Inhomogén lineáris rendszerek

$$\frac{d}{dt}\vec{y}(t) = A\vec{y}(t) + \vec{f}(t)$$

hom. lin: $\left.\begin{array}{l} \frac{d}{dt}\vec{y}_1(t) = A\vec{y}_1(t) \\ \frac{d}{dt}\vec{y}_2(t) = A\vec{y}_2(t) \end{array}\right\} \Longrightarrow \frac{d}{dt}\left(\alpha_1\vec{y}_1(t) + \alpha_2\vec{y}_2(t)\right) = A\left(\alpha_1\vec{y}_1(t) + \alpha_2\vec{y}_2(t)\right)$

Megoldások lineáris alteret alkotnak

inhom. lin: $\left.\begin{array}{l} \frac{d}{dt}\vec{y}_p = A\vec{y}_p + \vec{f} \\ \frac{d}{dt}\vec{y}_{hom} = A\vec{y}_{hom} \end{array}\right\} \Longrightarrow \frac{d}{dt}\left(\vec{y}_p + \vec{y}_{hom}\right) = A\left(\vec{y}_p + \vec{y}_{hom}\right) + \vec{f}$

Inhom. lin. általános megoldás = (Hom. lin ált. megold) + (egy inhom. lin. partikuláris megold.)

inhom. lin: Lineáris input-output reláció

$\left.\begin{array}{l} \frac{d}{dt}\vec{y}_1 = A\vec{y}_1 + \vec{f}_1 \\ \frac{d}{dt}\vec{y}_2 = A\vec{y}_2 + \vec{f}_2 \end{array}\right\} \Longrightarrow \frac{d}{dt}\left(\alpha_1\vec{y}_1 + \alpha_2\vec{y}_2\right) = A\left(\alpha_1\vec{y}_1 + \alpha_2\vec{y}_2\right) + \left(\alpha_1\vec{f}_1 + \alpha_2\vec{f}_2\right)$

input	output
$\vec{f}_1$	$\vec{y}_1$
$\vec{f}_2$	$\vec{y}_2$
$\alpha_1\vec{f}_1 + \alpha_2\vec{f}_2$	$\alpha_1\vec{y}_1 + \alpha_2\vec{y}_2$

Példa: rádioactív bomlás

$$\frac{d}{dt}y(t) = -2y(t) + f(t), \quad y(0) = y_0$$

Megoldás: $$y(t) = e^{-2t}y_0 + \int_0^t e^{-2(t-s)} f(s)\,ds$$

Bizonyítás: ① $y(0) = e^{-2\cdot 0}y_0 + \int_0^0 e^{-2(0-s)} f(s)\,ds = 1\cdot y_0 + 0 = y_0$ o.k.

② $\frac{d}{dt}y(t) = \frac{d}{dt}(e^{-2t})\cdot y_0 + \frac{d}{dt}\left[\int_0^t e^{-2(t-s)} f(s)\,ds\right]$

$= -2\cdot e^{-2t}y_0 + e^{-2(t-t)}f(t) + \int_0^t \frac{d}{dt}\left(e^{-2(t-s)}\right)\cdot f(s)\,ds$

$= -2e^{-2t}y_0 + f(t) + \int_0^t -2\cdot e^{-2(t-s)} f(s)\,ds =$

$= -2\left(e^{-2t}y_0 + \int_0^t e^{-2(t-s)} f(s)\,ds\right) + f(t) = -2y(t) + f(t)$ o.k.

Ugyanez általánosan:

$$\boxed{\begin{array}{l}\frac{d}{dt}\vec{y}(t) = A\vec{y}(t) + \vec{f}(t), \quad \vec{y}(0) = \vec{y}_0 \\ \text{Ekkor } \vec{y}(t) = e^{tA}\vec{y}_0 + \int_0^t e^{(t-s)A}\vec{f}(s)\,ds\end{array}}$$

$$\frac{d}{dt}\vec{y}(t) = \left(\frac{d}{dt}e^{tA}\right)\vec{y}_0 + \frac{d}{dt}\int_0^t e^{(t-s)A}\vec{f}(s)\,ds$$

$$= Ae^{tA}\vec{y}_0 + e^{(t-t)A}\vec{f}(t) + \int_0^t \frac{d}{dt}\left(e^{(t-s)A}\right)\cdot\vec{f}(s)\,ds$$

$$= Ae^{tA}\vec{y}_0 + \vec{f}(t) + \int_0^t Ae^{(t-s)A}\vec{f}(s)\,ds$$

$$= A\left(e^{tA}\vec{y}_0 + \int_0^t e^{(t-s)A}\vec{f}(s)\,ds\right) + \vec{f}(t) = A\vec{y}(t) + \vec{f}(t) \qquad \text{e.k.}$$

Egy variáns:

$$\vec{y}(-\infty) = \vec{0}, \quad \vec{y}(t) = \int_{-\infty}^t e^{(t-s)A}\vec{f}(s)\,ds = \int_{-\infty}^{\infty} G(t-s)\vec{f}(s)\,ds \qquad (t-s)+s=t$$

$$\text{ahol } G(t) = \begin{cases} e^{tA}, & \text{ha } t>0 \\ 0, & \text{ha } t<0\end{cases}$$

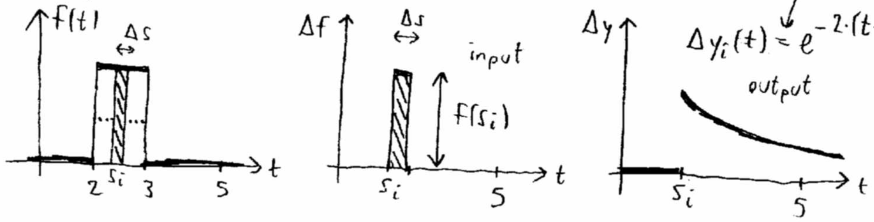
Tehát: Hom. DE $\longrightarrow e^{tA} \longrightarrow$ Inhom. DE megoldás

Feladat: $\frac{d}{dt}y = -2y + f(t)$, $y(0)=0$, $f(t) = \begin{cases} 1, & \text{ha } t\in[2,3] \\ 0 & \text{amúgy}\end{cases}$. Mennyi $y(5)$?

