Wave equation

$$(\partial_t^2 - \partial_x^2)\,\varphi(x,t) = 0$$

① Plane wave: $\varphi = e^{i(kx - \omega t)} \longrightarrow |k| = |\omega| \longrightarrow$ velocity $= \pm 1$

② Travelling wave: $\varphi(x,t) = f(x - vt) \longrightarrow v^2 f'' - f'' = 0 \longrightarrow v^2 = 1 \longrightarrow v = \pm 1$

③ $(\partial_t^2 - \partial_x^2) = (\partial_t + \partial_x)(\partial_t - \partial_x)$, $\left.\begin{array}{c}(\partial_t + \partial_x)\varphi = 0 \\ \text{OR} \\ (\partial_t - \partial_x)\varphi = 0\end{array}\right\} \to (\partial_t^2 - \partial_x^2)\varphi = 0$, $\varphi = f(x-t) + g(x+t)$

④ Initial value problem

$$(\partial_t^2 - \partial_x^2)\varphi = 0, \quad \varphi(0,x) = F(x), \quad \dot\varphi(0,x) = G(x) \qquad (F(x) = G(x) = 0, \text{ if } |x| \gg 1)$$

Problem: Compute f, g, if $\varphi(t,x) = f(x-t) + g(x+t)$!

Solution:

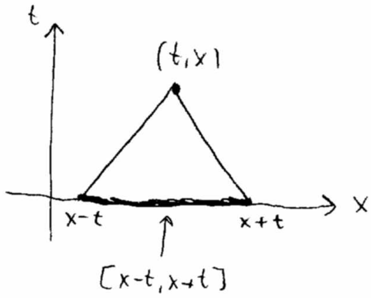
$$\left.\begin{array}{l} f(x) + g(x) = F(x) \\ -f'(x) + g'(x) = G(x) \\ -f(x) + g(x) = \int_{-\infty}^{x} G(y)\,dy \end{array}\right\} \longrightarrow \begin{array}{l} g(x) = \frac{1}{2}\left[F(x) + \int_{-\infty}^{x} G(y)\,dy\right] \\ f(x) = \frac{1}{2}\left[F(x) - \int_{-\infty}^{x} G(y)\,dy\right] \end{array}$$

if F, g solution, then
$f+c, g-c$ are solutions, too.

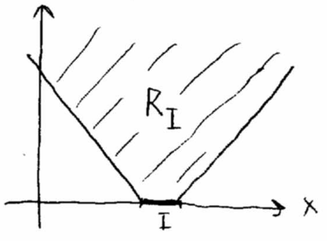
Consequently

$$\varphi(t,x) = f(x-t) + g(x+t) = \frac{1}{2}\left[F(x-t) - \int_{-\infty}^{x-t} G(y)\,dy\right] + \frac{1}{2}\left[F(x+t) + \int_{-\infty}^{x+t} G(y)\,dy\right]$$

$$= \frac{1}{2}\left[F(x-t) + F(x+t)\right] + \int_{x-t}^{x+t} G(y)\,dy = \frac{1}{2}\left[\varphi(0,x-t) + \varphi(0,x+t)\right] + \frac{1}{2}\int_{x-t}^{x+t} \dot\varphi(0,y)\,dy$$



$\varphi(t,x)$ depends only on
the values of $\varphi, \dot\varphi$
in the interval $[x-t, x+t]$

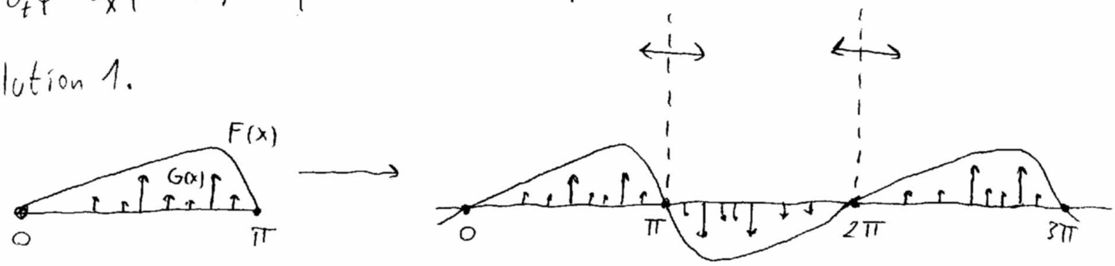


The initial conditions on I influenc the
value of $\varphi(t,x)$ only in the region R_I.

