1* VIII

Egységugrás, impulzus, frekvencia válaszok

Példa: $y'+3y=f(t)$

Impulzus válasz, Green függvény:

$G(t)=0$, ha $t<0$, $G'(t)+3G(t)=\delta(t) \longrightarrow G(t)=\begin{cases}0 & t<0\\ e^{-3t} & t>0\end{cases}$

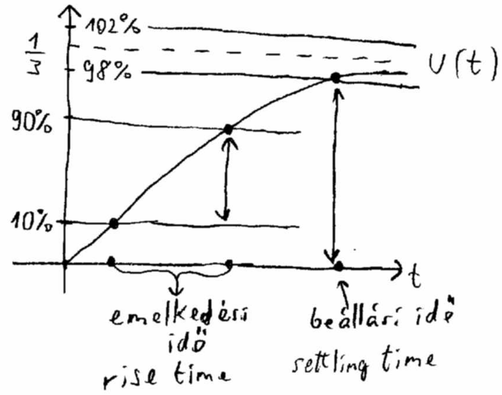
egységugrás válasz:

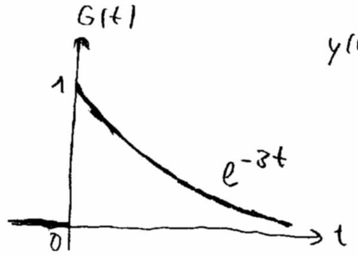
$U(t)=0$, ha $t<0$, $U'(t)+3U(t)=\Theta(t) \longrightarrow U(t)=\begin{cases}0 & t<0\\ \frac{1}{3}(1-e^{-3t}) & t>0\end{cases}$

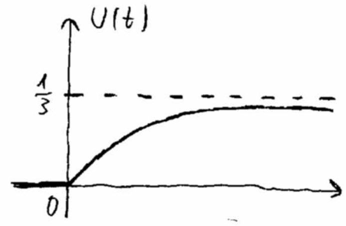
$y(0)=0$, $y'+3y=1$ DE megoldása

Ekkor: $\boxed{U'(t)=G(t)}$,

hiszen $(U')'+3U'=\Theta'=\delta=G'+3G$







$\frac{1}{3}(1-e^{-3t})=\int_0^t e^{-3s}\,ds$

frekvencia válasz

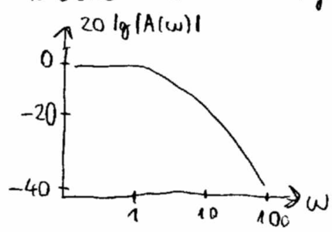
$$y'+3y=f(t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(p)e^{ipt}dp$$

$$\left(A(\omega)\cdot e^{-i\omega t}\right)'+3\left(A(\omega)\cdot e^{-i\omega t}\right)=e^{-i\omega t} \longrightarrow A(\omega)=\frac{1}{-i\omega+3} \qquad \omega>0$$

Transzfer függvény: $H(s)=\frac{1}{s+3}=A(is)$

$\mathrm{Arg}(5\cdot e^{2i})=2$

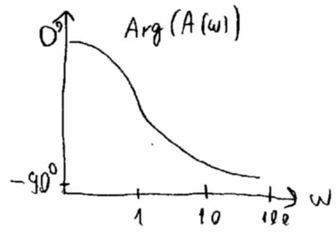
Bode ábra: $\lg(\omega)\longleftrightarrow 20\lg|A(\omega)|$, $\lg(\omega)\longleftrightarrow \mathrm{Arg}(A(\omega))$



Formálisan: $y'+3y=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(p)e^{ipt}dp$

$$\longrightarrow y(t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(p)\frac{1}{+ip+3}e^{ipt}dp$$

Alul definiált, mivel $y(t)$ csak a homogén egyenlet megoldása erejéig van meghatározva.



Laplace transzformáció.

$\boxed{2^{*}_{VIII}}$

$$\left(\mathcal{L}(f)\right)(s)=\int_0^\infty e^{-st}f(t)\,dt=F(s),\quad f:[0,\infty)\to\mathbb{C}$$

$\mathcal{L}$ lineáris: $\mathcal{L}(\alpha_1 f_1+\alpha_2 f_2)=\alpha_1 F_1+\alpha_2 F_2$

Ha $f(t)$ folytonos és $\left(f(t)<Me^{\alpha t}\right.$, ha $\left.t>k\right)$, akkor $F(s)$ definiált $Re(s)>\alpha$-ra

Inverz Laplace tr: $\quad t>0$

$$\left(\mathcal{L}^{-1}(F)\right)(t)=\frac{1}{2\pi i}\int_{-i\infty+a}^{+i\infty+a}F(s)e^{st}\,ds=f(t),$$

ahol a elégnagy, hogy $F(s)$ definiálva legyen a $-i\infty+a\ldots+i\infty+a$ kontúron

Formális bizonyítás, $a=0$.

