Long term behaviour of dynamical systems

(1) Discrete time, 1 dimension

Given x_0, $f(x)$, construct the recursive sequence

$$x_0,\ x_1 = f(x_0),\ x_2 = f(x_1) = f(f(x_0)) = f^2(x_0),\ x_3 = f(x_2) = f^3(x_0),\ \dots$$

Possible behaviours:

(1) $\lim_{n\to\infty} x_n = x_{fix}$, convergence to $f(x_{fix}) = x_{fix}$ fixpoint

(2) periodic sequence: $x_n = x_{n+p}$ for some $p \in \mathbb{N}$

(3) convergence to periodic sequence: $\lim_{n\to\infty} (x_n - x_{n+p}) = 0$ for some $p \in \mathbb{N}$

(4) everything else

Characterization

(a) attractor A:

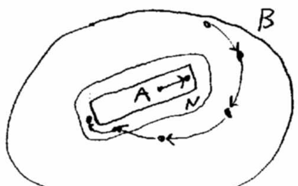
① $f(A) \subset A$ forward invariance

② $\exists B$: B open, $B \supset A$, (basin of attraction)

for $\forall N \supset A$, N open exist T, such that

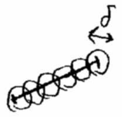
$f^t(b) \in N$ for $\forall b \in B$, $t > T$

③ A is the smallest possible, (no $\tilde{A} \subset A$ with ①, ②)



strange attractor: attractor with fractal structure

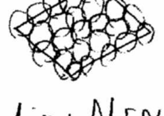
Remark: noninteger dimension (Hausdorff)



cover A with balls of radius δ. Minimal number: $N_\delta \sim \frac{1}{\delta^{dim}}$

dimension: $dim = \lim_{\delta\to\infty} \left(-\frac{\ln N_\delta}{\ln \delta}\right)$

1 dim: $N_\delta \sim \frac{1}{\delta}$



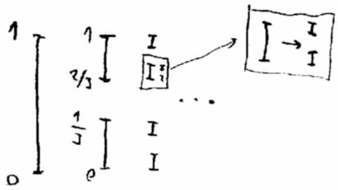
2 dim: $N_\delta \sim \frac{1}{\delta^2}$

$N_\delta \sim \frac{1}{\delta^2} + \frac{1}{\delta} \approx \frac{1}{\delta^2}$

$dim = 2$

Remark (cont.)

Noninteger dim: example: $C = \{0.c_1c_2c_3c_4\ldots \mid c_i = 0 \text{ or } 2\}$ (Cantor set; base 3 floating point)



To cover C by balls (interval) of diameter $\frac{1}{3^n} = \delta$,

we need $N_{1/3^n} = 2^n$, so $\dim = \lim_{n\to\infty}\left(-\frac{\ln 2^n}{\ln(1/3^n)}\right) = \frac{\ln(2)}{\ln(3)} \approx$

Characterization (3): Lyapunov exponent

Measures sensitivity to the initial condition

$x_0 \longrightarrow f(x_0) = x_1 \longrightarrow f(x_1) = x_2 \longrightarrow \ldots$

$x_0 + \delta x_0 \longrightarrow f(x_0+\delta x) \approx f(x_0) + f'(x_0)\delta x_0 = x_1 + \delta x_1 \longrightarrow f(x_1+\delta x_1) \approx f(x_1) + f'(x_1)\cdot f'(x_0)\,\delta x_0 = x_2 + \delta x_2$

So $x_n + \delta x_n \approx x_n + f'(x_0) f'(x_1) \cdot \ldots \cdot f'(x_{n-1}) \cdot \delta x_0$

Lyapunov exponent $\lambda = \lim_{n\to\infty} \frac{1}{n} \sum_{\ell=0}^{n-1} \ln|f'(x_\ell)| = \lim_{n\to\infty} \frac{1}{n} \lim_{\delta x_0 \to 0} \ln \frac{|\delta x_n|}{|\delta x_0|}$

We expect $\delta x_n \sim e^{\lambda \cdot n} \cdot \delta x_0$

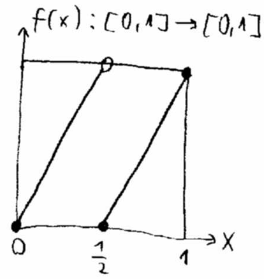
Example:

(a) $f(x) = \frac{2}{3}x$, $x \in [0,1]$, $x_n = x_0 \cdot \left(\frac{2}{3}\right)^n$, $\delta x_n = \left(\frac{2}{3}\right)^n \cdot \delta x$

