Hullámegyenlet

$$\left(\partial_t^2 - \partial_x^2\right) \varphi(x,t) = 0$$

① Síkhullám: $\varphi = e^{i(kx-\omega t)} \longrightarrow |k| = |\omega| \longrightarrow$ sebesség $= \pm 1$

② Haladóhullám: $\varphi(x,t) = f(x-vt) \longrightarrow v^2 f'' - f'' = 0 \longrightarrow v^2 = 1 \longrightarrow v = \pm 1$

③ $\left(\partial_t^2 - \partial_x^2\right) = (\partial_t + \partial_x)(\partial_t - \partial_x)$, $\left.\begin{matrix}(\partial_t+\partial_x)\varphi = 0 \\ \text{OR} \\ (\partial_t - \partial_x)\varphi = 0\end{matrix}\right\} \rightarrow (\partial_t^2 - \partial_x^2)\varphi = 0$, $\varphi = f(x-t) + g(x+t)$

④ Kezdeti-érték probléma:

$(\partial_t^2 - \partial_x^2)\varphi = 0$, $\varphi(0,x) = F(x)$, $\varphi_t'(0,x) = G(x)$ $\quad$ ($F(x) = G(x) = 0$, ha $|x| \gg 1$)

Feladat. Mennyi f, g, ha $\varphi(t,x) = f(x-t) + g(x+t)$?

Megoldás:

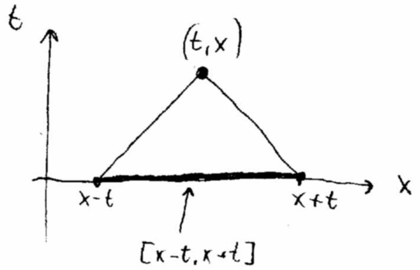
$$\left.\begin{aligned} f(x) + g(x) &= F(x) \\ -f'(x) + g'(x) &= G(x) \\ -f(x) + g(x) &= \int_{-\infty}^{x} G(y)\,dy \end{aligned}\right\} \rightarrow \begin{aligned} g(x) &= \frac{1}{2}\left[F(x) + \int_{-\infty}^{x} G(y)\,dy\right] \\ f(x) &= \frac{1}{2}\left[F(x) - \int_{-\infty}^{x} G(y)\,dy\right] \end{aligned}$$

f, g megoldás $\rightarrow f+c, g-c$ is megoldás

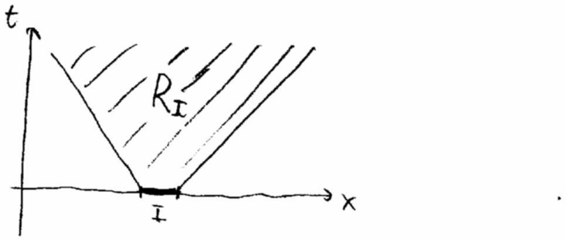
Tehát:

$$\varphi(t,x) = f(x-t) + g(x-t) = \frac{1}{2}\left[F(x-t) - \int_{-\infty}^{x-t} G(y)\,dy\right] + \frac{1}{2}\left[F(x+t) + \int_{-\infty}^{x+t} G(y)\,dy\right]$$

$$= \frac{1}{2}\left[F(x-t) + F(x+t)\right] + \int_{x-t}^{x+t} G(y)\,dy = \frac{1}{2}\left[\varphi(0,x-t) + \varphi(0,x+t)\right] + \frac{1}{2}\int_{x-t}^{x+t} \dot\varphi(0,y)\,dy$$



$\varphi(t,x)$ csak az $I = [x-t, x+t]$ intervallumon levő $\varphi, \dot\varphi$ adatoktól függ.

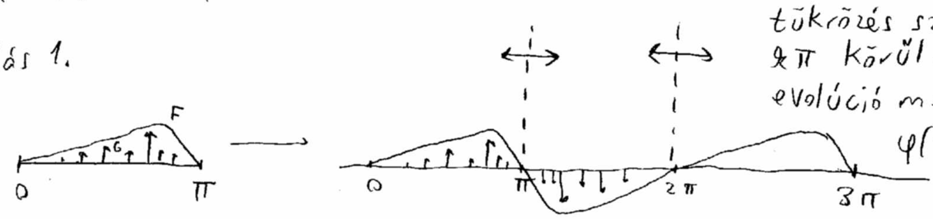


A kezdeti feltétel I-n csak az R_I régióban levő pontokmál befolyásolja φ értékét.

Kifeszített húr rezgése

$$\partial_t^2\varphi - \partial_x^2\varphi = 0, \quad \varphi(0,x) = F(x), \quad \dot\varphi(0,x) = G(x), \quad \varphi(t,0) = \varphi(t,\pi) = 0$$

Megoldás 1.



tükrözés szimmetria $k\pi$ körül.
evolúció megőrzi a $\varphi(t,k\pi)=0$ feltételt

Megoldás 2. Fourier sor (szinusz tr.)

$$\varphi(t,x) = f_n(t)\cdot\sin(nx) \longrightarrow \ddot f_n(t) = -n^2 f_n(t) \longrightarrow f_n(t) = c_n\cdot\cos(nt) + s_n\cdot\sin(nt)$$

Szinusz tr.: $F(x) = \sum_{n=1}^{\infty} \hat F_n \sqrt{\frac{2}{\pi}}\sin(nx)$, $\hat F_n = \left(\sqrt{\frac{2}{\pi}}\sin(nx), F(x)\right) = \int_0^\pi \sqrt{\frac{2}{\pi}}\sin(nx) F(x)\,dx$

$G(x) = \sum_{n=1}^{\infty} \hat G_n \sqrt{\frac{2}{\pi}}\sin(nx)$, $\hat G_n = \left(\sqrt{\frac{2}{\pi}}\sin(nx), G(x)\right) = \int_0^\pi \sqrt{\frac{2}{\pi}}\sin(nx) G(x)\,dx$

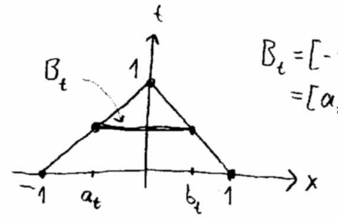
Ha $\varphi(t,x) = \sum_{n=1}^{\infty}\left[c_n\cdot\cos(nt) + s_n\cdot\sin(nt)\right]\cdot\sqrt{\frac{2}{\pi}}\sin(nx)$

$\dot\varphi(t,x) = \sum_{n=1}^{\infty}\left[-nc_n\sin(nt) + ns_n\cos(nt)\right]\cdot\sqrt{\frac{2}{\pi}}\cos(nx)$, akkor

$c_n = \hat F_n$, $s_n = \frac{\hat G_n}{n}$, tehát $\varphi(t,x) = \sum_{n=1}^{\infty}\left(\hat F_n\cos(nt) + \frac{\hat G_n}{n}\sin(nt)\right)\sqrt{\frac{2}{\pi}}\sin(nx)$

Terjedési sebesség

$$\varphi(t,x)=\frac{1}{2}\left[\varphi(0,x-t)+\varphi(0,x+t)+\int_{x-t}^{x+t}\dot\varphi(0,y)\,dy\right],$$ így $\varphi(t,x)$ csak $\varphi(0,x),\dot\varphi(0,x)$ -nek az $[x-t,x+t]$ intervallumban felvett értékeitől függ.