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}F(s)e^{st}\,ds \underset{s=ip}{=} \frac{1}{2\pi i}\int_{-\infty}^{\infty}F(ip)e^{ipt}\cdot i\,dp,\qquad F(ip)=\int_0^\infty e^{-ipt}f(t)\,dt=\int_{-\infty}^{\infty}e^{-ipt}\Theta(t)f(t)\,dt$$

Tehát $\tilde{F}(p)=F(ip)$ és $\Theta(t)f(t)$ egymás Fourier transzformáltjai, a következő normalizációval: $\hat{g}(p)=\int_{-\infty}^{\infty}e^{-ipt}g(t)\,dt,\quad g(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{g}(p)e^{ipt}\,dp.$

Így. ha $\hat{g}(p)=\tilde{F}(p)$, akkor $\frac{1}{2\pi i}\int_{-\infty}^{\infty}F(ip)e^{ipt}\cdot i\,dp=\frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{F}(p)e^{ipt}\,dp=\Theta(t)f(t)$

$F(s)$ analitikus (vagyis konvergens Taylor sorba fejthető)

Példák:

(1) $\mathcal{L}(1)=\left(\mathcal{L}(1)\right)(s)=\int_0^\infty e^{-st}\cdot 1\,dt=\frac{1}{-s}e^{-st}\Big|_0^\infty=-\frac{1}{s}\left(\overset{0 \text{ ha } Re(s)>0}{e^{-s\cdot\infty}}-e^{-s\cdot 0}\right)=\frac{1}{s}$

(2) $\mathcal{L}(t)=\int_0^\infty e^{-st}\cdot t\,dt=\int_0^\infty\left(\frac{e^{-st}}{-s}\right)'\cdot t\,dt=\frac{e^{-st}}{-s}\cdot t\Big|_0^\infty-\int_0^\infty\frac{e^{-st}}{-s}\cdot 1\,dt=\frac{1}{s^2}$

(3) $\mathcal{L}(e^{at})=\int_0^\infty e^{-st}\cdot e^{at}\,dt=\int_0^\infty e^{-(s-a)t}\,dt=\frac{1}{s-a}$

(4) $\mathcal{L}(\cos(at))=\int_0^\infty\frac{e^{iat}+e^{-iat}}{2}\cdot e^{-st}\,dt=\frac{1}{2}\left(\frac{1}{s-ia}+\frac{1}{s+ia}\right)=\frac{s}{s^2+a^2}$

(5) $\mathcal{L}(\sin(at))=\int_0^\infty\frac{e^{iat}-e^{-iat}}{2i}\cdot e^{st}\,dt=\frac{1}{2i}\left(\frac{1}{s-ia}-\frac{1}{s+ia}\right)=\frac{a}{s^2+a^2}$

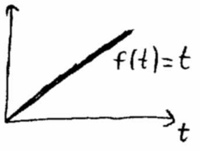
(6) $\mathcal{L}(t^n) = \frac{n!}{t^{n+1}}$, parciális integrálás, $n = 1, 2, 3, \ldots$

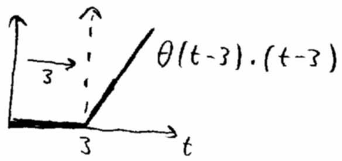
(7) $\mathcal{L}\left(\Theta(t-a) f(t-a)\right) = \int_0^\infty \Theta(\underbrace{t-a}_{=\tau}) e^{-st} f(t-a)\, dt = \int_0^\infty e^{-s(\tau+a)} f(\tau)\, d\tau$

$$= e^{-sa} \int_0^\infty e^{-s\tau} f(\tau)\, d\tau = \underline{e^{-sa} F(s)}$$

példa: $\mathcal{L}(t) = \frac{1}{s^2}$, legyen $f(t) = \begin{cases} 0, \text{ ha } t \in [0,3] \\ t-3, \text{ ha } t \geq 3 \end{cases} = \Theta(t-3)\cdot(t-3)$

ekkor $F(s) = e^{-s\cdot 3} \cdot \frac{1}{s^2}$





(8) Laplacetr: deriválás $\longrightarrow$ szorzás

$$\mathcal{L}(f') = \int_0^\infty e^{-st} f'(t)\, dt = e^{-st} f(t)\Big|_0^\infty - \int_0^\infty \left(-s \cdot e^{-st}\right) \cdot f(t)\, dt$$

$$= \left(e^{-s\cdot\infty} f(\infty) - e^{-s\cdot 0} f(0)\right) + s \int_0^\infty e^{-st} f(t)\, dt = sF(s) - f(0)$$

$$\boxed{\mathcal{L}(f') = sF(s) - f(0)}$$

(9) $\mathcal{L}(f'') = s\,\mathcal{L}(f') - f'(0) = s\left(sF(s) - f(0)\right) - f'(0)$

$$= s^2 F(s) - s f(0) - f'(0)$$

(10) $\mathcal{L}(f''') = s^3 F(s) - s^2 f(0) - s f'(0) - f''(0)$, stb.

(11) Legyen $g(t) = \int_0^t f(t)\, dt$, vagyis $g(0) = 0$, $g'(t) = f(t)$.