$\lambda = \lim_{n\to\infty} \frac{1}{n} \sum_{\ell=0}^{n-1} \ln\left|\left(\frac{2}{3}\right)\right| = \ln\left(\frac{2}{3}\right) = \lim_{n\to\infty} \frac{1}{n} \lim_{\delta x_0 \to 0} \left(\frac{2}{3}\right)^n < 0$

$\uparrow$ $\delta x_n \to 0$

(b)



$f(0.b_1b_2b_3\ldots) = f(0.b_2b_3b_4\ldots)$ (binary floating point)

$f'(x) = 2$ except at $x = \frac{1}{2}$ $\longrightarrow \lambda = \ln 2$

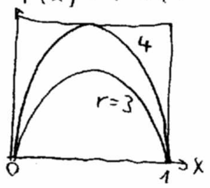
$x_0 - \tilde{x}_0 = (0.\underbrace{00000}_{k} d_1 d_2 d_3 \ldots) = \delta x_0$

$\delta x_L = 2^L \cdot \delta x_0$

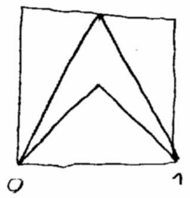
$L < k$ $\quad x_L - \tilde{x}_L = (0.\underbrace{000}_{k-L} d_1 d_2 d_3 \ldots) = \delta x_L$

Basic examples:

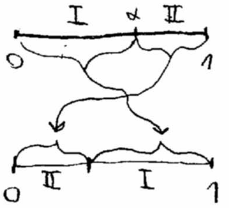
① Logistic map $f(x) = r\,x(1-x)$, $x \in [0,1]$, $r \in [0,4]$

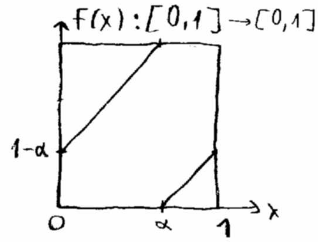


② Tent map

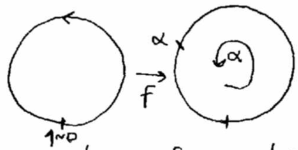


③ Interval exchange





f preserves measure:
Length(interval) = Length(f(interval))



Remark: Ergodic theory of measure preserving transformations

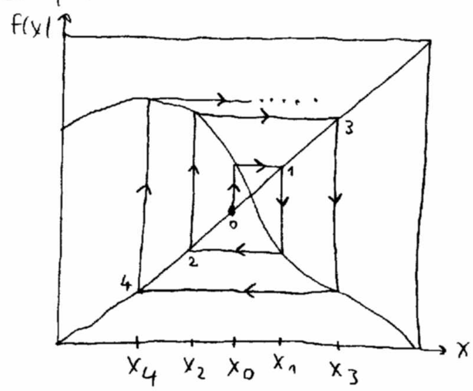
Let $T: M \to M$ be a measure preserving transformation (i.e. $\mu(T(A)) = \mu(A)$)

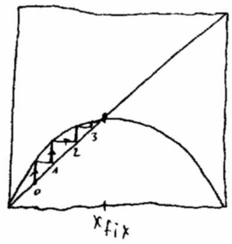
Assume that T is ergodic, i.e. $T^{-1}(B) = B$ implies either $\mu(B) = 0$ or 1.

Then $\int f(x)\, d\mu(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^k(x))$ for almost all $x \in M$. $(\mu(M) = 1)$

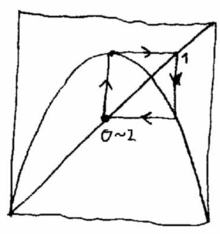
Remark: Hamiltonian dynamical systems are measure preserving.
But "energy conservation" ⟶ "not ergodic".

Graphical iteration:

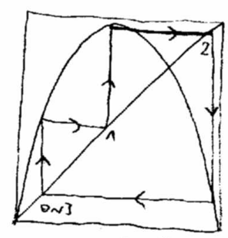




convergence to a fixed point



periodic orbit of period 2



period 3

Theorem (Sharkovskii): Let $f: [0,1] \to [0,1]$ be continuous.

Make this table:

3	5	7	9	..
2·3	2·5	[2·7]	2·9	...
$2^2 \cdot 3$	$2^2 \cdot 5$	$2^2 \cdot 7$	$2^2 \cdot 9$	
$2^3 \cdot 3$	$2^3 \cdot 5$	$2^3 \cdot 7$	$2^3 \cdot 9$	
⋮	⋮	⋮	⋮	

.... 2^n 4   2   1

Then if (for example) there is an orbit with period 2·7, there exist orbits with periods

[2·7]	2·9	...
$2^2 \cdot 7$	$2^2 \cdot 9$	...
$2^3 \cdot 7$	$2^3 \cdot 9$	...
⋮	⋮	

.... 8   4   2   1

Analysis of discrete dyn. sys.