$B_t=[-1+t,1-t]$
$=[a_t,b_t]$

Tétel: Legyen $\varphi(0,x)=\dot\varphi(0,x)$, ha $x\in[-1,1]$. Ekkor $\varphi(t,x)=0$, ha $x\in B_t$.

Bizonyítás:

Legyen $E(t)=\frac{1}{2}\int_{B_t}\varphi_t^2+\varphi_x^2\,dx$. Ekkor

$$\frac{d}{dt}E(t)=\underbrace{\int_{B_t}\varphi_t\cdot\varphi_{tt}+\varphi_x\cdot\varphi_{xt}\,dx}-\frac{1}{2}\Big[\big(\varphi_t^2(t,a_t)+\varphi_x^2(t,a_t)\big)+\big(\varphi_t^2(t,b_t)+\varphi_x^2(t,b_t)\big)\Big]$$

$$\partial_x(\varphi_x\varphi_t)=\varphi_x\cdot\varphi_{xt}+\varphi_{xx}\varphi_t$$

$$\int_{B_t}\varphi_t\cdot\underbrace{(\varphi_{tt}-\varphi_{xx})}_{=0}+\partial_x(\varphi_x\varphi_t)\,dx=\int_{B_t}\varphi_t\cdot(\varphi_{tt}-\varphi_{xx})\,dx+\big[\varphi_x(t,b_t)\cdot\varphi_t(t,b_t)-$$
$$-\varphi_x(t,a_t)\cdot\varphi_t(t,a_t)\big]$$

$$\frac{d}{dt}E(t)=-\frac{1}{2}\big(\varphi_t^2(t,a_t)+\varphi_x^2(t,a_t)\big)+\big[\varphi_x(t,b_t)\varphi_t(t,b_t)-\varphi_x(t,a_t)\varphi_t(t,a_t)\big]-\frac{1}{2}\big(\varphi_t^2(t,b_t)+\varphi_x^2(t,b_t)\big)$$
$$=-\frac{1}{2}\Big[\big(\varphi_t(t,a_t)-\varphi_x(t,a_t)\big)^2+\big(\varphi_t(t,b_t)+\varphi_x(t,b_t)\big)^2\Big]\le 0.$$

Mivel $E(t)\ge 0$, $E(0)=0$, így $E(t)=0$.

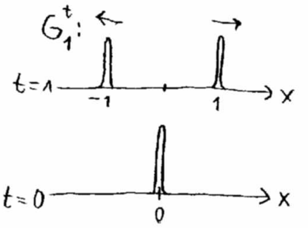
Megjegyzés: $\rho=(\varphi_t^2+\varphi_x^2)\cdot\frac{1}{2}$: energy density, $j=-\varphi_x\varphi_t$ energy density current

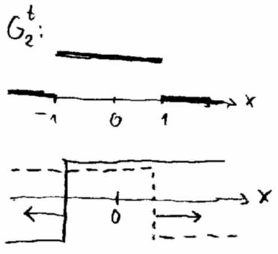
$$\frac{\partial\rho}{\partial t}+\operatorname{div}j=0.=(\varphi_{tt}\cdot\varphi_t+\varphi_{xt}\cdot\varphi_x)-\partial_x(\varphi_x\varphi_t)=\underbrace{(\varphi_{tt}\cdot\varphi_t+\varphi_{xt}\cdot\varphi_x)-(\varphi_{xx}\varphi_t+\varphi_x\varphi_{xt})}\to\varphi_t\underbrace{(\varphi_{tt}-\varphi_{xx})}_{=0}=0$$

Kontinuitási egyenlet $\longrightarrow \dfrac{d}{dt}\displaystyle\int_{-\infty}^{\infty}\rho(t,x)\,dx=0$

Megjegyzés: $\varphi(0,x)=F(x)$, $\dot\varphi(0,x)=V(x)$, $\varphi(t,x)=(G_1^t*F)(x)+(G_2^t*V)(x)$,

ahol $G_1^t(x)=\frac{1}{2}\big(\delta(x+t)+\delta(x-t)\big)$, $G_2^t(x)=\frac{1}{2}\chi_{[-t,t]}(x)$



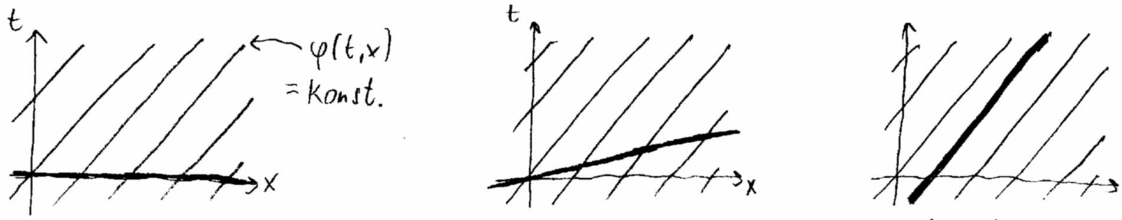


Karakterisztikus függvény: $\chi(x)=1$, ha $x\in[-t,t]$, 0 amúgy

Karakterisztika

Hol specifikálható a kezdeti feltétele egy PDE-nek?

Példa: transzport egyenlet: $\partial_t \varphi(t,x) + \partial_x \varphi(t,x) = 0 \longrightarrow \varphi(t,x) = f(x-t)$



$\varphi(0,x) = f_1(x)$ O.K. $\qquad \varphi(\tfrac{1}{3}x, x) = f_2(x)$ O.k. $\qquad \varphi(x,x) = f_3(x)$ Nem jó, mivel $\varphi(x,x) = f(0) \neq f_3(x)$

Hogyan olvasható ez le a PDE-ből?

L: lin. diff. op: $L = \sum_{|\alpha| \leq k} a_\alpha \partial^\alpha \longrightarrow$ karakterisztikus forma:

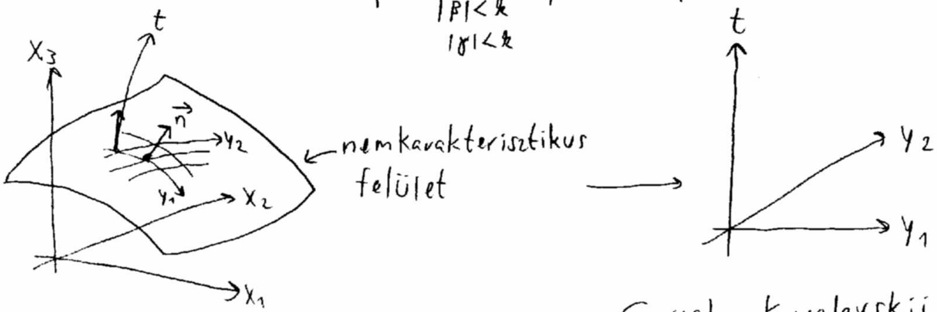
$$\chi_L(\vec{x}, \vec{p}) = \sum_{|\alpha| = k} a_\alpha(\vec{x}) \, \vec{p}^{\,\alpha}$$

(multiindex jelölés: $\partial^{(1,2,5)} = \partial_{x_1}^1 \partial_{x_2}^2 \partial_{x_3}^5$, $\underbrace{(7,6,8)}_{\vec{p}}{}^{\overbrace{(1,2,5)}^{\alpha}} = 7^1 \cdot 6^2 \cdot 8^5$)

$\vec{p}$ karakterisztikus vektora L-nek az $\vec{x}$ pontban, ha $\chi_L(\vec{x}, \vec{p}) = 0$.