Megoldás:

$$y(5) = \int_0^5 e^{-2(5-s)}f(s)\,ds = \int_2^3 e^{-2(5-s)}\cdot 1\,ds = e^{-10}\cdot\int_2^3 e^{2s}\,ds = \frac{1}{2}\left(e^{-4} - e^{-6}\right)$$

Interpretáció:



ha $t > s_i$

$$\Delta y_i(t) = e^{-2\cdot(t-s_i)}\cdot f(s_i)\Delta s = G(t-s_i)f(s_i)\Delta s$$

$$y(t) \underset{\Delta s\to 0}{=} \sum_i \Delta y_i(t) = \sum_i e^{-2(t-s_i)}f(s_i)\Delta s = \int_0^t e^{-2(t-s)}f(s)\,ds = \int_{-\infty}^{\infty} G(t-s)f(s)\,ds$$

$= (G*f)(t)$. $\quad *$: konvolúció. $\quad$ Micsoda $G(t)$? (retardált Green függvény)

$G(t)$ tulajdonságai: $t\neq 0 \longrightarrow \frac{d}{dt}G(t) = -2G(t)$

$t<0 \longrightarrow G(t) = 0$

$t\approx 0 \longrightarrow G(0^+) = G(0^+) - G(0^-) = 1$

Mit old meg $G(t)$?

$f(s_i)\cdot\Delta s$ input $\longrightarrow$ output: $G(t-s_i)f(s_i)\Delta s$

$s_i=0$, $f(0)\Delta s$ input $\longrightarrow$ output: $G(t)f(0)\Delta s$

$f(0)\Delta s=1$ input $\longrightarrow$ output: $G(t)$

Tehát $G(t)$ a rendszer válasza egy olyan $f(t)$ bemenetre, amely f függvény csak a $t=0$ pillanatban nem nulla, de $\int_{-\infty}^{\infty} f(t)dt=1$. Ilyen $f(t)$ függvény persze nincs, de közelíthető pl. a $\delta_\varepsilon(t)=\begin{cases}\frac{1}{\varepsilon}, \text{ha } t\in[0,\varepsilon]\\ 0, \text{amúgy}\end{cases}$, $\varepsilon\to 0$ függvényekkel.

Ekkor megoldhatjuk a $\frac{d}{dt}G_\varepsilon(t)=\delta_\varepsilon(t)$, $G_\varepsilon(-\infty)=0$ DE-t, és azt kapjuk, hogy $\lim_{\varepsilon\to 0} G_\varepsilon(t)=\begin{cases}0, \text{ha } t<0\\ e^{-2t}, \text{ha } t>0\end{cases}$, tehát $\lim_{\varepsilon\to 0} G_\varepsilon(t)=G(t)$, ha $t\neq 0$.

Mivel G-t úgyis csak a $\int G(t-s)f(s)ds$ integrálban használjuk, így nem baj, hogy egy pontban ($t=0$) nincs definiálva.

Tehát $G(t)$ megoldása a képzeletbeli

$$\frac{d}{dt}G(t)=-2G(t)+\delta(t),\quad G(-\infty)=0 \quad \text{diff. egy.-nek,}$$

ahol a nemlétező Dirac-delta $\delta(t)$ függvény tulajdonságai:

$$\delta(t)=0, \text{ ha } t\neq 0,\quad \int_{-\infty}^{\infty}\delta(t)dt=1.$$

Kívánság: $\delta_\varepsilon\to\delta$ ahogy $\varepsilon\to 0$, de a limit legyen független a $\delta_\varepsilon(t)$ választásoktól.

Megoldás (Laurent Schwarz): (disztribúciók, általánosított függvények)

Legyen $\mathcal{D}=\{\varphi(x)\mid\varphi \text{ sima, nulla egy véges intervallumon kívül}\}$ (teszt függvények)

Bilineáris párosítás $\mathcal{D}$-n: $\langle\psi,\varphi\rangle=\int_{-\infty}^{\infty}\psi(x)\varphi(x)dx=L_\psi(\varphi)$.

Vagyis $L_\psi:\mathcal{D}\to\mathbb{C}$ lineáriáris leképezés $\mathcal{D}$-ből $\mathbb{C}$-be. Viszont nem minden $\mathcal{D}\to\mathbb{C}$ lin. leképezés írható fel ilyen alakban:

$\delta:\mathcal{D}\to\mathbb{C}$, $\delta(\varphi)=\varphi(0)$ lineáris, de $\delta\neq L_\psi$ bármely $\psi\in\mathcal{D}$-re.

Viszont $\langle\delta_\varepsilon,\varphi\rangle=\int_{-\infty}^{\infty}\delta_\varepsilon(t)\varphi(t)dt=\int_0^\varepsilon\frac{1}{\varepsilon}\varphi(t)dt\longrightarrow\varphi(0)$, tehát $\delta_\varepsilon\to\delta$, ha $\varepsilon\to 0$.

csalás, $\delta_\varepsilon\notin\mathcal{D}$

Tehát formálisan $\langle\delta_\varepsilon,\varphi\rangle\to\varphi(0)=\langle\delta,\varphi\rangle=\int_{-\infty}^{\infty}\delta(t)\cdot\varphi(t)dt$,

ahol $\delta(t)$ "tulajdonságai": $\delta(t)=0$, ha $t\neq 0$.

$$\int_{-\infty}^{\infty}\delta(t)dt=1.$$

Disztribúciók ($\mathcal{D}'$):

Lineáris leképezések $\mathcal{D} \to \mathbb{C}$ (lineáris funkcionálok)

Dirac-delta: $\delta(\varphi) = \varphi(0)$, formálisan $\delta(\varphi) = \langle \delta, \varphi \rangle = \int_{-\infty}^{\infty} \delta(t)\varphi(t)\,dt = \varphi(0)$

Deriválás: $\varphi, \psi \in \mathcal{D}$: $\langle \varphi, \psi' \rangle = \int \varphi(t)\psi'(t)\,dt = -\int \varphi'(t)\psi(t)\,dt = -\langle \varphi', \psi \rangle$.