① Find fixpoints: solve $\vec{f}(\vec{x}_{fix}) = \vec{x}_{fix}$

② Study the linearization around the fixedpoints:

$$\vec{F}(\vec{x}_{fix} + \vec{\Delta x}) \approx \vec{0} + Jac(\vec{x}_{fix}) \cdot \vec{\Delta x}, \quad Jac = \frac{\partial \vec{f}}{\partial \vec{x}}, \quad Jac_{ij} = \frac{\partial f_i}{\partial x_j}$$

Fixpoint's stability: compute the λ_i eigenvalues of $Jac(\vec{x}_{fix})$.

all $|\lambda_i| < 1 \rightarrow$ stable

exist $|\lambda_i| > 1 \rightarrow$ unstable

③ period two orbit: $\vec{f}(\vec{f}(\vec{x}_p)) = \vec{x}_p$. Try to repeat ①, ②.

three $\vec{f}^{\,3}(\vec{x}_p) = x_p$,

etc.

④ Measure Lyapunov exponent

⑤ ???

Example: (logistic map) $f(x) = r x (1-x)$, $r \in [0,4]$, $f: [0,1] \rightarrow [0,1]$

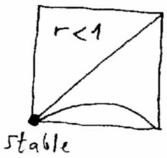
① Fixed point: $r x_{fix}(1 - x_{fix}) = x_{fix}$

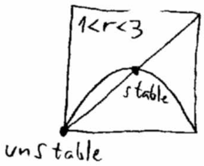
$$\longrightarrow x_{fix} = 0, \quad x_{fix} = 1 - \frac{1}{r}$$

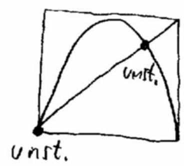
② Stability: $f'(x) = r(1-2x)$

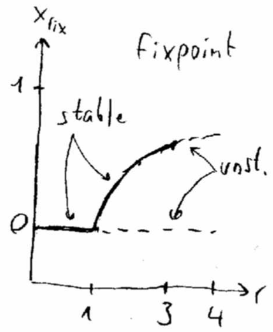
$|r \cdot (1 - 2 \cdot 0)| < 1$ $\qquad$ $\left| r\left(1 - 2\left(1 - \frac{1}{r}\right)\right)\right| < 1$

$0 \leq r < 1$ $\qquad$ $1 < r < 3$ $\qquad$ $3 < r < 4$





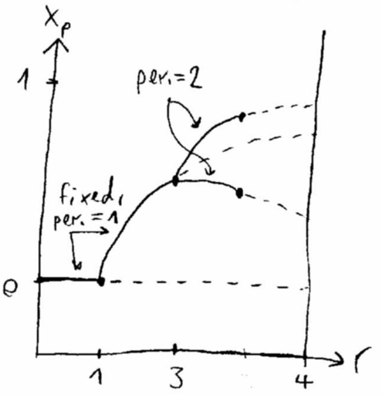




③ Period two: $f^2(x) = f(f(x)) = r[rx(1-x)][1 - rx(1-x)] = x$

Solutions: $x = 0$, $x = 1 - \frac{1}{r}$ (period 1 $\rightarrow$ period 2)

$$x = \frac{1 + r \pm \sqrt{-3 - 2r + r^2}}{2r} \quad \text{period 2, exists for } r > 3$$



$r = 0..1$ stable fixed point $x = 0$

$r = 0..3$ stable fixed point $x = 1 - \frac{1}{r}$

$r = 3..3.43$ stable period 2 cycle at $x = \frac{1 + r \pm \sqrt{-3 - 2r + r^2}}{2r}$

$r = 3.43...x$ stable period 2^2 cycle

2^3

⋮

Analysis of cont. dyn. sys.

$$\frac{d}{dt}\vec{x}(t) = \vec{f}(\vec{x}(t))$$

① Find fixedpoints: solve $\vec{f}(\vec{x}_{fix}) = \vec{0}$

② Study the linearized DE around the fixed points:

$$\vec{f}(x_{fix} + \vec{\Delta x}) \approx \vec{0} + \underbrace{Jac(\vec{x}_{fix})}_{A}\vec{\Delta x}, \quad Jac = \frac{\partial \vec{f}}{\partial \vec{x}}, \quad Jac_{ij} = \frac{\partial f_i}{\partial x_j}$$

$$\frac{d}{dt}\vec{\Delta x}(t) = A\vec{\Delta x}.$$

stability of the fixed point:

if A is diagonalizable and its eigenvalues are $\lambda_1, \ldots, \lambda_n$,

then: all $|Re\,\lambda_i| < 0 \longrightarrow$ stable

exists $|Re\,\lambda_i| > 0 \longrightarrow$ unstable

④ Measure Lyapunov exponent

ⓐ Solve DE with initial conditions: $\vec{x}(0)$, $\vec{x}(0) + \vec{\delta x}(0)$

solutions are $\vec{x}(t)$, $\tilde{\vec{x}}(t) = \vec{x}(t) + \vec{\delta x}(t)$