Egy (hiper)felület karakterisztikus, ha a normálvektora minden pontban karakterisztikus vektor. Ha pedig sehol sem a felületen, akkor a felület nem karakterisztikus.

Motiváció: $$L\varphi + \sum_{\substack{|\beta| < k \\ |\gamma| < k}} a_\beta(\vec{x}, \partial^\gamma \varphi) = 0$$



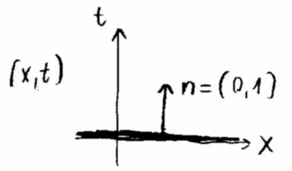
$$\partial_t^k \psi = b(t, \vec{y}, \partial_{\vec{y}}^\beta \partial_t^l \psi),$$

ahol $l < k$, $|\beta| + l \leq k$

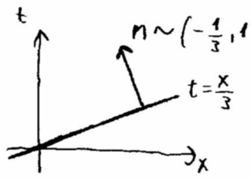
Cauchy-Kowalevskii: b, $\psi^l(0, \vec{y})$, $l < k$ analitikus $\longrightarrow$ lokálisan létezik analitikus megoldás

Példák:

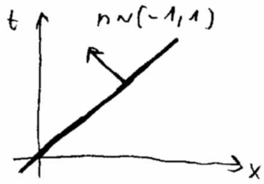
(1) $(\partial_t + \partial_x)\varphi = 0$, $L = 1\cdot\partial_t + 1\cdot\partial_x$, $\chi_L(\underbrace{(x,t)}_{\vec{x}}, \underbrace{(p_1,p_2)}_{\vec{p}}) = 1\cdot p_1 + 1\cdot p_2$



$\chi_L((x,t),(0,1)) = 1\cdot 0 + 1\cdot 1 \neq 0$
nemkarakterisztikus felület



$\chi_L((x,t),(-\frac{1}{3},1)) =$
$= 1\cdot(-\frac{1}{3}) + 1\cdot 1 \neq 0$

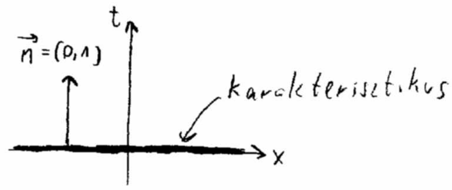


$\chi_L((x,t),(-1,1)) = 1\cdot(-1) + 1\cdot 1 = 0$
karakterisztikus felület, nem biztos, hogy a kezdeti feltétel megadható rajta

(2) $\partial_t\varphi = \partial_{xx}\varphi$. $\partial_{xx}\varphi = \partial_t\varphi$, $L = [\partial_x, \partial_t]\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}\partial_x\\ \partial_t\end{bmatrix}$, $\chi_L((x,t),(p_1,p_2)) = [p_1,p_2]\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}p_1\\ p_2\end{bmatrix}$

(p_1,p_2) karakterisztikus $\longleftrightarrow p_1^2 = 0$, $\vec{p} = (0, p_2)$

karakterisztikus felület: $\vec{n} \sim (0,1)$

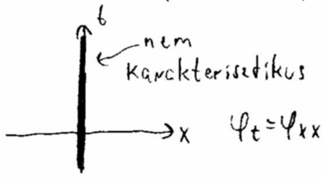


Probléma lehet, ha φ és φ_t-t a $t=0$ karakterisztikus felületen adjuk meg.

(1) ha $\varphi(x,0)$ adott, akkor $\dot{\varphi}(x,0) = \partial_{xx}\varphi(x,0)$, tehát nem adható meg tetszőlegesen.

(2) viszont tipikusan $\varphi(x,0)$ adott, a PDE megoldása csak $t>0$-ra létezik, $\longrightarrow$ nem analitikus t-ben.

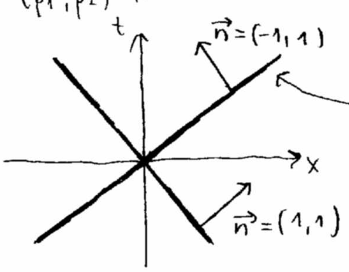
(3)



A (lokálisan) analitikusan megoldható $\varphi_t = \varphi_{xx}$, $\varphi(0,t) = f(t)$, $\varphi_x(0,t) = g(t)$ kezdeti feltétel szinte sosem fordul elő.

(3) $\partial_{tt}\varphi = \partial_{xx}\varphi$, $(\partial_{xx} - \partial_{tt})\varphi = 0$, $L = [\partial_x, \partial_t]\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}\partial_x\\ \partial_t\end{bmatrix}$, $\chi_L((x,t),(p_1,p_2)) = [p_1,p_2]\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}p_1\\ p_2\end{bmatrix}$

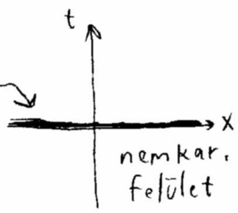
(p_1,p_2) karakterisztikus $\longleftrightarrow p_1^2 - p_2^2 = 0 \longleftrightarrow |p_1| = |p_2|$



karakterisztikus felületek

illegális kezdeti feltétel:
$\varphi(x,x) = f(x)$, $-1\cdot\varphi_x'(x,x) + 1\cdot\varphi_t'(x,x) = g(x)$

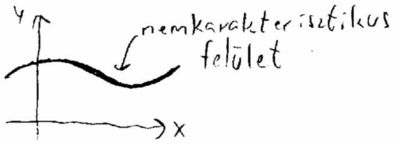
legális kezdeti feltétel:
$\varphi(x,0) = f(x)$, $\varphi_t'(x,0) = g(x)$



(4) $(\partial_{xx}+\partial_{yy})\varphi(x,y)=0$

$L = [\partial_x \ \partial_y]\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}\partial_x\\ \partial_y\end{bmatrix}$ $\quad \chi_L((x,y),(p_1,p_2)) = [p_1 \ p_2]\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}p_1\\ p_2\end{bmatrix} = p_1^2+p_2^2$

$p_1^2+p_2^2>0$, ha $\vec{p}\neq\vec{0}$, nincs nemtriviális karakterisztikus vektor, bármely felület nemkarakterisztikus.