Ekkor $\mathcal{L}(g') = \mathcal{L}(f) = sG(s) - g(0) = sG(s)$,

így $G(s) = \frac{1}{s} F(s)$

Tehát $\mathcal{L}\left(\int f\right) = \frac{1}{s} F(s)$

(12) $\mathcal{L}(t \cdot f(t)) = \int_0^\infty e^{-st}\, t \cdot f(t)\, dt = -\frac{d}{ds} \int_0^\infty e^{-st} f(t)\, dt = -\frac{d}{ds} F(s) = -F'(s)$

Állandó együtthatós lineáris DE

$\boxed{4^{*}_{\mathrm{VII}}}$

Példa: $y'(t)+3y(t)=5, \quad y(0)=6.$

Megoldás: $\big\downarrow \mathcal{L}$

$$\underbrace{\left(sY(s)-y(0)\right)}_{\mathcal{L}(y')}+3\underbrace{Y(s)}_{\mathcal{L}(y)}=5\cdot\underbrace{\frac{1}{s}}_{\mathcal{L}(1)}, \quad (s+3)Y(s)=\frac{5}{s}+6$$

$$(s+3)Y(s)=\frac{5}{s}+6, \qquad Y(s)=\underbrace{\frac{1}{s+3}}_{\substack{\text{transzfer}\\ \text{függvény}}}\Big(\underbrace{\frac{5}{s}+6}_{\mathcal{L}(\text{input})}\Big)$$

Megjegyzés: $y'(t)+3y(t)=5, y(0)=6, \ t>0$ elcserélhető erre:

$$y(t)=0, \text{ ha } t<0, \quad y'(t)+3y(t)=\underbrace{\theta(t)\cdot 5}_{\substack{\text{input pozitív}\\ t\text{-re}}}+\underbrace{6\cdot\delta(t)}_{y(0^+)-\underbrace{y(0^-)}_{=0}=y(0^+)=\underbrace{6\cdot 1}_{\substack{y(0)=6\\ \text{kezdeti feltétel}}}}$$

Tehát formálisan $\frac{5}{s}+6=\mathcal{L}\left(5+6\cdot\delta(t)\right)$,

hiszen $\mathcal{L}(\delta(t))=\int_{-0}^{\infty}e^{-st}\delta(t)\,dt=\lim_{a\to 0^+}\int_{0}^{\infty}e^{-st}\delta(t-a)\,dt=$

nem definiált

$$=\lim_{a\to 0^+}e^{-sa}=1$$

$$Y(s)=\frac{5+6s}{s(s+3)}=\frac{5}{3}\cdot\frac{1}{s}+\frac{13}{3}\underbrace{\frac{1}{3+s}}_{\frac{1}{s-(-3)}}$$

$\mathcal{L}^{-1}\big\downarrow$

$$y(t)=\frac{5}{3}\cdot 1+\frac{13}{3}e^{-3t}$$

Példa:

$$y'' + 4y' + 5y = 3 + t, \quad y(0) = 2, \; y'(0) = 7$$

Megoldás: $\quad Y(s) = \mathcal{L}(y(t))$

$$\left(s^2 Y(s) - s \cdot 2 - 7\right) + 4\left(sY(s) - 2\right) + 5Y(s) = \frac{3}{s} + \frac{1}{s^2}$$

$$Y(s) = \frac{1}{s^2 + 4s + 5}\left(\frac{3}{s} + \frac{1}{s^2} + [2s + 15]\right)$$

$$= -\frac{4}{25} \cdot \frac{1}{s} + \frac{1}{5} \cdot \frac{1}{s^2} + \left(\frac{27}{25} - \frac{631}{100} i\right) \cdot \frac{1}{s - (-2+i)} + \left(\frac{27}{25} + \frac{631}{100} i\right) \cdot \frac{1}{s - (-2-i)}$$

$$y(t) = -\frac{4}{25} \cdot 1 + \frac{1}{5} \cdot t + \left(\frac{27}{25} - \frac{631}{100} i\right) \cdot e^{(-2+i)t} + \left(\frac{27}{25} - \frac{631}{100} i\right) \cdot e^{(-2-i)t}$$

Megjegyzés:

$$y'' + 4y' + 5y = (3+t)\theta(t) + 2 \cdot \delta'(t) + 7\delta(t), \quad y(t) = 0 \text{ ha } t < 0$$

$$\mathcal{L}(\delta(t)) = 1, \quad \mathcal{L}(\delta'(t)) = \int_0^\infty e^{-st} \delta'(t) = -\frac{d}{dt} e^{-st}\Big|_{t=0} = s$$

Megjegyzés:

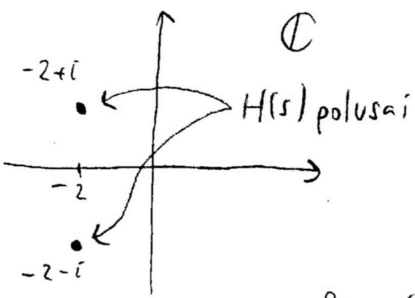
$y(t) = 0$, ha $t < 0$; $\quad \frac{d^2}{dt^2} y(t) = \delta'(t) \longrightarrow y(t) = \theta(t)$, hiszen $\left[\left[\theta(t)\right]'\right]' = \left[\delta(t)\right]' = \delta'(t)$

$y(t) = 0$, ha $t < 0$, $\quad \frac{d^2}{dt^2} y(t) = \delta(t) \longrightarrow y(t) = k(t)$, hiszen $k'' = \theta' = \delta$