Lyapunov exponent: $\lambda = \lim\limits_{t\to\infty} \frac{1}{t} \lim\limits_{|\vec{\delta x}(0)|\to 0} \ln \frac{|\vec{\delta x}(t)|}{|\vec{\delta x}(0)|}$

Remark: λ can depend on $\vec{x}(0)$. Moreover, if $\vec{x}(0) + \vec{\delta x}(0) = \vec{x}(\varepsilon)$ for some ε, then $\vec{x}(t)$ and $\tilde{\vec{x}}(t)$ provides the same trajectories, so λ will be measured as 0 in most of the cases.

ⓑ A more accurate way to measure the Lyapunov exponent:

① Solve DE with init cond $\vec{x}(0) = \vec{x}_0$, solution: $\vec{x}_p(t)$

② Find the linearized DE around the trajectory $\vec{x}_p(t)$:

$$\frac{d}{dt}\left(\vec{x}_p(t) + \vec{\delta x}(t)\right) = \vec{f}(\vec{x}_p(t) + \vec{\delta x}(t)) \approx \vec{f}(\vec{x}_p(t)) + Jac(\vec{x}_p(t)) \cdot \vec{\delta x}(t)$$

$$\longrightarrow \frac{d}{dt}\vec{\delta x}(t) = Jac(\vec{x}_p(t))\vec{\delta x}(t)$$

③ Solve: $\frac{d}{dt}U(t) = Jac(\vec{x}_p(t))\,U(t)$, $U(0) = E$ ← identity matrix

Then $\lambda = \lim\limits_{t\to\infty} \frac{1}{t} \|U(t)\|$ ← norm $\|B\| = \max\limits_{|\vec{v}|=1} |B\vec{v}| = \left[\text{maximal eigenvalue of } (B^T B)\right]^{1/2}$

Lyapunov exponent (cont. dynamics)

6 dyn

$\frac{d}{dt}\vec{x}_p(t) = \vec{f}(\vec{x}_p(t))$, $\frac{d}{dt}(\vec{x}_p(t) + \vec{\delta x}(t)) = \vec{f}(\vec{x}_p(t) + \vec{\delta x}(t))$

Solve it for a given $\vec{x}_p(0)$. ⟶ Solve it for $\vec{\delta x}(t)$ for a given very small $\vec{\delta x}(0)$.

If $\|\vec{\delta x}(t)\| \sim e^{\lambda t}$, $\lambda = \lim_{t\to\infty} \frac{1}{t} \lim_{\|\vec{\delta x}(0)\|\to 0} \log \|\vec{\delta x}(t)\|$ ⟵ Lyapunov expomet.

Exercise 1: Lorenz equation

$$\frac{d}{dt}\vec{S} = \frac{d}{dt}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \sigma(y-x) \\ x(\rho - z) - y \\ xy - \beta z \end{pmatrix}, \quad \sigma, \rho, \beta = 10,\ 28,\ 8/3$$

(gives nice chaotic behaviour)

Try to estimate the Lyapunov expont for this system!

(a) Pick an $\vec{S}_{01}$ then a small $\vec{\delta S}_{01}$ solve the DE with inidial conditions: $\vec{S}(0) = \vec{S}_0$, $\tilde{\vec{S}}(0) = \vec{S}_0 + \vec{\delta S}_0$.
Plot $\|\vec{S}(t) - \tilde{\vec{S}}(t)\|$, and find the range of t, where this difference is small.
In that region plot $\log\|\vec{S}(t) - \tilde{\vec{S}}(t)\|$, then estimate the straight line approximation's slope (linear regression). ⟵ Lyapunov expoment

(b) Study the sensitivity of your result on the choices of $\vec{S}_0$ and $\delta\vec{S}_0$

Exercise 2: Repeat Ex ① for the anharmonic oscillator:

$$\frac{d}{dt}\vec{S} = \frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ -x_1 + [(x_2 - x_3) + (x_2 - x_3)^3] \\ -x_2 - [(x_2 - x_3 + (x_2 - x_3)^3] \end{bmatrix}$$

Here you should find that λ is dependent on the inidial condition $\vec{S}(0)$.