Megjegyzés:

$\partial_t - \partial_{xx}$ parabolikus

$\partial_t^2 - \partial_x^2$ hiperbolikus operátorok

$\partial_x^2 + \partial_y^2$ elliptikus

ezek tipikus előfordulása:

parabolikus } → evolúciós egyenletek ∞ / véges terjedési sebesség
hiperbolikus }

elliptikus : statikai problémák

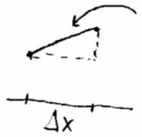
Variációszámítás

Példa:

erőegyensúly: $\varphi''(x) = -f(x)$

$\varphi(0) = \varphi(1) = 0$

hossznövekedés:



$\sqrt{\Delta x^2 + \varphi'^2 \Delta x^2} - \sqrt{\Delta x} \approx \Delta x \cdot \frac{1}{2}(\varphi')^2$,

ha $\varphi' \approx 0$. (hiszen $\sqrt{1+x} \approx 1 + \frac{1}{2}x$)

Energiaváltozás:

$$E[\varphi] = \int_0^1 \frac{1}{2}\left(\frac{d\varphi}{dx}\right)^2 - f(x)\varphi(x)\, dx$$

Két ekvivalens feladat:

① Oldd meg: $\varphi''(x) = -f(x)$, $\varphi(0) = \varphi(1) = 0$

② Minimalizáld $E[\varphi] = \int_0^1 \frac{1}{2}\left(\frac{d\varphi}{dx}\right)^2 - f(x)\varphi(x)\, dx$ -t a $\varphi(0) = \varphi(1) = 0$ feltétel mellett

Minimalizáció: (1 dim) ① kritikus pontok keresése: $h'(x) = 0$

② kritikus pontok típusa: $h''(x) < 0$ (max), $h''(x) > 0$ (min)

∞ dim:

$$S[\varphi] = \int_0^1 L(\varphi(x), \varphi'(x), x)\, dx, \quad \varphi(0) = a, \quad \varphi(1) = b$$

$\varphi_c(x)$ kritikus pont: $S[\varphi_c + \delta\varphi] \approx S[\varphi_c]$ elsőrendben $\delta\varphi$-ben.

$$S[\varphi_c + \delta\varphi] = \int_0^1 L(\varphi_c + \delta\varphi, \varphi_c' + \delta\varphi', x)\, dx \approx S[\varphi_c] + \int_0^1 \delta\varphi(x) \cdot \frac{\partial L}{\partial \varphi}\bigg|_{\varphi=\varphi_c} + \delta\varphi'(x) \cdot \frac{\partial L}{\partial \varphi'}\bigg|_{\varphi=\varphi_c} dx$$

$$= S[\varphi_c] + \int_0^1 \left(\frac{\partial L}{\partial \varphi} - \frac{d}{dx}\frac{\partial L}{\partial \varphi'}\right)\bigg|_{\varphi=\varphi_c} dx + \underbrace{\underbrace{\delta\varphi(x)}_{=0,\text{ ha } x=0 \text{ vagy } 1} \cdot \frac{\partial L}{\partial \varphi'}\bigg|_0^1}_{=0}$$

Tehát: $\varphi_c(x)$ kritikus pont $\longrightarrow$ φ_c kielégíti a $\frac{d}{dx}\frac{\partial L}{\partial \varphi'} - \frac{\partial L}{\partial \varphi} = 0$

Euler-Lagrange egyenletet.

$\vec{\varphi}(x)$: $\boxed{\frac{d}{dx}\frac{\partial L}{\partial \varphi_i'} - \frac{\partial L}{\partial \varphi_i} = 0}$, $i = 1..\dim\vec{\varphi}$

Térelméleti variáns: $S = \int_{\mathbb{R}^n} L(\varphi, \partial_i\varphi, \vec{x})\, d\vec{x}$

$$\frac{\partial}{\partial x_i}\frac{\partial L}{\partial(\partial_i\varphi)} - \frac{\partial L}{\partial \varphi} = 0$$

Példák:

(1) $L(\varphi,\varphi',x) = \frac{1}{2}\varphi'^2 - f(x)\cdot\varphi(x)$

$$\frac{\partial L}{\partial \varphi} = -f(x), \quad \frac{\partial L}{\partial \varphi'} = \frac{1}{2}\cdot 2\varphi' = \varphi'$$

EL: $\frac{d}{dx}\varphi' - (-f) = 0 \qquad \frac{d^2}{dx^2}\varphi(x) + f(x) = 0$ Poisson egyenlet

(2) $L(\varphi,\varphi',x) = (\varphi')^3\cdot e^x - \varphi'\cdot\varphi^2 + \varphi^3 + \sin x$

$$\frac{\partial L}{\partial \varphi'} = 3(\varphi')^2\cdot e^x - \varphi^2, \quad \frac{\partial L}{\partial \varphi} = -\varphi'\cdot 2\varphi + 3\varphi^2$$

EL: $\frac{d}{dx}\left[3(\varphi')^2\cdot e^x - \varphi^2\right] - \left[-\varphi'\cdot 2\varphi + 3\varphi^2\right] = 0$

$$\left[6\varphi'\cdot\varphi''\cdot e^x - 2\varphi\cdot\varphi'\right] - \left[-\varphi'\cdot 2\varphi + 3\varphi^2\right] = 0$$

(3) $L(x(t),\dot{x}(t),t) = \frac{1}{2}\dot{x}^2 - V(x)$ = kinetikus − potenciális (energia)

$$\frac{\partial L}{\partial x} = -V'(x), \quad \frac{\partial L}{\partial \dot{x}} = \dot{x}$$

EL: $\frac{d}{dt}\dot{x} - (-V'(x)) = 0, \qquad \frac{d^2}{dt^2}x(t) = -\frac{\partial V(x(t))}{\partial x(t)}$ ← potenciális energia gradiense

(4) $L(x_1,x_2,\dot{x}_1,\dot{x}_2,t) = \sqrt{\dot{x}_1^2+\dot{x}_2^2}$ geodetikus görbe $\vec{x}(0)=\vec{a}$ és $\vec{x}(1)=\vec{b}$ között

↑ egyenes vonal

$$\frac{\partial L}{\partial x_i} = 0, \quad \frac{\partial L}{\partial \dot{x}_i} = \frac{2\dot{x}_i}{2\sqrt{\dot{x}_1^2+\dot{x}_2^2}}$$

EL:
$$\left.\begin{aligned} \frac{d}{dt}\left(\frac{\dot{x}_1}{\sqrt{\dot{x}_1^2+\dot{x}_2^2}}\right) - 0 = 0 &= \frac{1}{(\dot{x}_1^2+\dot{x}_2^2)^{3/2}}\dot{x}_2\left(-\dot{x}_1\ddot{x}_2 + \dot{x}_2\ddot{x}_1\right) \\ \frac{d}{dt}\left(\frac{\dot{x}_2}{\sqrt{\dot{x}_1^2+\dot{x}_2^2}}\right) - 0 = 0 &= \frac{1}{(\dot{x}_1^2+\dot{x}_2^2)^{3/2}}\dot{x}_1\left(\dot{x}_1\ddot{x}_2 - \dot{x}_2\ddot{x}_1\right) \end{aligned}\right\} \rightarrow \dot{\vec{x}} \sim \ddot{\vec{x}}$$

Megoldás: $\vec{x}(t)$ egyenes vonal, tetszőleges parametrizáció

Megjegyzés: $S[\vec{r}(t)] = \int|\dot{\vec{r}}(t)|\,dt$ ívhossz invariáns az $\vec{r}(t) \longrightarrow \vec{r}(f(t))$ reparametrizációra nézve. Így nincs egyértelmű megoldása az EL egyenleteknek, viszont mindegyik megoldás ugyanazt a görbét reprezentálja.

(5) $L(x_1,x_2,\dot x_1,\dot x_2,t)=\frac{1}{2}(\dot x_1^2+\dot x_2^2)$

$$\frac{\partial L}{\partial x_1}=\frac{\partial L}{\partial x_2}=0,\quad \frac{\partial L}{\partial \dot x_1}=\dot x_1,\quad \frac{\partial L}{\partial \dot x_2}=\dot x_2$$

EL: $\frac{d}{dt}\dot x_1-0=0$, $\frac{d}{dt}\dot x_2-0=0$, $\quad \ddot x_1=0,\quad \ddot x_2=0$

egyenletes mozgás.

