Complex functions $\mathbb{C} \to \mathbb{C}$

Motivation: Inverse Laplace tr: $f(t) = \frac{1}{2\pi i} \int_{-i\infty+a}^{i\infty+a} F(s)\, e^{st}\, ds$

Green function: $G'(t) + 3G(t) = \delta(t)$, $\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{ip+3} e^{ipt}\, dp = G(t)$

Hard to understand these without complex analysis.

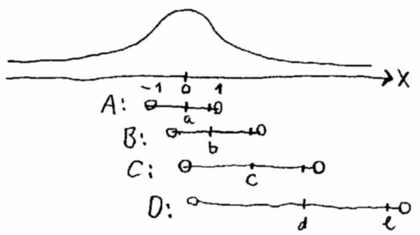
① Real analytic functions $\to$ Taylor series $\to$ Complex functions.

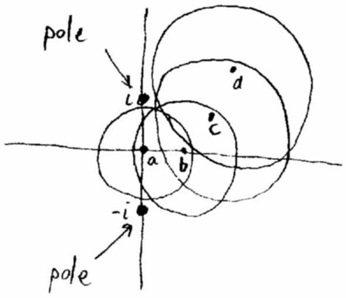
ⓐ $\exp: \mathbb{R} \to \mathbb{R}$, $\exp(x) = 1 + x + \frac{x^2}{2!} + \dots \to \exp: \mathbb{C} \to \mathbb{C}$, $e^x = 1 + x + \frac{x^2}{2!} + \dots$

∞ radius of convergence $\to e^x$ complex analytic on $\mathbb{C}$

ⓑ $f(x) = \frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \dots$ radius of convergence $= 1$. Why?

$$\frac{1}{1+x^2} = -\frac{i}{2} \frac{1}{x-i} + \frac{i}{2} \frac{1}{x+i}$$

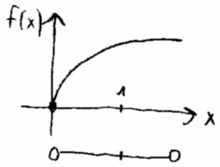


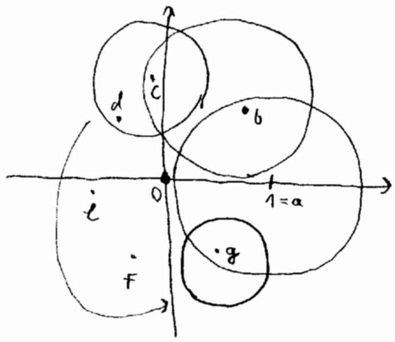


extends f to $\mathbb{C} \setminus \{\pm i\}$

Taylor series defines $f(x)$ on $A = (-1, 1)$, pick b, compute Taylor series around it, now f is defined on $A \cup B$, etc.
(Analytic continuation)

ⓒ $f(x) = \sqrt{x} = 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \dots$





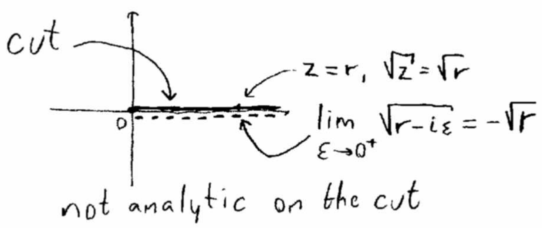
Analytic cont. produces two different values for $\sqrt{z}$.

Taylor series around a:
$z = r e^{i\varphi} \to \sqrt{z} = \sqrt{r} \cdot e^{i\varphi/2}$

around g:
$\to \sqrt{z} = \sqrt{r} \cdot e^{i\left(\frac{\varphi}{2} + \frac{2\pi}{2}\right)} = -\sqrt{r} \cdot e^{i\varphi/2}$

① Keep only one: $\varphi \in [0, 2\pi)$

$$\sqrt{z} = \sqrt{r} \cdot e^{i\varphi/2}$$

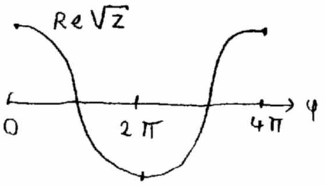


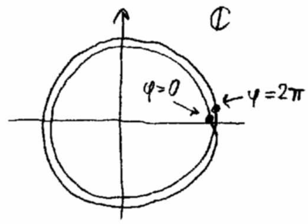
Remark: Better to use $(-\infty, 0]$ for the cut.

(ii) Double the complex plane.

$\varphi \in [0, 2\cdot 2\pi) = [0, 4\pi)$

$\sqrt{z} = \sqrt{r e^{i\varphi}} = \sqrt{r} \cdot e^{i\varphi/2}$





$\varphi = 0, 2\pi, 4\pi$ are the same points on the original complex plane.