Egy $f \in \mathcal{D}'$ deriváltjának a definíciója:

$$\langle f', \varphi \rangle = -\langle f, \varphi' \rangle \quad \text{bármely } \varphi\text{-re.}$$

Pl. $\langle \delta', \varphi \rangle = -\langle \delta, \varphi' \rangle = -\varphi'(0)$, $\langle \delta'', \varphi \rangle = -\langle \delta', \varphi' \rangle = \langle \delta, \varphi'' \rangle = \varphi''(0)$

$f \in \mathcal{D}' \longrightarrow f', f''$, stb, $\varphi f = \varphi(t) f(t)$ értelmes,

de $f \cdot g$ nem feltétlenül létezik.

Példa: $\delta_\varepsilon(t) = \begin{cases} \frac{1}{\varepsilon}, & t \in [0,\varepsilon] \\ 0 & \text{amúgy} \end{cases}$, $\tilde{\delta}_\varepsilon(t) = \begin{cases} 1/\varepsilon, & t \in [-\varepsilon, 0) \\ 0 & \text{amúgy} \end{cases}$.

$\delta_\varepsilon, \tilde{\delta}_\varepsilon \to \delta$, de $\delta_\varepsilon \cdot \tilde{\delta}_\varepsilon = 0$, viszont $\langle \delta_\varepsilon \cdot \delta_\varepsilon, \varphi \rangle = \int_0^\varepsilon \frac{1}{\varepsilon^2} \varphi(t)\,dt \to \varphi(0) \cdot \infty$

Viszont $\delta_2(x,y) = \delta(x)\delta(y) = \delta(\vec{r})$ értelmes:

$$\langle \delta_2, \varphi(x)\psi(y) \rangle = \iint \delta(x)\delta(y)\varphi(x)\psi(y)\,dx\,dy = \int \delta(x)\varphi(x)\,dx \cdot \int \delta(y)\psi(y)\,dy = \varphi(0)\cdot\psi(0) = (\varphi\cdot\psi)(0,0)$$

Most már tudunk nem deriválható függvényeket is deriválni:

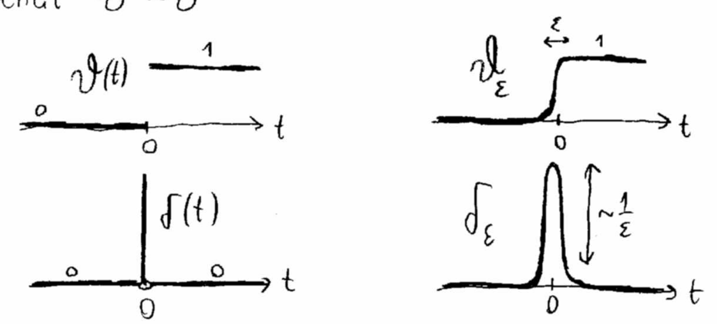
Heaviside theta, egységugrás: $\vartheta(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}$.

Állítás: $\vartheta'(t) = \delta(t)$.

Bizonyítás: $\langle \vartheta', \varphi \rangle = -\langle \vartheta, \varphi' \rangle = -\int_{-\infty}^{\infty} \vartheta(t)\varphi'(t)\,dt = -\int_0^\infty \varphi'(t)\,dt = -\varphi(t)\Big|_0^\infty$

$$= -\big(\varphi(\infty) - \varphi(0)\big) = \varphi(0) = \langle \delta, \varphi \rangle$$

Tehát $\vartheta' = \delta$

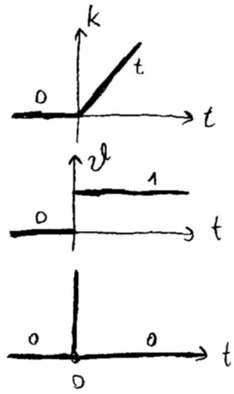


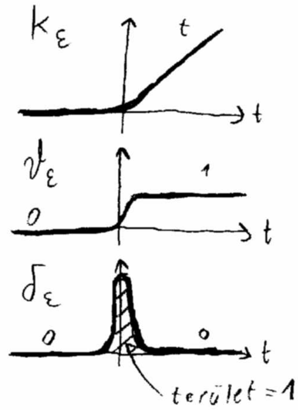
ϑ, δ és a $\vartheta_\varepsilon, \delta_\varepsilon$ közelítései vizuális képe.

$$k(t)=\begin{cases} t & t>0 \\ 0 & t<0 \end{cases}$$

$$\vartheta(t)=\begin{cases} 1 & t>0 \\ 0 & t<0 \end{cases}$$

$$\delta(t):\quad t\neq 0 \quad \delta(t)=0, \qquad \int_{-\infty}^{\infty}\delta(t)\,dt=1$$





5^*_{VIII}

$\varepsilon \to 0$

$$\boxed{k'=\vartheta, \quad \vartheta'=\delta, \quad k''=\delta}$$

Feladat: $\frac{d}{dt}y=-3y+f(t)$, $y(-\infty)=0$, $f(-\infty)=0$ $\quad (f(t)=0$, ha $t\ll 0)$

