Test 1. Diff.Eq. 2015.04.15.

1a. (1+1+1+2)

~~$y' = \frac{3}{1+y^2} - 1.$~~ $y' = (1-y)(2+y)$

Find the fixed points of the DE!

Write down the linearized DE around the fixed points!

If $y(0) = 0$, how much is

$\lim_{x\to\infty} y(x) =$ $\qquad\qquad$ $\lim_{x\to-\infty} y(x) =$

Sketch the solution curves of the DE!

1b. (2+3)

$$\frac{d}{dt}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} y_2^2 - 4 \\ y_1 - 2 \\ y_3 - 1 \end{pmatrix}.$$

Find the fixed pints of the DE!

Write down the linearized DE around the fixed points!

2. (3+4+1+2)

a)

$$y' = f(x, y) = y^3 x^2$$

How much are y'' ~~and y'''~~? Compute the ~~third~~ second-order Taylor polynom of y around $x = 2$, if $y(2) = 1$!

b) Apply the Euler and the Heun methods with $\Delta x = 0.1$ timestep for the following DE.

$$\frac{d}{dt}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} y_2 + 1 \\ y_1^2 - y_2^2 \end{pmatrix}. \qquad \begin{pmatrix} y_1(1) \\ y_2(1) \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

What are their predictions for ~~$y(2.1)$?~~ $\bar{y}(2.1)$?

Euler:

Heun:

c) Solve it: $y' = 3y$, $y(7) = 9$.

d) Express the solution of $y' = e^{3t^2}$, $y(7) = 6$ with the help of definite integrals.

3. (5+2+3)

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} -4y_1 + 3y_2 \\ 3y_1 - 4y_2 \end{pmatrix} = A\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \qquad \begin{pmatrix} y_1(1) \\ y_2(1) \end{pmatrix} = \begin{pmatrix} 6 \\ 8 \end{pmatrix}$$

Find the eigenvalues and eigenvectors of A !

Write down the general solution!

Compute

$$\begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} \quad !$$

4 ~~3~~. ((2 + 3) + 5)

a) Compute the Euler-Lagrange equations For L and M !

$$L = (2y' - 1)^2 - y^4 y', \qquad M = y_1' y_2' + (y_2')^2 + y_1' y_2 - y_1 y_2.$$

b) Let $S[u] = \int_0^3 (y'(x) - 1)^2 + x^2 y(x)\, dx$ where u is defined on $[0, 3]$ and vanishes at the endpoints. Let V be defined on $[0, 3]$, assume that it vanishes at the endpoints and is continuous. Assume also that elements of V are piecewise affine on the $[0, 1]$, $[1, 2]$, $[2, 3]$ intervals. Let ϕ_1 and ϕ_2 be a basis of V, such that $\phi_1(1) = \phi_2(2) = 1$ and ~~$\phi_2(2) = \phi_1(1) = 0$.~~ $\varphi_2(1) = \varphi_1(2) = 0$ Let $u_h = c_1\phi_1 + c_2\phi_2$. Compute the $S[u_h] = s(c_1, c_2)$ two variable function! (For the computation of the $xy(x)$ term in the integral use some approximate method!)

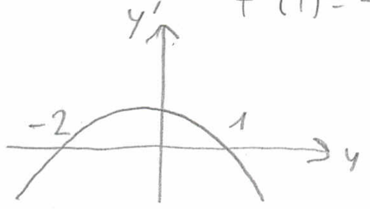
Test 1. DE. April 15

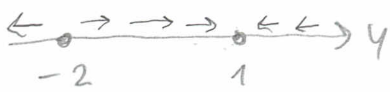
1a $y' = (1-y)(2+y) = -y^2 - y + 2 \qquad \dfrac{d(-y^2-y-2)}{dy} = -2y-1$

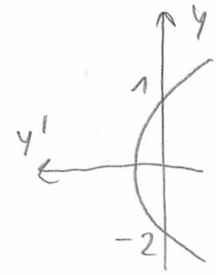
Fixedpoints: $y_1 = 1$ (1) $\qquad y_2 = -2$

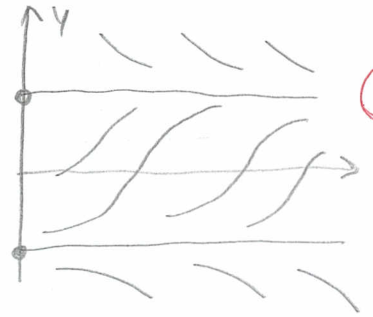
Lin. DE: $\dfrac{d}{dt}(y-1) = \dfrac{d}{dt}\Delta y = -3\Delta y$ (1) $\qquad \dfrac{d}{dt}(y-(-2)) = \dfrac{d}{dt}\Delta y = 3\Delta y$

$f'(1) = -2 \cdot 1 - 1 = -3 \qquad f'(-2) = -2\cdot(-2) - 1 = 3$









(2)

$y(0)=0, \quad \lim\limits_{t\to\infty} y(t) = 1, \quad \lim\limits_{t\to-\infty} y(t) = -2$ (1)