Megjegyzés: ugyanazokat a geodetikus görbéket (egyenes szakaszokat) generálja, mint (4), de most a parametrizáció arányos az ívhosszal.

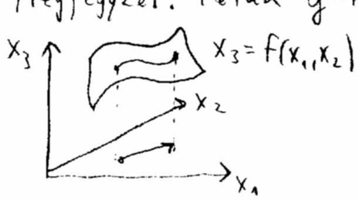
(6) $L(\vec x,\dot{\vec x},t)=\frac{1}{2}\sum_{i\gamma} g_{i\gamma}(\vec x)\,\dot x_i\dot x_\gamma$ $\qquad g_{i\gamma}$: metrikus tenzor, $g_{i\gamma}=\delta_{i\gamma}$ az Euklédeszi térben

$$\frac{\partial L}{\partial \dot x_i}=\sum_\gamma g_{i\gamma}(\vec x)\,\dot x_\gamma \qquad \frac{\partial L}{\partial x_i}=\frac{1}{2}\sum_{kl}\frac{\partial g_{kl}}{\partial x_i}\dot x_k\dot x_l$$

EL: $$\frac{d}{dt}\left(\sum_\gamma g_{i\gamma}\dot x_\gamma\right)-\frac{1}{2}\sum_{kl}\frac{\partial g_{kl}}{\partial x_i}\dot x_k\dot x_l=0$$

$$\sum_\gamma g_{i\gamma}\ddot x_\gamma+\frac{\partial g_{i\gamma}}{\partial x_k}\cdot\dot x_k\dot x_\gamma-\frac{1}{2}\sum_{kl}\frac{\partial g_{kl}}{\partial x_i}\dot x_k\dot x_l=0$$

Megjegyzés: Példa g-re:



$$\Delta x_1^2+\Delta x_2^2+\Delta x_3^2=\Delta x_1^2+\Delta x_2^2+\left[(f'_{x_1}\cdot\Delta x_1)+(f'_{x_2}\cdot\Delta x_2)\right]^2$$

$$=[\Delta x_1,\Delta x_2]\underbrace{\begin{bmatrix}1+(f'_{x_1})^2 & f'_{x_1}\cdot f'_{x_2}\\ f'_{x_2}\cdot f'_{x_1} & 1+(f'_{x_2})^2\end{bmatrix}}_{g}\begin{bmatrix}\Delta x_1\\ \Delta x_2\end{bmatrix}$$

(7) Egykerekű bicikli. Fázistér: (x,y,α)

$S=\int_0^1 \frac{1}{2}\dot x^2+\frac{1}{2}\dot y^2+\frac{1}{2}\dot\alpha^2\,dt$, kényszer: $(\dot x,\dot y)\sim(\cos\alpha,\sin\alpha)\longleftrightarrow -\sin\alpha\cdot\dot x+\cos\alpha\cdot\dot y=0$

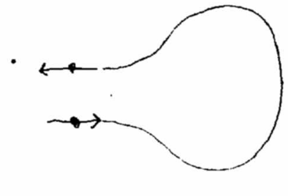
$$S=\int_0^1 \frac{1}{2}(\dot x^2+\dot y^2+\dot\alpha^2)+\lambda(-\sin\alpha\cdot\dot x+\cos\alpha\cdot\dot y)\,dt$$

EL: "λ": $-\sin\alpha\cdot\dot x+\cos\alpha\cdot\dot y=0$ kényszer

"x": $\frac{d}{dt}(\dot x-\lambda\sin\alpha)-0=0$

"y": $\frac{d}{dt}(\dot y+\lambda\cos\alpha)-0=0$

"α": $\frac{d}{dt}\dot\alpha-\lambda(-\cos\alpha\cdot\dot x-\sin\alpha\cdot\dot y)=0$



(8) Mágneses tér

$$L = \frac{1}{2}(\dot{x}_1^2 + \dot{x}_2^2) + A_1\dot{x}_1 + A_2\dot{x}_2 \,, \qquad A_i = A_i(x_1, x_2)$$

$$\frac{\partial L}{\partial \dot{x}_1} = \dot{x}_1 + A_1 \,, \quad \frac{\partial L}{\partial \dot{x}_2} = \dot{x}_2 + A_2 \,, \quad \frac{\partial L}{\partial x_1} = \frac{\partial A_1}{\partial x_1}\dot{x}_1 + \frac{\partial A_2}{\partial x_1}\dot{x}_2 \,, \quad \frac{\partial L}{\partial x_2} = \frac{\partial A_1}{\partial x_2}\dot{x}_1 + \frac{\partial A_2}{\partial x_2}\dot{x}_2$$

$$EL: \ \frac{d}{dt}(\dot{x}_1 + A_1) - \left(\frac{\partial A_1}{\partial x_1}\dot{x}_1 + \frac{\partial A_2}{\partial x_2}\dot{x}_2\right) = 0 = \left(\ddot{x}_1 + \frac{\partial A_1}{\partial x_1}\dot{x}_1 + \frac{\partial A_1}{\partial x_2}\dot{x}_2\right) - \left(\frac{\partial A_1}{\partial x_1}\dot{x}_1 + \frac{\partial A_2}{\partial x_1}\dot{x}_2\right)$$

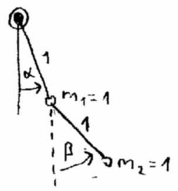
$$\ddot{x}_1 = \left(\frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2}\right)\cdot \dot{x}_2$$

$$\ldots \quad \ddot{x}_2 = -\left(\frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2}\right)\cdot \dot{x}_1$$

$$3d: \ \ddot{\vec{x}} = \underbrace{\vec{B} \times \dot{\vec{x}}}_{\text{Lorentz erő}} \,, \qquad \vec{B} = \operatorname{rot} \vec{A}$$

(9) Dupla inga

Állapottér: $S^1 \times S^1 \times \mathbb{R} \times \mathbb{R}$, $(\alpha, \beta, \dot{\alpha}, \dot{\beta})$



Potenciális energia:

$$V(\alpha, \beta) = -\left[\cos(\alpha) + (\cos(\alpha) + \cos(\beta))\right]$$

Kinetikus energia:

$$T = \frac{1}{2}\vec{v}_1^{\,2} + \frac{1}{2}\vec{v}_2^{\,2}$$

$$\vec{v}_1 = \frac{d}{dt}\left[\cos\alpha, \sin\alpha\right], \quad \vec{v}_2 = \frac{d}{dt}\left[\cos\alpha + \cos(\beta), \ \sin\alpha + \sin(\beta)\right].$$

$$L(\alpha, \beta, \dot{\alpha}, \dot{\beta}) = T - V = \dot{\alpha}^2 + \frac{1}{2}\dot{\beta}^2 + \cos(\alpha - \beta)\dot{\alpha}\dot{\beta} + 2\cos(\alpha) + \cos(\beta)$$

$$EL_\alpha: \ 2\ddot{\alpha} + \cos(\alpha - \beta)\ddot{\beta} + \sin(\alpha - \beta)\dot{\beta}^2 + 2\sin(\alpha) = 0$$

$$EL_\beta: \ \ddot{\beta} + \cos(\alpha - \beta)\ddot{\alpha} - \sin(\alpha - \beta)\dot{\alpha}^2 + \sin(\beta) = 0$$

Véges elem módszer (Finite elements method, FEM)