Megoldás:

(1) $y(t)=\int_{-\infty}^{t} e^{-3(t-s)}f(s)\,ds=\int_{-\infty}^{\infty}G(t-s)f(s)\,ds$, ahol $G(t)=\begin{cases} e^{-3t} & t>0 \\ 0 & t<0 \end{cases}$

(2) (a) Oldd meg: $\frac{d}{dt}G(t)=-3G(t)+\delta(t)$, $\quad G(-\infty)=0$

(1) $t\neq 0$: $\frac{d}{dt}G(t)=-3G(t) \longrightarrow G(t)=C\cdot e^{-3t}$ $\quad$ (c más lehet ha $t<0$, vagy $t>0$)

$G(-\infty)=0 \longrightarrow \boxed{G(t)=0, \text{ ha } t<0}$

(2) $t\approx 0$: $\frac{d}{dt}G(t)=-3G(t)+\delta(t)\approx\delta(t) \longrightarrow \left.\begin{array}{l} G(t)\approx\vartheta(t)+C \\ G(-\infty)=0 \end{array}\right\} \longrightarrow \boxed{G(0^+)=1}$

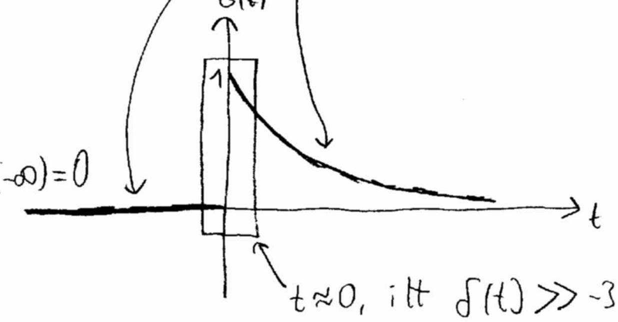
(3) $t>0$: $\frac{d}{dt}G(t)=-3G(t)$, $G(0^+)=1 \longrightarrow G(t)=e^{-3t}$

Tehát $G(t)=\begin{cases} 0, & \text{ha } t<0 \\ e^{-3t}, & \text{ha } t>0 \end{cases}$

Megjegyzés: $G'=-3G+\delta \iff$
$\iff \langle G',\varphi\rangle=-3\langle G,\varphi\rangle+\langle\delta,\varphi\rangle \quad \forall\varphi\in\mathcal{D}$
$\underbrace{-\langle G,\varphi'\rangle}$ $\quad -\int_0^\infty e^{-3t}\varphi'(t)\,dt=-3\int_0^\infty e^{-3t}\cdot\varphi(t)\,dt+\varphi(0)$

(b) Tehát
$$y(t)=\int_{-\infty}^{\infty}G(t-s)f(s)\,ds=\int_{-\infty}^{t}e^{-3(t-s)}f(s)\,ds$$

$t\neq 0$, itt $G(t)$ a homogén egyenlet megoldása



$G(-\infty)=0$

$G(0^+)-G(0^-)=\vartheta(0^+)-\vartheta(0^-)=1$,
így $G(0^+)=1$

$t\approx 0$, itt $\delta(t)\gg -3\cdot G(t)$, tehát $\frac{d}{dt}G(t)=-3G(t)+\delta(t)\approx\delta(t)$

$$\frac{d^2}{dt^2} y(t) = -4y(t) + f(t), \quad y(-\infty)=0, \quad f(t)=0, \text{ ha } t \ll 0$$

Megoldás:

① $$\frac{d^2}{dt^2} G(t) = -4G(t) + \delta(t), \quad G(t)=0, \text{ ha } t<0.$$

ⓐ $t \neq 0$, $G''(t) = -4G(t) \longrightarrow G(t) = C_1 \cos(2t) + C_2 \sin(2t)$

$(G(t)=0, \text{ ha } t<0) \longrightarrow G(0^-) = G'(0^-) = 0$

ⓑ $t \approx 0$: $G''(t) = -4G(t) + \delta(t) \approx \delta(t)$

$$K''(t) = \delta(t) \longrightarrow K(t) = (a+bt) + \begin{cases} t, & \text{ha } t>0 \\ 0, & \text{ha } t<0 \end{cases}$$

$$G \approx k, \; G(0^-) = G'(0^-) = 0 \longrightarrow k(t) = \begin{cases} t, & \text{ha } t>0 \\ 0, & \text{ha } t<0 \end{cases}$$

Tehát $G(0^+) = K(0^+) = 0$, $G'(0^+) = K'(0^+) = 1$

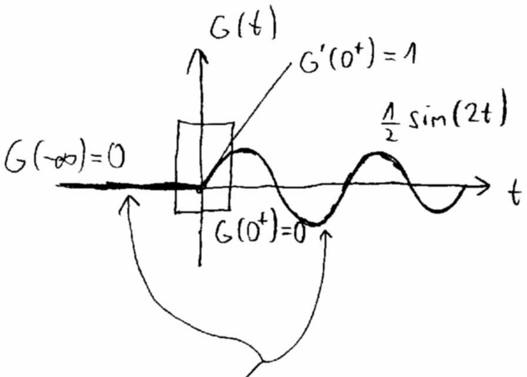
ⓒ $t>0$:

$$G(0^+)=0, \; G'(0^+)=1, \; G''(t) = -4G(t) \longrightarrow$$

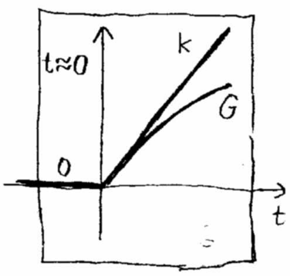
$$\longrightarrow G(t) = \begin{cases} 0, & \text{ha } t<0 \\ \frac{1}{2}\sin(2t), & \text{ha } t>0 \end{cases}$$