1b Fixpoints:
$$\left.\begin{aligned} y_2^2 - 4 &= 0 \\ y_1 - 2 &= 0 \\ y_3 - 1 &= 0 \end{aligned}\right\} \quad \begin{aligned} y_2 &= \pm 2 \\ y_1 &= 2 \\ y_3 &= 1 \end{aligned} \qquad P_1 = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}, \; P_2 = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$$ (2)

$$\text{Jac} = \begin{pmatrix} \frac{\partial}{\partial y_1}(y_2^2-4) & \frac{\partial}{\partial y_2}(y_2^2-4) & \frac{\partial}{\partial y_3}(y_2^2-4) \\ \frac{\partial}{\partial y_1}(y_1-2) & \frac{\partial}{\partial y_2}(y_1-2) & \frac{\partial}{\partial y_3}(y_1-2) \\ \frac{\partial}{\partial y_1}(y_3-1) & \frac{\partial}{\partial y_2}(y_3-1) & \frac{\partial}{\partial y_3}(y_3-1) \end{pmatrix} = \begin{pmatrix} 0 & 2y_2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ (1)

$$\text{Jac}(P_1) = \begin{pmatrix} 0 & -4 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{Jac}(P_2) = \begin{pmatrix} 0 & 4 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Lin. DE:
$$\frac{d}{dt}\begin{pmatrix} y_1 - 2 \\ y_2 - (-2) \\ y_3 - 1 \end{pmatrix} = \frac{d}{dt}\overline{\Delta y} = \begin{pmatrix} 0 & -4 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\overline{\Delta y} \qquad \frac{d}{dt}\overline{\Delta y} = \begin{pmatrix} 0 & 4 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\overline{\Delta y}$$

(1) $\qquad$ (1)

2a

$$y' = y^3 x^2$$

$$y'' = \left(\frac{\partial}{\partial x} + y^3x^2 \cdot \frac{\partial}{\partial y}\right)(y^3x^2) = y^3 \cdot 2x + y^3x^2 \cdot 3y^2x^2 =$$

$$= 2xy^3 + 3y^5x^4$$

$$y(2) = 1$$

$$y'(2) = 1^3 \cdot 2^2 = 4$$

$$y''(2) = 2 \cdot 2 \cdot 1^3 + 3 \cdot 1^5 \cdot 2^4 = 52$$

$$y(2+\Delta x) \approx 1 + 4\Delta x + \frac{52}{2!}\Delta x^2$$

or $$y(x) \approx 1 + 4\cdot(x-2) + \frac{52}{2!}(x-2)^2$$

3

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b

Euler: $$\bar{y}(2.1) \approx \begin{pmatrix} 3 \\ 2 \end{pmatrix} + \begin{pmatrix} 2+1 \\ 3^2 - 2^2 \end{pmatrix} \cdot 0.1 = \begin{pmatrix} 3.3 \\ 2.5 \end{pmatrix}$$ 3

Heun: $$\bar{y}(2.1) \approx \begin{pmatrix} 3 \\ 2 \end{pmatrix} + \frac{1}{2}\left[\begin{pmatrix} 2+1 \\ 3^2 - 2^2 \end{pmatrix} + \begin{pmatrix} 2.5+1 \\ 3.3^2 - 2.5^2 \end{pmatrix}\right] \cdot 0.1$$

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c

$$y_{gen} = C \cdot e^{3t}, \quad y_{part} = e^{3(t-7)} \cdot 9 = 9e^{-21} \cdot e^{3t}$$ 2

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d

$$y(t) = 6 + \int_7^t e^{3s^2}\,ds$$ 2

③ $A = \begin{pmatrix} -4 & 3 \\ 3 & -4 \end{pmatrix}$ $\det(A-\lambda E) = \begin{vmatrix} -4-\lambda & 3 \\ 3 & -4-\lambda \end{vmatrix} = 0$

$(-4-\lambda)^2 - 3^2 = \lambda^2 + 8\lambda + 7 = 0$

$\lambda_1 = -1$, $\lambda_2 = -7$ ②

$\begin{pmatrix} -4-(-1) & 3 \\ 3 & -4-(-1) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 & 3 \\ 3 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \longrightarrow x = y, \bar{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

③

$\begin{pmatrix} -4-(-7) & 3 \\ 3 & -4-(-7) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \longrightarrow x = -y, \bar{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\bar{y}_{gen} = c_1 e^{-1\cdot t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-7\cdot t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ ②

$\begin{pmatrix} 6 \\ 8 \end{pmatrix} = c_1 e^{-1}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-7}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \longleftrightarrow \begin{array}{l} 6 = c_1 e^{-1} + c_2 e^{-7} \\ 8 = c_1 e^{-1} - c_2 e^{-7} \end{array}$

② $c_1 = 7e$, $c_2 = -e^7$

$\begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = 7e \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} + (-e^7)\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ ①

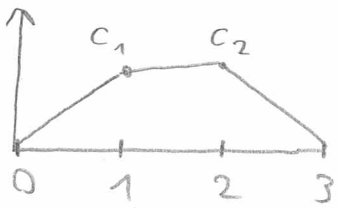
④ ⓐ $\frac{d}{dx}\frac{\partial L}{\partial y'} - \frac{\partial L}{\partial y} = 0$

L: $\frac{d}{dx}\left(2\cdot(2y'-1)\cdot 2 + y^4\right) - 4y^3\cdot y' = 0$ 2

M: $\frac{d}{dx}\left(y_2' + y_2\right) - (-y_2) = 0$

$\frac{d}{dx}\left(y_1' + 2y_2'\right) - (y_1' - y_1) = 0$ 3

ⓑ



$$S[u_h] \approx \left(\overset{y'}{\frac{c_1}{1}} - 1\right)^2 \cdot \overset{\Delta x}{1} + \left(\frac{c_2 - c_1}{1} - 1\right)^2 \cdot 1 + \left(\frac{-c_2}{1} - 1\right)^2 \cdot 1 +$$

$$+ \frac{1}{2}\left(0^2 \cdot 0 + \underset{x^2}{1^2} \cdot \underset{y(x)}{c_1}\right) \cdot \underset{\Delta x}{1} + \frac{1}{2}\left(1^2 \cdot c_1 + 2^2 \cdot c_2\right) \cdot 1 +$$

$$+ \frac{1}{2}\left(2^2 \cdot c_2 + 3^2 \cdot 0\right) \cdot 1$$

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