$11'_{\mathrm{IV}}$

Probléma:

1. Oldd meg: $\varphi''(x) = -f(x)$, $\varphi(0) = \varphi(1) = 0$.
2. Variációszámítás: Ekvivalens probléma:

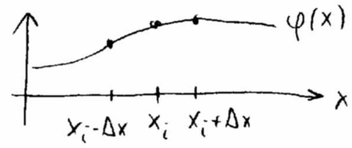
   Minimalizáld: $\int_0^1 \frac{1}{2}[\varphi'(x)]^2 - f(x)\varphi(x)\,dx$

Ekvivalencia: Euler-Lagrange: $S[\varphi] = \int_0^1 \frac{1}{2}\varphi'^2 - f\varphi\,dx \longleftrightarrow \frac{d}{dx}\varphi' + f = 0$

Közelítő, numerikus megoldás:

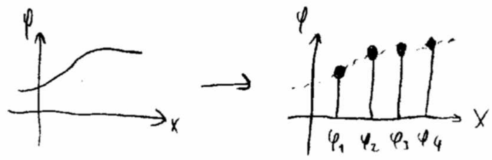
1. Véges differenciák

$$\varphi''(x_i) \approx \frac{1}{\Delta x^2}\Big(\varphi(x_i+\Delta x) - 2\varphi(x_i) + \varphi(x_i - \Delta x)\Big) \approx -f(x_i)$$



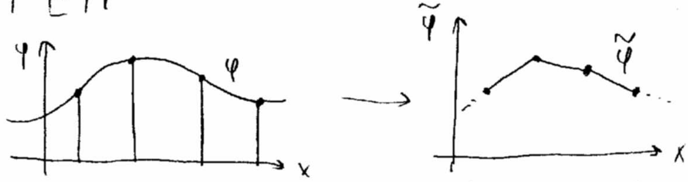
Probléma: $\varphi \in \mathcal{F}un([0,1]) \longrightarrow \vec{\varphi} \in \mathbb{R}^N$

∞ dim $\longrightarrow$ N dim



$\varphi \longrightarrow [\varphi_1, \varphi_2, \varphi_3, \varphi_4]$

2. FEM



$\varphi \in \mathcal{F}un([0,1]) \longrightarrow \tilde{\varphi} \in V \subset \mathcal{F}un([0,1])$

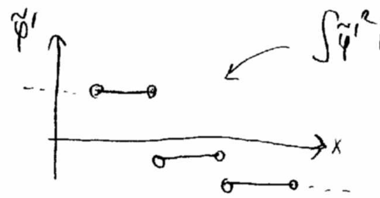
∞ dim   véges dim

PDE: $\varphi'' = -f$ $\tilde{\varphi}'' = -\tilde{f}$ "értelmetlen"

Minimalizáció:

$$\min_{\varphi \in \mathcal{F}un} \int \frac{1}{2}\varphi'^2 - f\varphi\,dx \qquad \min_{\tilde{\varphi} \in V} \int \frac{1}{2}\tilde{\varphi}'^2 - \tilde{f}\tilde{\varphi}\,dx \quad \text{"értelmes feladat"}$$

$\int \tilde{\varphi}'^2$ létezik



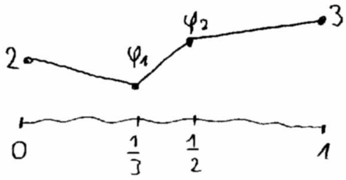
$\frac{d}{dx}: \longrightarrow \neq 0$

$\frac{d^2}{dx^2}: \longrightarrow 0$

"Feladat": Legyen $S[\varphi]=\int_0^1 [\varphi'(x)]^2 + x\varphi(x)\,dx$, $\varphi(0)=2$, $\varphi(1)=3$.

Minimalizáld S-t, ha $\varphi\in V$, ahol V azon függvények halmaza, amelyek darabonként affinak az $[0,1/3]$, $[\frac{1}{3},\frac{1}{2}]$, $[\frac{1}{2},1]$ szakaszokon, továbbá $\varphi(0)=2$, $\varphi(1)=3$.

"Megoldás":



φ': $\frac{\varphi_1-2}{1/3}$ $\quad\frac{\varphi_2-\varphi_1}{\frac{1}{2}-\frac{1}{3}}$ $\quad\frac{3-\varphi_2}{1-1/2}$

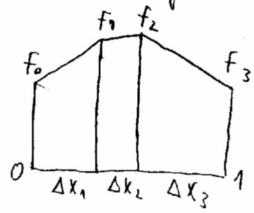
$\dim V=2$

$$S[(\varphi_1,\varphi_2)] = \left(\frac{\varphi_1-2}{1/3}\right)^2\cdot\frac{1}{3} + \overset{\varphi'^2}{\left(\frac{\varphi_2-\varphi_1}{1/6}\right)^2}\cdot\overset{\Delta x}{\frac{1}{6}} + \left(\frac{3-\varphi_2}{1/2}\right)^2\cdot\frac{1}{2}$$

$$+\int_0^{1/3} x\cdot\left(2+\frac{\varphi_1-2}{1/3}x\right)dx + \int_{1/3}^{1/2} x\left(\varphi_1+\frac{\varphi_2-\varphi_1}{1/6}\cdot\left(x-\frac{1}{3}\right)\right)dx +$$

$$+\int_{1/2}^{1} x\left(\varphi_2+\frac{3-\varphi_2}{1/2}\left(x-\frac{1}{2}\right)\right)dx$$

Mivel csak közelítő megoldást keresünk, nincs sok értelme az integrálok pontos kiszámításának, pl. a trapéz módszert használva:



$$S[(\varphi_1,\varphi_2)]\approx\left(\frac{\varphi_1-2}{1/3}\right)^2\cdot\frac{1}{3}+\left(\frac{\varphi_2-\varphi_1}{1/6}\right)^2\cdot\frac{1}{6}+\left(\frac{3-\varphi_2}{1/2}\right)^2\cdot\frac{1}{2}+$$

$$+\frac{1/3}{2}\cdot 0\cdot 2+\underbrace{\frac{1/3+1/6}{2}}_{\frac{\Delta x_1+\Delta x_2}{2}}\cdot\underset{x}{\frac{1}{3}}\cdot\underset{\varphi(x)}{\varphi_1}+\frac{1/6+1/2}{2}\cdot\frac{1}{2}\varphi_2+\underset{\frac{\Delta x_3}{2}}{\frac{1/2}{2}}\cdot\underset{x}{1}\cdot\underset{\varphi(x)}{3}$$

$$T=\Delta x_1\frac{f_1+f_0}{2}+\Delta x_2\frac{f_2+f_1}{2}+\Delta x_3\frac{f_3+f_2}{2}$$
$$=\frac{\Delta x_1}{2}f_0+\frac{(\Delta x_1+\Delta x_2)}{2}f_1+$$
$$+\frac{\Delta x_2+\Delta x_3}{2}f_2+\frac{\Delta x_3}{2}f_3$$

$$=[\varphi_1,\varphi_2]\underbrace{\begin{bmatrix}l_{11} & l_{12}\\ l_{21} & l_{22}\end{bmatrix}}_{\substack{l_{12}=l_{21},\\ L=L^T}}\begin{bmatrix}\varphi_1\\ \varphi_2\end{bmatrix}+[m_1\ m_2]\begin{bmatrix}\varphi_1\\ \varphi_2\end{bmatrix}+c=\vec{\varphi}^T L\vec{\varphi}+\vec{m}^T\vec{\varphi}+c$$

$$\tilde{S}=\vec{\varphi}^T L\vec{\varphi}+\vec{m}^T\vec{\varphi}+c$$

$$\frac{\partial\tilde{S}}{\partial\varphi_i}=\sum_j 2L_{ij}\varphi_j+m_i,\quad \operatorname{grad}\tilde{S}=2L\vec{\varphi}+\vec{m},$$

Kritikus pont: $\vec{\varphi}_{crit}=-\frac{1}{2}L^{-1}\vec{m}$

$\vec{\varphi}_{crit}$ globális minimum, ha L pozitív definit, vagyis csak pozitív sajátértékei vannak. (Mivel $L=L^T$, a sajátértékek valósak.)