② Tehát

$$y(t) = \int_{-\infty}^{t} \frac{1}{2} \sin(2(t-s)) f(s) ds = \int_{-\infty}^{\infty} G(t-s) f(s) ds$$



$$\frac{d^2}{dt^2} G(t) = -4G(t), \quad \text{ha } t \neq 0$$



$K(t) = G(t) = 0$, ha $t<0$

$K(t) \approx G(t)$, ha $t>0$, $t \approx 0$

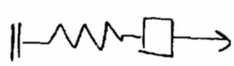
$K(0^-)=0$, $K(0^+) - K(0^-) = 0$

$K'(0^-)=0$, $K'(0^+) - K'(0^-) = 1$

$\longrightarrow K(0^+)=0$, $K'(0^+)=1$

$\longrightarrow G(0^+)=0$, $G'(0^+)=1$

$\longrightarrow G(t) = \frac{1}{2}\sin(2t)$, ha $t>0$

 $y'' + 4y = f(t)$

$$\frac{d}{dt}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix} \qquad \left(\text{esetünkben } f_1(t)=0,\ f_2(t)=f(t)\right)$$

$A = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix}$, $\det(A-\lambda E) = \lambda^2+4=0$, $\lambda_{1,2} = \pm 2i$, $\lambda_1 = 2i$, $\vec{v}_1 = \begin{bmatrix} 1 \\ 2i \end{bmatrix}$, $\lambda_2 = -2i$, $\vec{v}_2 = \begin{bmatrix} 1 \\ -2i \end{bmatrix}$

$$e^{tA} = \begin{bmatrix} 1 & 1 \\ 2i & -2i \end{bmatrix}\begin{bmatrix} e^{2it} & 0 \\ 0 & e^{-2it} \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 2i & -2i \end{bmatrix}^{-1} = \begin{bmatrix} \cos(2t) & \frac{1}{2}\sin(2t) \\ -2\sin(2t) & \cos(2t) \end{bmatrix}$$

① $y(t) = f(t) = 0$, ha $t \ll 0$

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \int_{-\infty}^{t} e^{(t-s)A}\vec{f}(s)\,ds = \int_{-\infty}^{t}\begin{bmatrix} \cos(2(t-s)) & \frac{1}{2}\sin(2(t-s)) \\ -2\sin(2(t-s)) & \cos(2(t-s)) \end{bmatrix}\begin{bmatrix} \overset{0}{f_1(s)} \\ \underset{f(t)}{f_2(s)} \end{bmatrix} ds$$

$$y(t) = y_1(t) = \int_{-\infty}^{t} \cos(2(t-s))\cdot 0 + \frac{1}{2}\sin(2(t-s))\cdot f(t)\,dt$$

$$\vec{y}(t) = \int_{-\infty}^{t} e^{(t-s)A}\vec{f}(s)\,ds = \int_{-\infty}^{\infty} G(t-s)\vec{f}(s)\,ds, \qquad G(t) = \begin{cases} 0, & \text{ha } t<0 \\ e^{tA}, & \text{ha } t>0 \end{cases}$$

$$G(t) = \left[\boxed{\begin{matrix} G_{11}(t) \\ G_{21}(t) \end{matrix}}\ \boxed{\begin{matrix} G_{12}(t) \\ G_{22}(t) \end{matrix}}\right]$$

$$\frac{d}{dt}G(t) = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix}G(t) + \delta(t)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \text{hiszen } G(0^+) - G(0^-) = E \longrightarrow \frac{d}{dt}G(t)\Big|_{t=0} = \delta(t)\cdot E$$

$$\frac{d}{dt}\begin{bmatrix} G_{11}(t) \\ G_{21}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix}\begin{bmatrix} G_{11}(t) \\ G_{21}(t) \end{bmatrix} + \begin{bmatrix} \delta(t) \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} G_{11}(0^+) \\ G_{21}(0^+) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\frac{d}{dt}\begin{bmatrix} G_{12}(t) \\ G_{22}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix}\begin{bmatrix} G_{12}(t) \\ G_{22}(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \delta(t) \end{bmatrix}, \qquad \begin{bmatrix} G_{12}(0^+) \\ G_{22}(0^+) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

② $\vec{y}(0) = \vec{y}_0$,

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = e^{tA}\vec{y}_0 + \int_0^t e^{(t-s)A}\vec{f}(s)\,ds = G(t)\vec{y}_0 + \int_0^{\infty} G(t-s)\vec{f}(s)\,ds$$

Megjegyzés: $[\vec{y}(0) = \vec{y}_0,\ t>0]$ elcserélhető: $[\vec{y}(-\infty) = \vec{0},\ \tilde{\vec{f}}(t) = \vartheta(t)\vec{f}(t) + \vec{y}_0\cdot\delta(t)]$

Megjegyzés: Ha $y''+4y=f$, akkor f y_2-t befolyásolja, míg elképzelhető, hogy az állapottér $(y,y') = (y_1,y_2)$ koordinátáiból csak $y=y_1$-et tudjuk direkt módon megmérni. Mátrix alakban:

$$\frac{d}{dt}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}[f(t)], \qquad [z_1(t)] = [1, 0]\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = [y(t)]$$

$$\frac{d}{dt}\vec{y} = A\vec{y} + B\vec{u}(t), \qquad \vec{z}(t) = C\vec{y}$$

B: input mátrix
C: output mátrix

$\frac{d}{dt}\vec{y} = A\vec{y} + \vec{F}(t)$, $\vec{F}(t) = \vec{y}(t) = \vec{0}$, ha $t << 0$.

Ekkor $\vec{y}(t) = \int_{-\infty}^{t} e^{(t-s)A}\vec{F}(s)ds = \int_{-\infty}^{\infty} G(t-s)\vec{F}(s)ds$

Minek G?