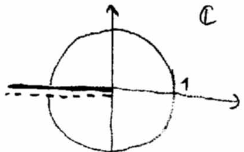
$\varphi = 0, 4\pi$ are the same points on the doubled complex plane

Advantage: Now $\sqrt{z}$ is analytic except at $z=0$.

disadvantage: Need to learn Riemann surface theory.

(d) logarithm: $\log z = \log(r \cdot e^{i\varphi}) = \log(r) + i\varphi$

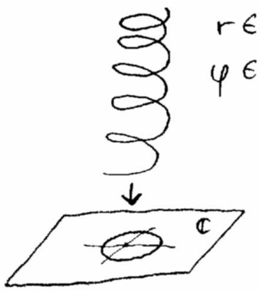
(i) $\varphi \in (-\pi, \pi]$, $\mathrm{Log}(z)$



$\mathrm{Log}(-1) = i\pi$

$\lim_{\varepsilon \to 0^+} \mathrm{Log}(-1 - i\varepsilon) = -i\pi$

(ii)



$r \in (0, \infty)$

$\varphi \in (-\infty, \infty)$, $\log(r \cdot e^{i\varphi}) = \log(r) + i\varphi$

infinite covering of $\mathbb{C} \setminus \{0\}$

Derivation

Real analysis: $C^0 \supset C^1 \supset C^2 \ldots \supset C^n \supset \ldots \supset C^\infty \supset$ Analytic functs.

C^0: continuous; C^2: twice differentiable; C^∞: smooth

Complex analysis: $C^1 :=$ Holomorf $=$ Analytic

Linear approximation:

$f: \mathbb{C} \to \mathbb{C}$, $\quad f(z+\Delta z) \approx f(z) + f'(z)\Delta z$

$f(z) = f(x+iy) = u(x,y) + i v(x,y)$, $\quad x, y, u, v$ are real

$$\Delta_1 f = f(z+(\delta+0i)) - f(z) = f((x+\delta)+iy) - f(x+iy) \approx \left[u'_x(x,y) + i v'_x(x,y)\right]\cdot\delta$$

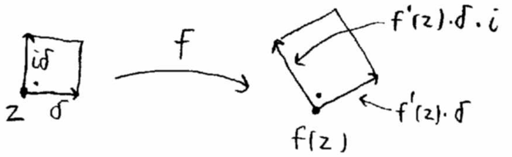
$$\Delta_2 f = f(z+(0+\delta i)) - f(z) = f(x+i(y+\delta)) - f(x+iy) \approx \left[u'_y(x,y) + i v'_y(x,y)\right]\cdot\delta$$

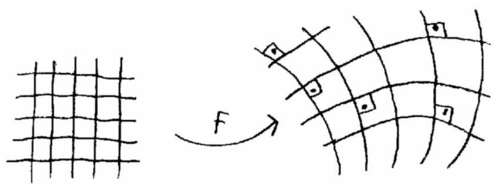
$$\underbrace{\Delta_2 f}_{\Delta z = i\cdot\delta} = i \underbrace{\Delta_1 f}_{\Delta z = \delta} \qquad i(u'_x + i v'_x) = (u'_y + i v'_y)$$

$$u'_x = v'_y, \quad -v'_x = u'_y$$

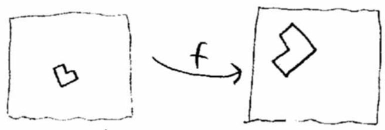
$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}$$

Cauchy-Riemann equations (CR)





Conformal mapping:



small shapes are approximately preserved, since $f'(z)\cdot \sim$ rotation + rescaling

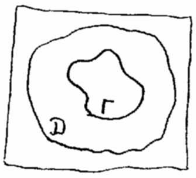
Remark: CR equation $\longrightarrow$ Laplace eq. for u, v

$$\frac{\partial^2 u}{\partial^2 x} + \frac{\partial^2 u}{\partial^2 y} = \frac{\partial}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial}{\partial y}\left(-\frac{\partial v}{\partial x}\right) = v_{xy} - v_{yx} = 0 \qquad \Delta u = 0$$

$$\frac{\partial^2 v}{\partial^2 x} + \frac{\partial^2 v}{\partial^2 y} = \frac{\partial}{\partial x}\left(-\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = -u_{xy} + u_{xy} = 0 \qquad \Delta v = 0$$

Cauchy integral formula

Theorem:



f holomorph in D, D open and simply connected, the closed curve Γ is inside D,

then $\oint_\Gamma f(z)dz = 0$.