$y'' = \delta$: $\quad y(0^-) = y(0^+), \; y'(0^+) - y'(0^-) = 1$

$y'' = \delta'$: $\quad y(0^+) - y(0^-) = 1$

Transzferfüggvény: $\quad H(s) = \dfrac{1}{s^2 + 4s + 5} = \dfrac{1}{(s - (-2-i))(s - (-2+i))}$



$\mathrm{Re}(-2+i) = \mathrm{Re}(-2-i) = -2 < 0$

stabil rendszer, $\lim\limits_{t \to \infty} e^{(-2 \pm i)t} = 0$

Rezgés körfrekvenciája: $\left|\mathrm{Im}(-2 \pm i)\right| = 1$,

frekvencia: $1 \cdot \dfrac{1}{2\pi} \approx 0.16$ Hertz

Green függvény — Transzfer függvény

$\boxed{6^{*}_{VIII}}$

$$y''(t)+4y'(t)+5y(t)=f(t)\cdot\Theta(t) \,\hat{=}\, f(t)$$

$y(t)=f(t)=0$, ha $t<0$ $\qquad$ $y(0)=y'(0)=0$

Retardált Green függvény

$$G(t)=\begin{cases}0, & t<0\\ e^{-2t}\sin t, & t>0\end{cases}$$

Transzfer függvény

$$H(s)=\frac{1}{s^2+4s+5}=\mathcal{L}(G(t))$$

Megoldás:

$$y(t)=\int_{-\infty}^{\infty}G(t-\tau)f(\tau)d\tau=(G*f)(t)$$

$$y(t)=\mathcal{L}^{-1}\left(H(s)\cdot \underbrace{F(s)}_{\mathcal{L}(f)}\right)$$

$G(t) \longleftrightarrow$ frekvencia válasz $\longleftrightarrow$ Transzfer függvény

$$\left(\frac{d^2}{dt^2}+4\frac{d}{dt}+5\right)G(t)=\delta(t)$$

$$\delta(t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{ipt}\cdot 1\,dp$$

$$G(t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{1}{-p^2+4ip+5}\cdot e^{ipt}dp$$

formális számolás!

$$\xrightarrow{p=-is}\frac{1}{-p^2+4ip+5}=\frac{1}{s^2+4s+5}$$

$$\left(\frac{d^2}{dt^2}+4\frac{d}{dt}+5\right)y(t)=f(t) \qquad \left(\frac{d^2}{dt^2}+4\frac{d}{dt}+5\right)y(t)=f(t)$$

$$\hat{y}(p)=\frac{1}{-p^2+4ip+5}\hat{f}(p) \quad ??? \qquad Y(s)=\frac{1}{s^2+4s+5}F(s)$$

Konvolúció a "t" időtartományon $\longleftrightarrow$ Szorzás a Fourier: frekvencia tartományon / Laplace: képzetes frek.

Konvolúció

Fourier transzformáció

$$\hat{f}(p)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ipt} f(t)\,dt, \qquad f(t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(p)\, e^{ipt}\,dp$$

Konvolúció: $(f*g)(t)=\int_{-\infty}^{\infty} f(t-\tau)\,g(\tau)\,d\tau$

Állítás: $\widehat{f*g}=\hat{f}\cdot\hat{g}\cdot\sqrt{2\pi}$

Bizonyítás:

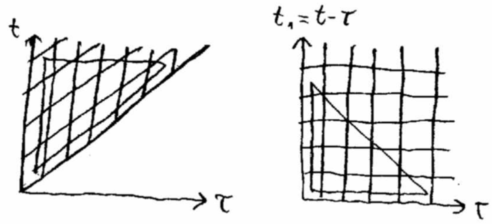
$$(\widehat{f*g})(p)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ipt}\left(\int_{-\infty}^{\infty} f(t-\tau)\,g(\tau)\,d\tau\right)dt \qquad \begin{pmatrix} t\\ \tau\end{pmatrix} \longleftrightarrow \begin{pmatrix} t-\tau\\ \tau\end{pmatrix}=\underbrace{\begin{pmatrix} 1 & -1\\ 0 & 1\end{pmatrix}}_{det=1}\begin{pmatrix} t\\ \tau\end{pmatrix}$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t_1)\,e^{-ipt_1}\cdot g(\tau)\,e^{-ip\tau}\,d\tau\,dt_1$$

$$=\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t_1)\,e^{-ipt_1}\,dt_1\right)\cdot\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(\tau)\,e^{-ip\tau}\,d\tau\right)\cdot\sqrt{2\pi}=\hat{f}(p)\cdot\hat{g}(p)\cdot\sqrt{2\pi}$$

Laplace tr:

$$(f*g)(t)=\int_0^t f(t-\tau)\,g(\tau)\,d\tau$$



Állítás: $\mathcal{L}(f*g)=\mathcal{L}(f)\cdot\mathcal{L}(g)$

Bizonyítás:

$$\mathcal{L}(f*g)=\int_0^{\infty} e^{-st}\left(\int_0^t f(t-\tau)\,g(\tau)\,d\tau\right)dt=\int_0^{\infty}\left(\int_0^t e^{-s(t-\tau)} f(t-\tau)\cdot e^{-s\tau} g(\tau)\,d\tau\right)dt$$