Legyen L eltolás invariáns differenciál operátor, $(LG)(\vec{x}) = \delta(\vec{x})$,

Ekkor $L\varphi(\vec{x}) = f(\vec{x})$ megoldása $\int d\vec{s}\, G(\vec{x}-\vec{s})f(\vec{s}) = \varphi(\vec{x})$

Példák: operátor	Green függvény (impulzusválasz)
$\partial_t + a$	$\theta(t)\, e^{-at}$
$\partial_t^2 + k^2$	$\theta(t)\frac{1}{k}\sin(kt)$
$\Delta = \partial_x^2 + \partial_y^2 + \partial_z^2$	$-\frac{1}{4\pi}(x^2+y^2+z^2)^{-1/2} = -\frac{1}{4\pi r}$
$\partial_t - \partial_x^2$	$\theta(t)\frac{1}{\sqrt{4\pi t}}e^{-x^2/4t}$

Feladat: $(\Delta\varphi)(x_1,x_2,x_3) = \exp(-(x_1^2+x_2^2+x_3^2)) = e^{-r^2}$. Adj meg egy φ megoldást!

Megoldás:

$$\varphi(x_1,x_2,x_3) = \iiint_{\mathbb{R}^3} ds_1 ds_2 ds_3 \, -\frac{1}{4\pi}\left((x_1-s_1)^2+(x_2-s_2)^2+(x_3-s_3)^2\right)^{-1/2} \cdot \exp\left(-(s_1^2+s_2^2+s_3^2)\right)$$

Feladat: $\partial_t\varphi = \partial_{xx}\varphi + f(t,x)$. Adj meg egy φ megoldást!

Megoldás:

$$(\partial_t - \partial_{xx})\left[\theta(t)\frac{1}{\sqrt{4\pi t}}e^{-x^2/4t}\right] = \delta(t)\delta(x)$$

$$\varphi(t,x) = \iint ds\, dy\; \theta(t-s)\frac{1}{\sqrt{4\pi(t-s)}}e^{-\frac{1}{4(t-s)}\cdot(x-y)^2}\cdot f(s,y)$$

Feladat: $\partial_t\varphi = \partial_{xx}\varphi$, $\varphi(0,x) = \frac{1}{1+x^2}$. Mennyi $\varphi(t,x)$, $t>0$?

Megoldás:

$$\varphi(t,x) = \int dy\; \theta(t-0)\frac{1}{\sqrt{4\pi(t-0)}}e^{-\frac{(x-y)^2}{4(t-0)}} = \int dy \frac{1}{\sqrt{4\pi t}}e^{-\frac{(x-y)^2}{4t}}$$

Időfüggő lineáris rendszer

$$\frac{d}{dt}\vec{y}(t) = A(t)\vec{y}(t) + \vec{F}(t), \quad \vec{y}(0) = \vec{y}_0.$$

① Számítsd ki a lineáris homogén rendszer evolúciós operátorát!

$$\vec{y}(t) = \Phi_{t,0}\,\vec{y}(0) \longrightarrow \frac{d}{dt}\vec{y}(t) = \left(\frac{d}{dt}\Phi_{t,0}\right)\cdot\vec{y}(0) = A(t)\,\Phi_{t,0}\,\vec{y}(0)$$

$$\longrightarrow \frac{d}{dt}\Phi_{t,0} = A(t)\,\Phi_{t,0}, \quad \Phi_{0,0} = E.$$

Megjegyzés: $\Phi_{t,0}$ ritkán számolható explicit módon ki.

Ha $[A(t), A(\tilde{t})] = 0$ bármely $t, \tilde{t}$-va, akkor $\Phi_{t,0} = \exp\left(\int_0^t A(\tilde{t})\,d\tilde{t}\right)$

② a $t_2 \leftarrow t_1$ evolúciós operátor:

$$\vec{y}(t_2) = \Phi_{t_2 t_1}\vec{y}(t_1), \qquad \Phi_{t_2,t_1} = \Phi_{t_2,0}\cdot\Phi_{0,t_1} = \Phi_{t_2,0}\,\Phi^{-1}_{t_1,0}$$

③ Inhomogén DE megoldása:

$$\frac{d}{dt}\vec{y}(t) = A(t)\vec{y}(t) + \vec{F}(t), \quad \vec{y}(0) = \vec{y}_0$$

$$\vec{y}(t) = \Phi_{t,0}\vec{y}_0 + \int_0^t \Phi_{t,s}\vec{F}(s)\,ds = \Phi_{t,0}\vec{y}_0 + \int_0^t \Phi_{t,0}\Phi^{-1}_{s,0}\vec{F}(s)\,ds$$

④ Green függvény:

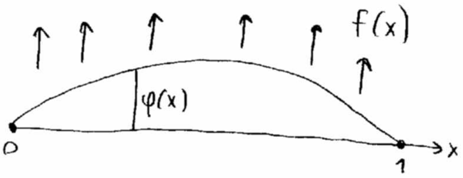
$$\frac{d}{dt}\vec{y}(t) = A(t)\vec{y}(t) + \vec{F}(t), \quad \vec{y}(t) = \vec{F}(t) = \vec{0}, \text{ ha } t \ll 0$$

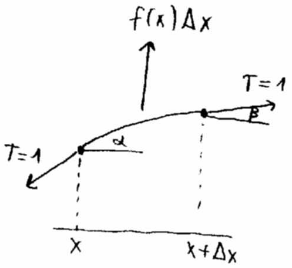
$$\vec{y}(t) = \int_{-\infty}^{t} \Phi_{t,s}\vec{F}(s)\,ds = \int_{-\infty}^{\infty} G(t,s)\vec{F}(s)\,ds$$

$$G(t,s) = 0, \text{ ha } t < s, \quad \left(\frac{\partial}{\partial t} - A(t)\right)G(t,s) = \delta(s-t)\,E, \quad G(t,s) = \Phi_{t,s}, \text{ ha } t > s.$$