Mi legyen, ha nincs, vagy nem ismerjük a Lagrange-függvényt? $\boxed{13^*_{\text{IX}}}$

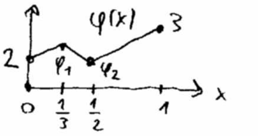
"Feladat": Keress közelítő megoldást az $\varphi'' = -f(x)$, $\varphi(0)=2$, $\varphi(1)=3$ DE-hez.

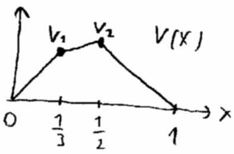
"Megoldás":

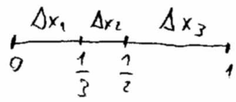
$$\varphi''(x) = -f(x) \quad \longleftrightarrow \quad \int_0^1 \overbrace{(\varphi''(x)+f(x))\, v(x)\, dx}^{=0} = 0 \quad \text{"bármely" } v(x)\text{-re.}$$

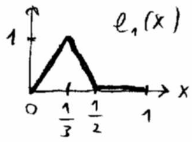
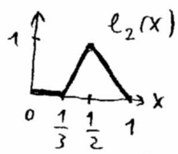
$$0 = \int_0^1 \varphi'' v + f v\, dx = \varphi' v \Big|_0^1 - \int_0^1 \varphi' v' - f v\, dx$$

ha $v(0)=v(1)=0$, akkor $\displaystyle\int_0^1 -\varphi' v' + f v\, dx = 0$

Keressük a közelítő megoldást itt:  Φ

$v(x)$ pedig legyen innen:  $\mathcal{V}$



Ezen $v(x)$ függvények bázisa:  

$v(x) = v_1 e_1(x) + v_2 e_2(x)$

A $\varphi \in \Phi$ közelítő megoldásra követeljük meg, hogy $\displaystyle\int_0^1 -\varphi' v' - f v = 0$ bármely v_1, v_2-re.

Tehát $\displaystyle\int_0^1 -\varphi' e_1' + f e_1\, dx = 0$, $\displaystyle\int_0^1 -\varphi' e_2' + f e_2\, dx = 0$.

Továbbá reméljük, hogy nem baj, ha az integrálokat csak közelítőleg számoljuk ki.

$$e_1:\ 0 = \int_0^{1/2} -\varphi' e_1' + f e_1\, dx \approx \frac{1}{3}\cdot\left(-\frac{\varphi_1-2}{1/3}\right)\cdot\frac{1}{1/3} + \frac{1}{6}\left(-\frac{\varphi_2-\varphi_1}{1/6}\right)\cdot\left(-\frac{1}{1/6}\right) +$$

$$+ \frac{1/3}{2} f(0)\cdot 0 + \frac{1/3+1/6}{2} f\left(\tfrac{1}{3}\right)\cdot 1 + \frac{1/6}{2} f\left(\tfrac{1}{2}\right)\cdot 0 \longleftarrow \text{trapéz módszer}$$

$$e_2:\ 0 = \int_{1/3}^{1} -\varphi' e_2' + f e_2\, dx \approx \frac{1}{6}\cdot\left(-\frac{\varphi_2-\varphi_1}{1/6}\right)\cdot\frac{1}{1/6} + \frac{1}{2}\left(-\frac{3-\varphi_2}{1/2}\right)\cdot\left(-\frac{1}{1/2}\right) +$$

$$+ \frac{1/6}{2} f\left(\tfrac{1}{3}\right)\cdot 0 + \frac{1/6+1/2}{2} f\left(\tfrac{1}{2}\right)\cdot 1 + \frac{1/2}{2} f(1)\cdot 0$$

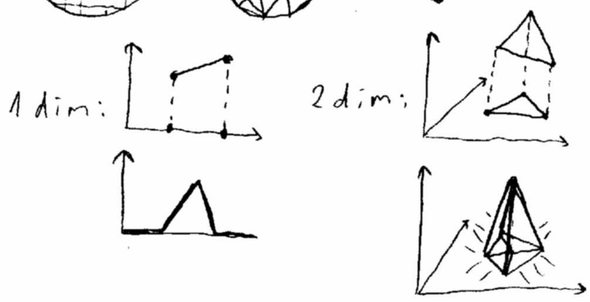
e_2' $\quad -\varphi'$ $\quad \Delta x$

$\frac{\Delta x_2 + \Delta x_3}{2}$ $\quad f(x)$ $\quad e_2(x)$

Megjegyzések

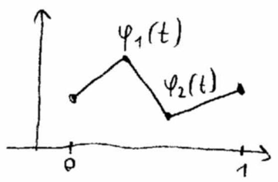
13a IX

(1) Minek ez az egész? 1dim: véges differencia sokkal egyszerűbb
2,3dim, komplikált alak, nehéz rácspontokkal közelíteni



(2) Dinamika

Példa: $\varphi_t(t,x) = \varphi_{xx}(t,x) + f(t,x)\varphi(t,x) + g(t,x)$



$$\varphi(t,x) \longrightarrow \vec{\varphi}(t) = \begin{pmatrix} \varphi_1(t) \\ \varphi_2(t) \end{pmatrix}$$

$$\int_0^1 \left(\varphi_t - \varphi_{xx} - f\varphi + g \right) v(x)\, dx = 0 \quad \text{"bármely" } v\text{-re}$$

$$\int_0^1 (\varphi_t - f\varphi + g) v + \varphi_x V_x \, dx = 0 \quad \text{bármely } v(x) = e_i(x)\text{-re}$$

$$\longrightarrow \frac{d}{dt}\vec{\varphi}(t) = A(t)\,\vec{\varphi}(t) + \vec{b}(t)$$

$A, \vec{b}$ kiszámolás véges elemekkel

véges dim. DE megoldás numerikus módszerekkel,
mint pl. Runge-Kutta

Összegzés

① Hullámegyenlet (1+1 dim) $\varphi_{tt} - \varphi_{xx} = 0$

Megoldás: $\varphi(t,x) = f(x-t) + g(x+t) = \frac{1}{2}\left[\varphi(0,x-t) + \varphi(0,x+t)\right] + \frac{1}{2}\int_{x-t}^{x+t} \dot{\varphi}(0,y)\,dy$

Síkhullám: $\varphi = e^{i(kx - \omega t)} \longrightarrow |k| = |\omega|$ terjedési sebesség $= 1$.