Vibration of a streched string

$$\partial_t^2\varphi - \partial_x^2\varphi = 0, \quad \varphi(0,x) = F(x), \quad \dot\varphi(0,x) = G(x), \quad \varphi(t,0) = \varphi(t,\pi) = 0$$

Solution 1.



The differential equation is invariant to the spatial reflections : $P_k : x - k\pi \longrightarrow -(x - k\pi)$

$P_k : G, F \longrightarrow -G, -F.$

Furthermore if φ is a solution of the PDE, then so is $-\varphi$.

The time evolution preserves the $\varphi(t,0) = \varphi(t,\pi) = 0$ conditions

Solution 2. Sine transform (Fourier series)

$$\varphi(t,x) = f_n(t)\cdot\sin(nx) \longrightarrow \ddot f_n(t) = -n^2 f_n(t) \longrightarrow f_n(t) = c_n\cdot\cos(nt) + s_n\cdot\sin(nt)$$

Sine-tr: $F(x) = \sum_{n=1}^{\infty} \hat F_n \sqrt{\frac{2}{\pi}} \sin(nx), \quad \hat F_n = \left(\sqrt{\frac{2}{\pi}}\sin(nx), F(x)\right) = \int_0^\pi \sqrt{\frac{2}{\pi}}\sin(nx) F(x)\,dx$

$G(x) = \sum_{n=1}^{\infty} \hat G_n \sqrt{\frac{2}{\pi}} \sin(nx), \quad \hat G_n = \left(\sqrt{\frac{2}{\pi}}\sin(nx), G(x)\right) = \int_0^\pi \sqrt{\frac{2}{\pi}}\sin(nx) G(x)\,dx$

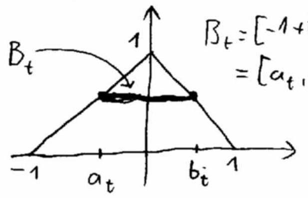
IF $\varphi(t,x) = \sum_{n=1}^{\infty} \left[c_n\cdot\cos(nt) + s_n\cdot\sin(nt)\right]\cdot\sqrt{\frac{2}{\pi}}\sin(nx)$

$\dot\varphi(t,x) = \sum_{n=1}^{\infty} \left[-n c_n\cdot\sin(nt) + n s_n\cdot\cos(nt)\right]\cdot\sqrt{\frac{2}{\pi}}\sin(nx)$, then

$c_n = \hat F_n, \quad s_n = \frac{\hat G_n}{n}, \quad$ so $\varphi(t,x) = \sum_{n=1}^{\infty}\left(\hat F_n\cos(nt) + \frac{\hat G_n}{n}\sin(nt)\right)\sqrt{\frac{2}{\pi}}\sin(nx)$

Propagation speed

$\varphi(t,x) = \frac{1}{2}\left[\varphi(0,x-t) + \varphi(0,x+t)\right] + \int_{x-t}^{x+t} \dot\varphi(0,y)\,dy \,]$, so $\varphi(t,x)$ depends only on the values of $\varphi(0,x), \dot\varphi(0,x)$ in the interval $[x-t, x+t]$



$B_t = [-1+t, 1-t] = [a_t, b_t]$

Theorem: Let $\varphi(0,x) = \dot\varphi(0,x) = 0$ if $x \in [-1,1]$. Then $\varphi(t,x) = 0$ if $x \in B_t$.

Proof: Let $E(t) = \frac{1}{2}\int_{B_t} \varphi_t^2 + \varphi_x^2\,dx$. Then

$$\frac{d}{dt}E(t) = \underbrace{\int_{B_t} \varphi_t\cdot\varphi_{tt} + \varphi_x\cdot\varphi_{xt}\,dx}_{\partial_x(\varphi_x\varphi_t) = \varphi_x\cdot\varphi_{xt} + \varphi_{xx}\varphi_t} - \frac{1}{2}\left[\left(\varphi_t^2(t,a_t) + \varphi_x^2(t,a_t)\right) + \left(\varphi_t^2(t,b_t) + \varphi_x^2(t,b_t)\right)\right]$$

$$\int_{B_t} \varphi_t\cdot\underbrace{(\varphi_{tt} - \varphi_{xx})}_{=0} + \partial_x(\varphi_x\varphi_t)\,dx = \int_{B_t} 0\,dx + \left[\varphi_x(t,b_t)\varphi_t(t,b_t) - \varphi_x(t,a_t)\varphi_t(t,a_t)\right]$$

$$\frac{d}{dt}E(t) = -\frac{1}{2}\left[\left(\varphi_t(t,a_t) - \varphi_x(t,a_t)\right)^2 + \left(\varphi_t(t,b_t) + \varphi_x(t,b_t)\right)^2\right] \leq 0$$

Since $E(t) \geq 0$, $E(0) = 0$, we obtain the desired $E(t) = 0$.