1 dim Poissom egyenlet

Kifeszített húr





$\sin\alpha \approx tg\,\alpha = \varphi'(x)$

$\sin\beta \approx tg\,\beta = \varphi'(x+\Delta x)$

$\approx \varphi'(x) + \varphi''(x)\Delta x$

$f(x)\Delta x + T\cdot(\sin\beta - \sin\alpha) \approx 0$

$\approx (f(x) + 1\cdot\varphi''(x))\Delta x \approx 0$

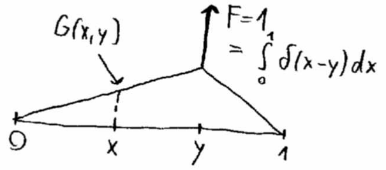
$\longrightarrow \boxed{\varphi''(x) = -f(x)}$

$\Delta\varphi = \partial_x^2\varphi$

Feladat: $\varphi''(x) = -f(x)$, $\varphi(0) = \varphi(1) = 0$. Mennyi φ?

Megoldás:

1. Green függvény $G(x,y)$:



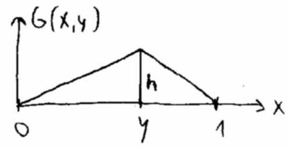
$G(x,y)$: Az y pontban ható egységnyi erő mekkora kitérést (választ) hoz létre az x pontban

a) $G(0,y) = G(1,y) = 0$

b) $\frac{d^2}{dx^2}G(x,y) = -\delta(x-y) \longrightarrow \frac{d^2}{dx^2}G(x,y) = 0$, ha $x\in[0,y)$, vagy $x\in(y,1]$

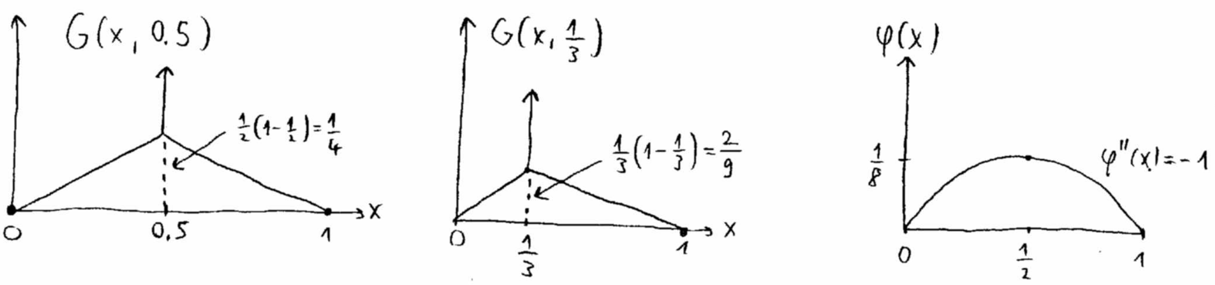
$\searrow G'(y+0,y) - G'(y-0,y) = -1$

$$\longrightarrow G(x,y) = \begin{cases} (1-y)x, & \text{ha } x<y \\ (1-x)y, & \text{ha } x>y \end{cases}$$



$\frac{h}{y} + \frac{h}{1-y} = 1 \rightarrow h = y(1-y)$

c) $$\varphi(x) = \int_0^1 G(x,y)\,f(y)\,dy$$



d) Ha $f(x) = 1$,

$$\varphi(x) = \int_0^1 G(x,y)\cdot 1\,dy = \int_0^x (1-x)y\,dy + \int_x^1 (1-y)x\,dy = \frac{1}{2}x(1-x)$$

Összegzés:

$$\frac{d}{dt}\vec{y}(t) = A\vec{y}(t) + \vec{F}(t)$$

ⓐ $\vec{y}(t) = \vec{F}(t) = \vec{0}$, ha $t \ll 0$

ⓑ $\vec{y}(0) = \vec{y}_0$, $t \geq 0$

Megoldás: ⓐ $\vec{y}(t) = \int_{-\infty}^{t} e^{(t-s)A}\vec{F}(s)ds = \int_{-\infty}^{\infty} G(t-s)\vec{F}(s)ds$

ⓑ $\vec{y}(t) = e^{tA}\vec{y}_0 + \int_0^t e^{(t-s)A}\vec{F}(s)ds = G(t)\vec{y}_0 + \int_0^{\infty} G(t-s)\vec{F}(s)ds$

Green függvény: $G(t) = \begin{cases} 0, & \text{ha } t<0 \\ e^{tA}, & \text{ha } t>0 \end{cases}$ $\quad \left(\frac{d}{dt} - A\right)G(t) = \delta(t)\cdot E$

Disztribúciók:

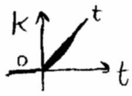
ⓐ $\mathcal{D}$ tesztfüggvények tere: sima, csak egy véges szakaszon nem nulla függvények

ⓑ $\mathcal{D}'$ disztribúciók: Lineáris leképezések: $\mathcal{D} \to \mathbb{C}$.