Remark: Similar to: $\operatorname{rot}\vec{V}(\vec{r}) = 0 \longleftrightarrow \vec{V} = \operatorname{grad}\varphi \longrightarrow \oint_\Gamma \vec{V}(\vec{r}(t))\, d\vec{r}(t) = 0$

Proof: Γ: $\quad \gamma(t) = \gamma_1(t) + \gamma_2(t)\cdot i, \qquad \vec{r}(t) = (\gamma_1(t), \gamma_2(t))$

$$\oint_\Gamma f(z)dz = \oint dt\ f(\gamma_1(t)+\gamma_2(t)\cdot i)\cdot(\dot\gamma_1(t)+\dot\gamma_2(t)\cdot i)$$

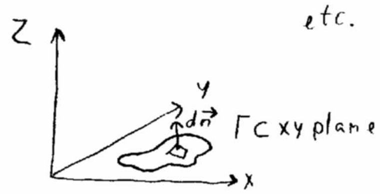
$$= \oint dt \Big[U(\gamma_1(t),\gamma_2(t))\cdot\dot\gamma_1(t) - V(\gamma_1(t),\gamma_2(t))\cdot\dot\gamma_2(t) \Big]$$

$$+ i \Big[U(\gamma_1(t),\gamma_2(t))\cdot\dot\gamma_2(t) + V(\gamma_1(t),\gamma_2(t))\cdot\dot\gamma_1(t) \Big]$$

$$= \oint dt \Big([U, -V, 0](\vec{r}(t)) \Big) \cdot \Big([\dot\gamma_1(t), \dot\gamma_2(t), 0] \Big)$$

$$+ i \oint dt \Big([V, U, 0](\vec{r}(t)) \Big) \cdot \Big([\dot\gamma_1(t), \dot\gamma_2(t), 0] \Big)$$

here $U(x,y) \longrightarrow U(x,y,z) = U(x,y)$, etc.



$$= \iint d\vec{n} \cdot \operatorname{rot}[U,-V,0] + i \iint d\vec{n} \cdot \operatorname{rot}[V,U,0]$$

$$= \iint dn\cdot[0,0,1]\cdot\Big[0, 0, \underbrace{\tfrac{\partial}{\partial x}(-V) - \tfrac{\partial}{\partial y}U}_{=0,\ \text{since } -V'_x = U'_y}\Big] + i \iint dn[0,0,1]\cdot\Big[0,0,\underbrace{\tfrac{\partial}{\partial x}U - \tfrac{\partial}{\partial y}V}_{=0,\ \text{since } U_x = V_y}\Big]$$

Stokes: $\oint \vec{V}(\vec{r}(t))\, d\vec{r}(t) =$
$= \iint \operatorname{rot}\vec{V}\, d\vec{n}$

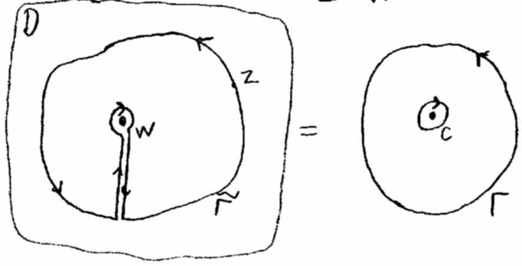
Remark: alternative proof:

1. Check for $f(z) = \alpha z + \beta$, Γ = triangle

2. 

$$\oint f(z)dz \approx \sum_\Delta \underbrace{\oint f(z)dz}_{\substack{\text{zero for linear } f(z),\\ \approx 0 \text{ if } f(z) \text{ can be approximated}\\ \text{by } \alpha z+\beta,\ \text{i.e. } f'(z) \text{ exists}}} \to 0$$

Apply this for $\dfrac{f(z)}{z-w}$: $\dfrac{f(z)}{z-w}$ holomorph inside the modified $\tilde{\Gamma}$ curve $\boxed{5_{\text{Comp}}}$



$$=0=\oint_\Gamma \frac{f(z)}{z-w}dz+\underbrace{\oint_C \frac{f(z)}{z-w}dz}$$

$z=\varepsilon\cdot e^{-it}$

C: circle of radius ε around w, $\varepsilon \to 0^+$

$$\approx f(w)\cdot\int_0^{2\pi}\frac{1}{\varepsilon\cdot e^{-it}}\cdot\frac{d(\varepsilon\cdot e^{-it})}{dt}\,dt$$

$$=-2\pi i\cdot f(w)$$

So $\boxed{\dfrac{1}{2\pi i}\oint_\Gamma \dfrac{f(z)}{z-w}dz=f(w)}$

Remark: This <u>does not</u> mean that given an "arbitrary" function $f(z)$ on the curve Γ we will get a holomorph function $f(w)$ for the inside points w of Γ, such that its value on Γ is given by $f(z)$