Rezgő húr: $\varphi(0,x) = F(x)$, $\dot{\varphi}(0,x) = G(x)$, $\varphi(t,0) = \varphi(t,\pi) = 0$

Megoldás: Szinusz tr.: $F(x) = \sum_{n=1}^{\infty} \hat{F}_n \sqrt{\frac{2}{\pi}} \sin(nx)$, $\hat{F}_n = \int_0^{\pi} \sqrt{\frac{2}{\pi}} \sin(nx) F(x)\,dx$, $\hat{G}$ hasonló.

$$\varphi(t,x) = \sum_{n=1}^{\infty} \left(\hat{F}_n \cos(nt) + \frac{\hat{G}_n}{n} \sin(nt)\right) \sqrt{\frac{2}{\pi}} \sin(nx)$$

② Variációszámítás: $S[\varphi] = \int_0^1 L(\varphi(x), \varphi'(x), x)\,dx$, $\varphi(0) = a$, $\varphi(1) = b$

φ_c kritikus pont: $S[\varphi_c + \delta\varphi] \approx S[\varphi_c]$

Euler-Lagrange: $\frac{d}{dx}\frac{\partial L}{\partial \varphi'} - \frac{\partial L}{\partial \varphi} = 0$

Példa: $L(\varphi(x), \varphi'(x), x) = \varphi(x)\cdot[\varphi'(x)]^3 + x\varphi(x) + x^2$

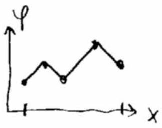
EL: $\frac{d}{dx}\left(\varphi(x)\cdot 3[\varphi'(x)]^2\right) - \left([\varphi'(x)]^3 + x\right) = 0$

③ Véges elem módszer

DE $\longleftrightarrow$ variációszámítás, minimalizáció ⓐ

$\nwarrow\!\searrow$ gyenge megoldás ⓑ

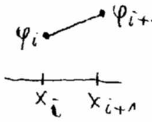
ⓐ Közelítő megoldás:



φ'' nehezen értelmezhető

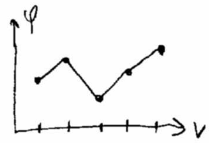
$S = \int L(\varphi', \varphi, x)\,dx$ értelmes

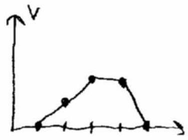
$L = \varphi'^2 - f\varphi$, $S \approx \ldots + \underbrace{(x_{i+1} - x_i)}_{\Delta x_i}\left[\left(\frac{\varphi_{i+1} - \varphi_i}{\Delta x_i}\right)^2 - \frac{f(x_i)\varphi(x_i) + f(x_{i+1})\varphi(x_{i+1})}{2}\right] + \ldots$ trapéz módszer

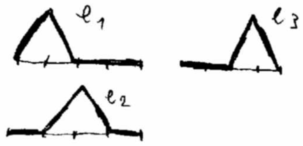


$S \approx \vec{\varphi}^T L \vec{\varphi} + \vec{m}^T \vec{\varphi} + c$, grad $S = 0 \longrightarrow \vec{\varphi}_{crit} = -\frac{1}{2} L^{-1} \vec{m}$

ⓑ $\varphi'' + f = 0 \longleftrightarrow \int_0^1 (\varphi'' + f) v\,dx = \int_0^1 -\varphi' v' + f v\,dx = 0$, ha $v(1) = v(0) = 0$

φ: 

V: 

bázis $\longrightarrow$ 

$\int_0^1 -\varphi' e_i + f e_i\,dx = 0$ bármely i-re

Minta feladatok

15* IX

(1) $\varphi_{tt}=\varphi_{xx}$, $\varphi(0,x)=3\sin(4x)$, $\dot\varphi(0,x)=5\cdot\sin(6x)$, $\varphi(t,0)=\varphi(t,\pi)=0$

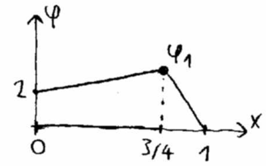
Mennyi $\varphi(t,x)$, $x\in[0,\pi]$?

Megoldás: $\varphi(t,x)=3\cos(4t)\sin(4x)+\frac{5}{6}\sin(6t)\cdot\sin(6x)$

(2) Legyen $L(x,\dot x,t)=\dot x^4+x^2+t^2$. Ha $x(1)=2$, $\dot x(1)=3$, akkor az Euler-Lagrange egyenlet mit jósol $\ddot x(1)$-re?

Megoldás: $0=\frac{d}{dt}\frac{\partial L}{\partial \dot x}-\frac{\partial L}{\partial x}=\frac{d}{dt}(4\dot x^3)-2x=12\dot x^2\cdot\ddot x-2x=0$

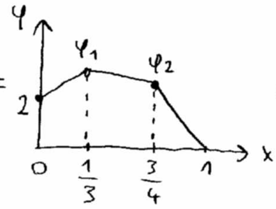
tehát ha $x(1)=2$, $\dot x(1)=3$, akkor $12\cdot 3^2\cdot\ddot x(1)-2\cdot 2=0 \longrightarrow \ddot x(1)=\frac{4}{12\cdot 3^2}=\frac{1}{27}$

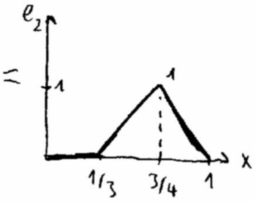
(3) Legyen $\varphi(x)=$ 

. Számold ki, hogy mennyi $\int_0^1 \varphi'^2+(1-x)\varphi\,dx$, ha az $\int_0^1(1-x)\varphi\,dx$ integrált a trapézmódszer segítségével számoljuk egyetlen $x=\frac{3}{4}$ osztóponttal.

Megoldás: $\int_0^1 \varphi'^2+(1-x)\varphi\,dx \approx$

$$\approx \frac{3}{4}\left(\frac{\varphi_1-2}{3/4}\right)^2+\frac{1}{4}\left(\frac{0-\varphi_1}{1/4}\right)^2+\frac{3}{4}\cdot\frac{(1-0)\cdot 2+(1-3/4)\cdot\varphi_1}{2}+\frac{1}{4}\,\frac{(1-\frac{3}{4})\cdot\varphi_1+(1-1)\cdot 0}{2}$$

(4) Legyen $\varphi(x)=$ 

, $e_2(x)=$ 

Mennyi $\int_0^1 (\varphi''-(1-x))\,e_2(x)\,dX$, ha megengedjük a parciális integrálás használatát ahhoz, hogy $\int_0^1 \varphi'' e_2\,dx$-t a $-\int_0^1 \varphi' e_2'\,dx$ alakba írjuk át? (Használj trapéz módszert az integrál kiszámítására.)

Megoldás:

$$\int_0^1(\varphi''-(1-x))\,e_2\,dx=\int_0^1 -\varphi' e_2'-(1-x)e_2\,dx=\left(\frac{3}{4}-\frac{1}{3}\right)\cdot\left(-\frac{\varphi_2-\varphi_1}{\frac{3}{4}-\frac{1}{3}}\right)\cdot\frac{1}{\frac{3}{4}-\frac{1}{3}}+\left(1-\frac{3}{4}\right)\left(-\frac{0-\varphi_2}{1-\frac{3}{4}}\right)\cdot\left(-\frac{1}{1-\frac{3}{4}}\right)$$

$$+\frac{\left(\frac{3}{4}-\frac{1}{3}\right)+\left(1-\frac{3}{4}\right)}{2}\cdot\varphi_2\cdot 1$$