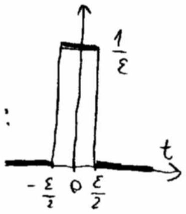
$$=\int_0^{\infty} e^{-st_1} f(t_1)\,dt_1\cdot\int_0^{\infty} e^{-s\tau} g(\tau)\,d\tau=\mathcal{L}(f)\cdot\mathcal{L}(g)$$

Megjegyzések:

8^{*}_{VIII}

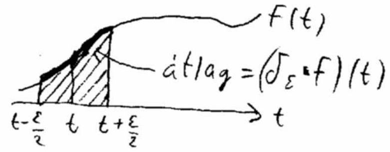
① $(f*g)(t)=\int_{-\infty}^{\infty} f(t-\tau)g(\tau)d\tau = \int_{-\infty}^{\infty} f(t_1)g(t-t_1)dt_1 = (g*f)(t) \qquad t_1=t-\tau$

② δ_ε:



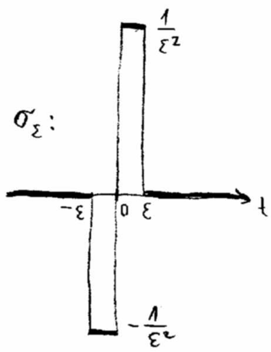
$$(\delta_\varepsilon * f)(t) = (f*\delta_\varepsilon)(t) = \int_{-\infty}^{\infty} f(t-\tau)\,\delta_\varepsilon(\tau)d\tau = \int_{-\varepsilon/2}^{\varepsilon/2} f(t-\tau)\cdot\frac{1}{\varepsilon}d\tau$$

$(\delta_\varepsilon * f)(t) = f$ átlagértéke a t körüli $\pm\frac{\varepsilon}{2}$ félszélességű intervallumon.



$\varepsilon\to 0,\ \delta_\varepsilon\to\delta, \qquad (\delta*f)(t) = (f*\delta)(t) = \int_{-\infty}^{\infty} f(t-\tau)\delta(\tau)d\tau = f(t), \quad \delta* = id$

σ_ε:



$$(\sigma_\varepsilon * f)(0) = (f*\sigma_\varepsilon)(0) = \frac{1}{\varepsilon}\left[\left(\int_0^{\varepsilon} f(t)\cdot\frac{1}{\varepsilon}dt\right) - \left(\int_{-\varepsilon}^{0} f(t)\cdot\frac{1}{\varepsilon}dt\right)\right]$$

$$\approx \frac{1}{\varepsilon}\left(f(\tfrac{\varepsilon}{2}) - f(-\tfrac{\varepsilon}{2})\right) \approx f'(0)$$

$(\delta' * f)(t) = (f*\delta')(t) = \int_{-\infty}^{\infty} f(t-\tau)\delta'(\tau)d\tau = -f'(t)$,

így $\lim_{\varepsilon\to 0^+}\sigma_\varepsilon = -\delta'$

③ Az $(f*g)(t)=\int_{-\infty}^{\infty} f(t-\tau)g(\tau)d\tau$ definíció asszimetrikusan kezeli f és g-t.

Alternatív def: Legyen $\{T_a \mid a\in\mathbb{R}\}$ az eltolások transzformációcsoportja: $T_a:\mathbb{R}\to\mathbb{R}$, $T_a(x)=x+a$

Így $T_aT_b = T_{a+b}$. Legyen $T(f) = \int_{-\infty}^{\infty} f(t)T_t\,dt$.

Ekkor $T(f)T(g) = \left(\int_{-\infty}^{\infty} f(t_1)T_{t_1}dt_1\right)\left(\int_{-\infty}^{\infty} g(t_2)T_{t_2}dt_2\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t_1)g(t_2)T_{t_1+t_2}\,dt_1\,dt_2$

$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t-t_2)g(t_2)\,dt_2\cdot T_t\,dt = \int_{-\infty}^{\infty}(f*g)(t)\cdot T_t\,dt$

Ugyanez elvégezhető: $T_{\vec a}:\mathbb{R}^n\to\mathbb{R}^n,\quad T_{\vec a}(\vec x) = \vec x + \vec a$

$T_{\vec a}:\mathbb{Z}^n\to\mathbb{Z}^n,\quad T_{\vec a}(\vec n) = \vec n + \vec a$

$T_\alpha: S^1\to S^1,\quad T_\alpha(\beta) = \beta+\alpha$ modulo 2π csoportokra is.

Pl. konvolúció a $\mathbb{Z}$-n értelmezett függvényeknél:

$\{a_n\}_{n=-\infty}^{\infty}$, $\{b_n\}_{n=-\infty}^{\infty}$ mindkét irányban végtelen sorozat.