ⓒ Dirac-delta δ: $\delta(\varphi) = \varphi(0) = \langle \delta, \varphi \rangle = \int_{-\infty}^{\infty} \delta(t)\varphi(t)dt$

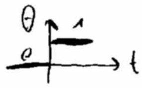
ⓓ $f \in \mathcal{D}' \to f' \in \mathcal{D}'$, $\langle f', \varphi \rangle = -\langle f, \varphi' \rangle = \int_{-\infty}^{\infty} f'(t)\varphi(t)dt = -\int_{-\infty}^{\infty} f(t)\varphi'(t)dt$

ⓔ $K(t) = \begin{cases} t, & t>0 \\ 0, & t<0 \end{cases}$



$K' = \theta$

$\theta(t) = \begin{cases} 1, & t>0 \\ 0, & t<0 \end{cases}$



$\theta' = \delta$

$\delta(t)$: $\delta(t) = 0$, ha $t \neq 0$; $\int_{-\infty}^{\infty} \delta(t)dt = 1$

$K'' = \delta$

Példák: ① $y' + 5y = F(t) \longrightarrow G'(t) + 5G(t) = \delta(t)$

$t<0 \to G(t) = 0$, $t \approx 0$: $G(t) \approx \theta(t)$, mivel $G'(t) = -5G(t) + \delta(t) \approx \delta(t)$

$G(0^+) - G(0^-) = 1 \to G(0^+) = 1$

$t>0$: $G(0^+) = 1$, $G'(t) + 5G(t) = 0 \to G(t) = \begin{cases} e^{-5t} & t>0 \\ 0 & t<0 \end{cases}$

② $y'' + 4y = F(t) \longrightarrow G''(t) + 4G(t) = \delta(t)$

$t<0 \to G(t) = 0$, $t \approx 0$: $G(t) \approx K(t)$, mivel $G''(t) = -4G(t) + \delta(t) \approx \delta(t)$

$G(0^+) - G(0^-) = 0$, $G'(0^+) - G'(0^-) = 1 \longrightarrow$

$\longrightarrow G(0^+) = 0$, $G'(0^+) = 1$, $t>0$: $G''(t) + 4G(t) = 0 \longrightarrow G(t) = \begin{cases} \frac{1}{2}\sin t & t>0 \\ 0 & t<0 \end{cases}$

Míntapéldák:

(1) $y'(t) = 3y(t) + f(t)$, $f(t) = \begin{cases} 4, \text{ ha } t \in [1,2] \\ 0 \text{ amúgy} \end{cases}$, $y(0) = 7$.

(a) Mennyi $y(8)$? (b) Mennyi a $G'(t) = 3G(t) + \delta(t)$ DE retardált megoldása?

(a) $y(t) = e^{3t} \cdot 7 + \int_0^t e^{3(t-s)} f(s)\,ds$

$y(8) = e^{3\cdot 8} \cdot 7 + \int_0^8 e^{3(8-s)} \cdot f(s)\,ds = e^{3\cdot 8} \cdot 7 + \int_1^2 e^{3(8-s)} \cdot 4\,ds$

(b) G retardált $\longrightarrow G(t) = 0$, ha $t < 0 \longrightarrow G(0^-) = 0$

$t \approx 0$ $G'(t) = 3G(t) + \delta(t) \approx \delta(t) \longrightarrow G(0^+) - G(0^-) = \theta(0^+) - \theta(0^-) = 1$

$\longrightarrow G(0^+) = 1$

$t > 0$ $G(0^+) = 1$, $G'(t) = 3G(t) \longrightarrow G(t) = \begin{cases} 0, \text{ ha } t < 0 \\ e^{3t}, \text{ ha } t > 0 \end{cases}$

Megoldás kifejezhető G-vel:

pl. $(y(t) = f(t) = 0$, ha $t \ll 0) \longrightarrow y(t) = \int_{-\infty}^t e^{3(t-s)} f(s)\,ds = \int_{-\infty}^{\infty} G(t-s) f(s)\,ds$

(2) $y''(t) = -9y(t) + f(t)$, $y(t) = f(t) = 0$, ha $t \ll 0$. Mennyi $y(t)$?

Mennyi a $G''(t) = -9G(t) + \delta(t)$ DE retardált megoldása?

(a) G retardált $\longrightarrow G(t) = 0$, ha $t < 0 \longrightarrow G(0^-) = G'(0^-) = 0$

$t \approx 0$ $G'' = -9G + \delta \approx \delta \longrightarrow G(t) \approx K(t) \longrightarrow G(0^+) = 0$, $G'(0^+) = 1 \longrightarrow$

$t > 0$ $G'' = -9G$, $G(0^+) = 0$, $G'(0^+) = 1 \longrightarrow G(t) = \begin{cases} 0, & t < 0 \\ \frac{1}{3}\sin(3t), & t > 0 \end{cases}$

(b) $y(t) = \int_{-\infty}^t \frac{1}{3}\sin(3(t-s)) f(s)\,ds = \int_{-\infty}^{\infty} G(t-s) f(s)\,ds$

(3) $\frac{d}{dt}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix}$. Oldd meg a $\frac{d}{dt} G(t) = \begin{bmatrix} -2 & 0 \\ 2 & -3 \end{bmatrix} + \delta(t)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ DE-t,

majd fejezd ki az $\begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$ kezdetiérték probléma megoldását G-vel!

Megoldás:

$$e^{tA} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} e^{-2t} & 0 \\ 0 & e^{-3t} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} e^{-2t} & 0 \\ 2e^{-2t} - 2e^{-3t} & e^{-3t} \end{bmatrix}$$

$$G(t) = \begin{cases} 0, \text{ ha } t < 0 \\ e^{tA}, \text{ ha } t > 0. \end{cases}$$

$$\vec{y}(t) = G(t)\begin{bmatrix} 4 \\ 5 \end{bmatrix} + \int_0^t G(t-s)\vec{F}(s)\,ds$$

$$= \begin{bmatrix} e^{-2t} & 0 \\ 2e^{-2t} - 2e^{-3t} & e^{-3t} \end{bmatrix}\begin{bmatrix} 4 \\ 5 \end{bmatrix} + \int_0^t \begin{bmatrix} e^{-2(t-s)} & 0 \\ 2e^{-2(t-s)} - 2e^{-3(t-s)} & e^{-3(t-s)} \end{bmatrix}\begin{bmatrix} f_1(s) \\ f_2(s) \end{bmatrix} ds$$