_n \hat{f}_n e_n.$$

Sol.

$$\hat{f}_n = \frac{1}{(e_n, e_n)} (e_n, f) = \frac{1}{\pi/2} \int_0^{\pi} \sin(nx) f(x)\, dx.$$

9. We claim that the set of functions $\cos(nx)$, $n = 0, 1, 2, \ldots$ is an orthogonal complete basis in $\mathcal{H} = L^2([0,\pi], dx)$. Construct an orthonormed basis in $\mathcal{H}$!

Sol. Orthonormed basis: $\frac{1}{\sqrt{\pi}}, \sqrt{\frac{2}{\pi}}\, n = 1, 2, \ldots$.

10. We claim that the set of functions $\sin(nx)$, $n = 1, 2, \ldots$ is an orthogonal complete basis in $\mathcal{H} = L^2([0,\pi], dx)$. Construct an orthonormed basis in $L^2([0,88], dx)$!

11. We claim that the set of functions $\sin(nx)$, $n = 1, 2, \ldots$ is an orthogonal complete basis in $\mathcal{H} = L^2([0,\pi], dx)$. Construct an orthonormed basis in $L^2([-77,88], dx)$!

Sol. Orthonormed basis: $\sqrt{\frac{2}{(88-(-77))}} \sin((\pi/(88-(-77))) \cdot n(x-(-77)), n = 1, 2, \ldots$

12. In the sequel the characteristic function χ_D is defined as $\chi_D(x) = 0$ if $x \notin D$, otherwise $\chi_D(x) = 1$. Let $f(x) = \chi_{[0.1]}(x)$, $f(x) = \sum_{n\in\mathbb{Z}} \hat{f}_n \frac{e^{inx}}{\sqrt{2\pi}}$. Compute $\hat{f}_5$!

Sol.

$$\hat{f}_5 = \int_{-\pi}^{\pi} \overline{\frac{1}{\sqrt{2\pi}} e^{i \cdot 5x}} f(x)\, dx = \frac{1}{\sqrt{2\pi}} \int_0^1 e^{-i \cdot 5x} \cdot 1\, dx = \left. \frac{e^{-i\cdot 5x}}{-i5} \right|_0^1 = \frac{i}{5}(e^{-i\cdot 5} - 1).$$

13. Let $f(x) = \chi_{[0.1]}(x) \in L^2(\mathbb{R}, dx)$. Compute $\hat{f}(2.3)$!

Sol.

$$\hat{f}(2.3) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\cdot 2.3x} f(x)\, dx = \frac{1}{\sqrt{2\pi}} \int_0^1 e^{-i\cdot 2.3x} \cdot 1\, dx = \left. \frac{e^{-i\cdot 2.3x}}{-i \cdot 2.3} \right|_0^1 = \frac{i}{2.3}(e^{-i\cdot 2.3} - 1).$$

14. If $\phi(x,t)$ satisfies $(\partial_{tt}^2 - 9\partial_{xx}^2)\phi = 0$ and $\phi(x,t) = e^{i(kx+\omega t)}$, then what is the relation between k and ω ?

Sol. $\partial_{tt}^2 e^{i(kx+\omega t)} = (i\omega)^2 e^{i(kx+\omega t)}$, $\partial_{xx}^2 e^{i(kx+\omega t)} = (ik)^2 e^{i(kx+\omega t)}$, so $\omega^2 - 9k^2 = 0 \implies |\omega| = 9|k|$.

15. If $\phi(x,t)$ satisfies $(\partial_{tt}^2 - 9\partial_{xx}^2 - \partial_{xt}^2)\phi = 0$ and $\phi(x,t) = e^{i(kx+\omega t)}$, then what is the relation between k and ω ?

16. $\partial_t \phi(t,x) = \partial_{xx}^2 \phi(t,x)$, $\phi(0,x) = \sin(4x) + 5\cos(6x)$. Compute $\phi(t,x)$!

Sol. $\sin(4x)$ and $\cos(6x)$ are eigenfunctions of the operator ∂_{xx}^2 :

$$\partial_{xx}^2 \sin(4x) = -4^2 \sin(4x), \quad \partial_{xx}^2 \cos(6x) = -6^2 \sin(4x).$$

So the solution is

$$\phi(t,x) = e^{-4^2 t} \sin(4x) + 5e^{-6^2 t} \cos(6x).$$