$(a*b)_n = \sum_k a_{n-k}\cdot b_k$

Lineáris rendszer

$\boxed{9^{*}_{VIII}}$

$$\frac{d}{dt}\vec{y} = A\vec{y} + \vec{F}(t), \quad \vec{y}(0) = \vec{y}_0$$

$$s\vec{Y}(s) - \vec{y}_0 = A\vec{Y}(s) + \vec{F}(s)$$

$$\vec{Y}(s) = (sE-A)^{-1}\left(\vec{F}(s) + \vec{y}_0\right)$$

Hogyan számoljuk ki $(sE-A)^{-1}$-et?

$$A\vec{v}_i = \lambda_i \vec{v}_i, \quad U = [\vec{v}_1 \dots \vec{v}_n], \quad A = UDU^{-1}, \quad D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

$$(sE-A)^{-1} = (UsEU^{-1} - UDU^{-1})^{-1} = (U[sE-D]U^{-1})^{-1} = U[sE-D]^{-1}U^{-1}$$

$$= U \begin{bmatrix} \frac{1}{s-\lambda_1} & & 0 \\ & \ddots & \\ 0 & & \frac{1}{s-\lambda_n} \end{bmatrix} U^{-1}$$

Ha $\vec{f}(t) = \vec{F}(s) = 0$, akkor:

$$\vec{y}(t) = e^{tA}\vec{y}_0 = Ue^{tD}U^{-1}\vec{y}_0 = U \begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{bmatrix} U^{-1}\vec{y}_0$$

$$\vec{Y}(s) = U \begin{bmatrix} \frac{1}{s-\lambda_1} & & 0 \\ & \ddots & \\ 0 & & \frac{1}{s-\lambda_n} \end{bmatrix} U^{-1}\vec{y}_0 \qquad \mathcal{L}(e^{\lambda_n t}) = \frac{1}{s-\lambda_n}$$

Példa: $\frac{d}{dt}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}[f(t)], \quad f(t) = \begin{cases} 4, & \text{ha } t \in [1,2] \\ 0 & \text{amúgy} \end{cases}, \quad \vec{y}(0) = \begin{bmatrix} 7 \\ 8 \end{bmatrix}$

Megoldás: $\mathcal{L}(f) = \int_1^2 e^{-st} \cdot 4\,dt = \frac{4}{-s}(e^{-2s} - e^{-1\cdot s}) = \frac{4}{s}(e^{-s} - e^{-2s})$

$$\begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix} = \begin{bmatrix} s+2 & 0 \\ -2 & s+3 \end{bmatrix}^{-1} \begin{bmatrix} \frac{4}{s}(e^{-s} - e^{-2s}) + 7 \\ 8 \end{bmatrix} = \frac{1}{(s+2)(s+3)} \begin{bmatrix} s+3 & 0 \\ +2 & s+2 \end{bmatrix} \begin{bmatrix} \frac{4}{s}(e^{-s} - e^{-2s}) + 7 \\ 8 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} (s+2)^{-1} & 0 \\ 0 & (s+3)^{-1} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} \frac{4}{s}(e^{-s} - e^{-2s}) + 7 \\ 8 \end{bmatrix}$$

Összegzés

① Egységugrás válasz: $LG=\delta$, $LU=\theta$, $U'=G$.

Pl: $\left(\frac{d^2}{dt^2}+4\right)G(t)=\delta(t)\longrightarrow G(t)=\begin{cases}0, & t<0\\ \frac{1}{2}\sin(2t), & t>0\end{cases}$

$\left(\frac{d^2}{dt^2}+4\right)U(t)=\theta(t)\longrightarrow U(t)=\begin{cases}0, & t<0\\ \frac{1}{4}(1-\cos(2t))=\int_0^t\frac{1}{2}\sin(2\tau)d\tau, & t>0\end{cases}$

② Frekvenciaválasz:

$$\left(\frac{d}{dt^2}+4\right)\left(A(\omega)e^{-i\omega t}\right)=e^{-i\omega t}\longrightarrow A(\omega)=\frac{1}{-\omega^2+4}$$

③ Laplace transzformáció

$$[\mathcal{L}(f(t))](s)=\int_0^\infty e^{-st}f(t)dt=F(s)\qquad [\mathcal{L}^{-1}(F(s))](t)=\frac{1}{2\pi i}\int_{-i\infty+a}^{i\infty+a}F(s)e^{s\cdot t}ds=f(t)$$

$\mathcal{L}(1)=\frac{1}{s}$ $\quad\mathcal{L}(e^{at})=\frac{1}{s-a}$ $\quad\mathcal{L}(t^n)=\frac{n!}{s^{n+1}}$ $\quad\mathcal{L}(\theta(t-a)f(t-a))=e^{-sa}F(s)$ $\quad a>0$

$\mathcal{L}(\delta(t))=1$ $\quad\mathcal{L}(\cos(at))=\frac{s}{s^2+a^2}$ $\quad\mathcal{L}\left(\int_0^t f(\tau)d\tau\right)=\frac{1}{s}F(s)$ $\quad\mathcal{L}(f')=sF(s)-f(0)$

$\mathcal{L}(\delta'(t))=s$ $\quad\mathcal{L}(\sin(at))=\frac{a}{s^2+a^2}$ $\quad\mathcal{L}(t\cdot f(t))=-F'(s)$ $\quad\mathcal{L}(f'')=s^2F(s)-sF(0)-f'(0)$