We can compute $f'(w)$: etc.

$$f'(w)=\frac{\partial}{\partial w}\left(\frac{1}{2\pi i}\oint_\Gamma \frac{f(z)}{(z-w)^2}dz\right)$$

$$f''(w)=\frac{\partial}{\partial w}f'(w)=2\cdot\frac{1}{2\pi i}\oint_\Gamma\frac{f(z)}{(z-w)^3}dz$$

$$\vdots$$

$$f^{(n)}(w)=\frac{n!}{2\pi i}\oint_\Gamma\frac{f(z)}{(z-w)^{n+1}}dz$$

$$\vdots$$

So f is infinitely differentiable, and in fact analytic,

since $f(w+a)=\dfrac{1}{2\pi i}\oint_\Gamma\dfrac{f(z)}{z-(w+a)}dz=\sum_{n=0}^{\infty}\dfrac{n!}{2\pi i}\dfrac{f(z)}{(z-w)^{n+1}}\cdot\dfrac{a^n}{n!}$

as $\dfrac{1}{(z-w)-a}=\sum_{n=0}^{\infty}\dfrac{a^n}{(z-w)^{n+1}}$.

So holomorph $\Rightarrow$ analytic

Green function

$\boxed{6 \text{ comp.}}$

Example: $G'(t) + 3G(t) = \delta(t)$

Retarded Green: $G_R(t) = \begin{cases} 0, & \text{if } t<0 \\ e^{-3t}, & \text{if } t>0 \end{cases}$, Advanced Green: $G_A(t) = \begin{cases} -e^{-3t}, & \text{if } t<0 \\ 0, & \text{if } t>0 \end{cases}$

Solution by Fourier transform:

$$\hat{f}(p) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ipt} f(t)\,dt \qquad f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(p) e^{ipt}\,dp$$

$$\hat{\delta}(p) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ipt} \delta(t)\,dt = \frac{1}{\sqrt{2\pi}}, \qquad \delta(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ipt}\,dp$$

potential problem: $\hat{\delta}(p) = \left\langle \frac{e^{-ipt}}{\sqrt{2\pi}}, \delta(t) \right\rangle$, $\frac{e^{-ipt}}{\sqrt{2\pi}}$ is not a test function

smooth, zero outside of a finite interval

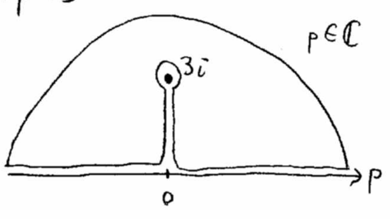
$$G(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{G}(p) e^{ipt}\,dp,$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{G}(p) \cdot ip \cdot e^{ipt}\,dp + 3 \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{G}(p) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ipt}\,dp$$

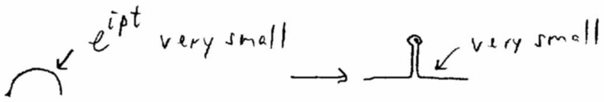
$$\hat{G}(p) = \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{ip+3}, \qquad G(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{ipt}}{ip+3}\,dp$$

What is $G(t)$? G_A or G_R? Answer: depends on the specification of the improper integral.

$\frac{e^{ipt}}{ip+3}$ holomorph if $p \neq 3i$

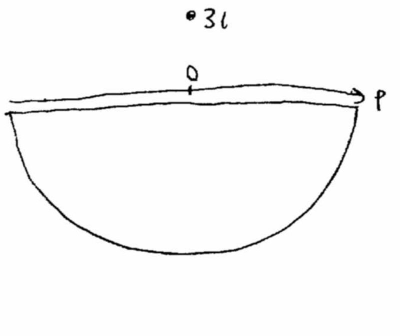


$t>0$

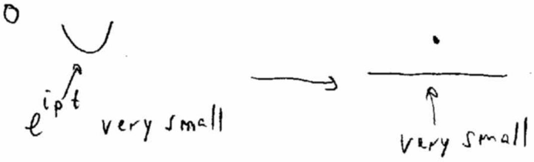


$G(t)=0$ if $t>0$

advanced Green function



$t<0$



$G(t)=0$ if $t<0$

retarded Green