④ DE megoldása

ⓐ $y'+2y=\theta(t-3)$, $y(0)=4\longrightarrow sY(s)-4+2\cdot Y(s)=\frac{e^{-3s}}{s}\longrightarrow Y(s)=\frac{1}{s+2}\left(4+\frac{e^{-3s}}{s}\right)\longrightarrow$

$\longrightarrow\mathcal{L}^{-1}\left(\frac{4}{s+2}\right)=4e^{-2t}$, $\mathcal{L}^{-1}\left(\frac{1}{s+2}\cdot\frac{e^{-3s}}{s}\right)=\mathcal{L}^{-1}\left(e^{-3s}\left[\frac{1/2}{s}-\frac{1/2}{s+2}\right]\right)=\frac{1}{2}\theta(t-3)-\frac{1}{2}\theta(t-3)\cdot e^{-2(t-3)}$

$\longrightarrow y(t)=4\cdot e^{-2t}+\frac{1}{2}\theta(t-3)[1-e^{-2(t-3)}]$

ⓑ $y''+2y'+y=(t+1)^2$, $y(0)=4$, $y'(0)=5\longrightarrow[s^2Y(s)-4s-5]+2[sY(s)-4]+Y(s)=\frac{2!}{s^3}+\frac{2}{s^2}+\frac{1}{s}\longrightarrow$

$\longrightarrow Y(s)=\frac{1}{s^2+2s+1}\left[4s+13+\frac{2}{s^3}+\frac{2}{s^2}+\frac{1}{s}\right]=\frac{\mathrm{poly}(s)}{(s+1)^2s^3}=\frac{A}{(s+1)^2}+\frac{B}{(s+1)}+\frac{C}{s^3}+\frac{D}{s^2}+\frac{E}{s}\longrightarrow$

$\rightarrow\mathcal{L}^{-1}\left(\frac{1}{(s+1)^2}\right)=\mathcal{L}^{-1}\left(-\left[\frac{1}{s+1}\right]'\right)=t\cdot e^{-1\cdot t}\longrightarrow y(t)=Ate^{-t}+Be^{-t}+\frac{C}{2}t^2+Dt+E$

ⓒ $\frac{d}{dt}\begin{bmatrix}y_1\\y_2\end{bmatrix}=\begin{bmatrix}5&2\\2&5\end{bmatrix}\begin{bmatrix}y_1\\y_2\end{bmatrix}+\begin{bmatrix}1\\t\end{bmatrix}$, $\vec{y}(0)=\begin{bmatrix}6\\8\end{bmatrix}\longrightarrow s\cdot\begin{bmatrix}Y_1(s)\\Y_2(s)\end{bmatrix}-\begin{bmatrix}6\\8\end{bmatrix}=\begin{bmatrix}5&2\\2&5\end{bmatrix}\begin{bmatrix}Y_1(s)\\Y_2(s)\end{bmatrix}+\begin{bmatrix}1/s\\1/s^2\end{bmatrix}$

$$\begin{bmatrix}Y_1(s)\\Y_2(s)\end{bmatrix}=\begin{bmatrix}s-5&-2\\-2&s-5\end{bmatrix}^{-1}\begin{bmatrix}6+1/s\\8+1/s^2\end{bmatrix}=\left[\boxed{\begin{matrix}1\\1\end{matrix}}\boxed{\begin{matrix}1\\-1\end{matrix}}\right]\begin{bmatrix}(s-7)^{-1}&0\\0&(s-3)^{-1}\end{bmatrix}\begin{bmatrix}1&1\\1&-1\end{bmatrix}^{-1}\begin{bmatrix}6+1/s\\8+1/s^2\end{bmatrix}$$

⑤ konvolúció:

Fourier: $(f*g)(t)=\int_{-\infty}^{\infty}f(t-\tau)g(\tau)d\tau$ $\qquad$ Laplace: $(f*g)(t)=\int_0^t f(t-\tau)g(\tau)d\tau$

Fourier tr. $\mathcal{F}$: $\mathcal{F}(f)=\hat{f}$, $\mathcal{F}(g)=\hat{g}$ $\qquad$ Laplace tr. $\mathcal{L}$:

$\mathcal{F}(f*g)=\sqrt{2\pi}\,\mathcal{F}(f)\cdot\mathcal{F}(g)$ $\qquad$ $\mathcal{L}(f*g)=\mathcal{L}(f)\cdot\mathcal{L}(g)$

Mintafeladatok

① keresd meg az $y''+9y=f(t)$ DE retardált impulzus és egységugrás válaszait!

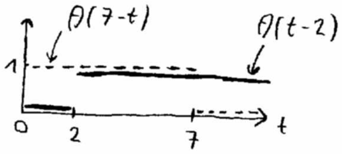
Megoldás: $G''+9G=\delta \longrightarrow G(t)=\begin{cases}0, \text{ ha } t<0\\ \frac{1}{3}\sin(3t), \text{ ha } t>0\end{cases}$

$$U''+9U=\theta \longrightarrow U'=G \rightarrow U(t)=\int_{-\infty}^{t} G(\tau)d\tau=\begin{cases}0, \text{ ha } t<0\\ \frac{1}{9}(1-\cos(3t)), \text{ ha } t>0\end{cases}$$

(1.b) keresd meg a frekvenciaválaszt is!

Megoldás: $\left[A(\omega)e^{-i\omega t}\right]''+9\left[A(\omega)e^{-i\omega t}\right]=e^{-i\omega t} \longrightarrow A(\omega)=\frac{1}{-\omega^2+9}$

② Mennyi $\mathcal{L}\left(\theta(t-2)\theta(7-t)\cdot e^{3t}\right)$?



Megoldás: $=\int_2^7 e^{-st}\cdot e^{3t}dt=\frac{1}{3-s}\left.e^{(3-s)t}\right|_2^7$

$$=\frac{1}{3-s}\left(e^{(3-s)\cdot 7}-e^{(3-s)\cdot 2}\right)=$$

③ $y'-5y=-7-15t$, $y(0)=3$. Mennyi $Y(s)$? Mennyi $Y(s)$ parciális tört felbontása? Mennyi $y(t)$?

Megoldás: $sY(s)-3-5Y(s)=-\frac{7}{s}-\frac{15}{s^2} \longrightarrow Y(s)=\frac{1}{s-5}\left(3-\frac{7}{s}-\frac{15}{s^2}\right)=\frac{3s^2-7s-15}{(s-5)s^2}=$

$$=\frac{A}{s-5}+\frac{B}{s^2}+\frac{C}{s} \longrightarrow Y(s)=\frac{1}{s-5}+\frac{3}{s^2}+\frac{2}{s} \longrightarrow y(t)=1\cdot e^{+5t}+3t+2$$

④ $y''-3y'+2y=8$, $y(0)=9$, $y'(0)=8$. Ismételd meg ③-at!

Megoldás: $\left[s^2Y(s)-9s-8\right]-3\left[sY(s)-9\right]+2Y(s)=\frac{8}{s} \longrightarrow Y(s)=\frac{1}{s^2-3s+2}\left[9s+19+\frac{8}{s}\right]=$

$$=\frac{9s^2+19s+8}{(s-1)(s-2)s}=\frac{A}{s-1}+\frac{B}{s-2}+\frac{C}{s} \longrightarrow Y(s)=\frac{2}{s-1}+\frac{3}{s-2}+\frac{4}{s} \longrightarrow y(t)=2\cdot e^{t}+3e^{2t}+4$$

⑤ $\frac{d}{dt}\begin{bmatrix}y_1\\y_2\end{bmatrix}=\begin{bmatrix}2&0\\-2&3\end{bmatrix}\begin{bmatrix}y_1\\y_2\end{bmatrix}+\begin{bmatrix}4\\5\end{bmatrix}$, $\vec{y}(0)=\begin{bmatrix}6\\7\end{bmatrix}$. Mennyi $\vec{Y}(s)$?

Megoldás: $s\begin{bmatrix}Y_1(s)\\Y_2(s)\end{bmatrix}-\begin{bmatrix}6\\7\end{bmatrix}=\begin{bmatrix}2&0\\-2&3\end{bmatrix}\begin{bmatrix}Y_1(s)\\Y_2(s)\end{bmatrix}+\begin{bmatrix}4/s\\5/s\end{bmatrix} \longrightarrow \begin{bmatrix}Y_1(s)\\Y_2(s)\end{bmatrix}=\begin{bmatrix}s-2&0\\2&s-3\end{bmatrix}^{-1}\begin{bmatrix}6+4/s\\7+5/s\end{bmatrix}=$

$$=\begin{bmatrix}1&0\\2&1\end{bmatrix}\begin{bmatrix}(s-2)^{-1}&0\\0&(s-3)^{-1}\end{bmatrix}\begin{bmatrix}1&0\\2&1\end{bmatrix}^{-1}\begin{bmatrix}6+4/s\\7+5/s\end{bmatrix}$$

$\left(\text{mert } \begin{bmatrix}2&0\\-2&3\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix}=2\cdot\begin{bmatrix}1\\2\end{bmatrix},\ \begin{bmatrix}2&0\\-2&3\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}=3\cdot\begin{bmatrix}0\\1\end{bmatrix}\right)$

⑥ $f(t)=t$, $g(t)=1-t$. Mennyi $(f*g)(t)$? Mennyi $\mathcal{L}(f*g)$?

$(f*g)(t)=\int_0^t (t-\tau)(1-\tau)d\tau=\int_0^t \tau^2-(1+t)\tau+t\,d\tau=\frac{1}{2}t^2-\frac{1}{6}t^3$. $\mathcal{L}(f*g)=\frac{1}{2}\cdot\frac{2!}{s^3}-\frac{1}{6}\cdot\frac{3!}{s^4}=$

$$=\left(\frac{1}{s^2}\right)\cdot\left(\frac{1}{s}-\frac{1}{s^2}\right)$$

⑦ $\theta(7-t)\theta(t-2)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(p)e^{ipt}dp$. Mennyi $\hat{f}(2)$?

$$\hat{f}(2)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-i\cdot 2\cdot t}\,\theta(7-t)\theta(t-2)dt=\frac{1}{\sqrt{2\pi}}\int_2^7 e^{-i\cdot 2\cdot t}\cdot 1\,dt=\frac{1}{\sqrt{2\pi}\cdot(-i\cdot 2)}\left(e^{-14i}-e^{-4i}\right)$$