Remark: $\rho = \frac{1}{2}(\varphi_t^2 + \varphi_x^2)$ is the energy density, $\vec{j} = -\varphi_x\varphi_t$: energy density current.

$$\frac{\partial\rho}{\partial t} + \operatorname{div}\vec{j} = 0 = (\varphi_{tt}\cdot\varphi_t + \varphi_{xt}\cdot\varphi_x) - \partial_x(\varphi_x\varphi_t) = (\varphi_{tt}\varphi_t + \varphi_{xt}\cdot\varphi_x) - (\varphi_{xx}\varphi_t + \varphi_x\varphi_{xt})$$

$$\rightarrow \varphi_t\underbrace{(\varphi_{tt} - \varphi_{xx})}_{=0} = 0$$

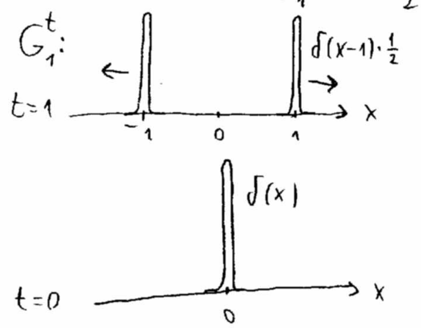
Continuity equation $\longrightarrow \frac{d}{dt}\int_{-\infty}^{\infty}\rho(t,x)\,dx = 0$

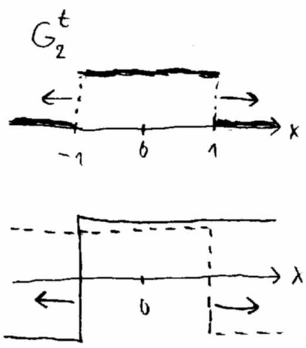
Remark: $\varphi(0,x) = F(x)$, $\dot\varphi(0,x) = V(x)$, $\varphi(t,x) = (G_1^t * F)(x) + (G_2^t * V)(x)$

where $G_1^t(x) = \frac{1}{2}\left(\delta(x+t) + \delta(x-t)\right)$, $G_2^t(x) = \frac{1}{2}\chi_{[-t,t]}(x)$

characteristic function of $[-t,t]$:

$$\chi(x) = \begin{cases} 1 & \text{if } x \in [-1,1] \\ 0 & \text{otherwise} \end{cases}$$





Huygens principle: $G_1^t(\vec{x})$ is concentrated at $|\vec{x}| = t$.

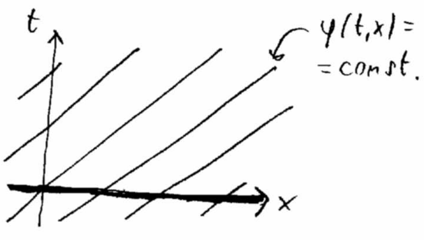
True in 1+1, 1+3 ... dimensions, false in 1+2, 1+4 .. dimensions

Characteristics

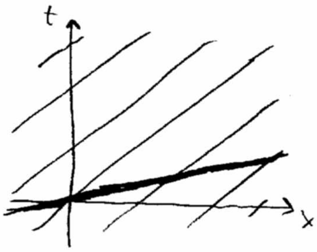
Where can we specify the initial conditions for a PDE?

Example (transport equation):

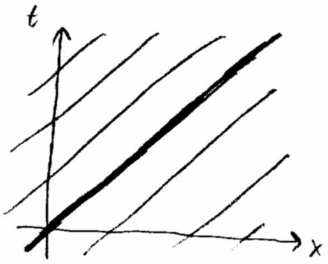
$$\partial_t \varphi(t,x) + \partial_x \varphi(t,x) = 0 \longrightarrow \varphi(t,x) = f(x-t)$$



$\varphi(0,x) = f_1(x)$ O.k.



$\varphi(\frac{1}{3}x, x) = f_2(x)$ O.k.



$\varphi(x,x) = f_3(x)$ Not O.k.
since $\varphi(x,x) = f(0) \neq f_3(x)$

How to read this off from the PDE?

L: lin. diff. op: $L = \sum_{|\alpha| \leq k} a_\alpha \partial^\alpha \longrightarrow$ characteristic form:

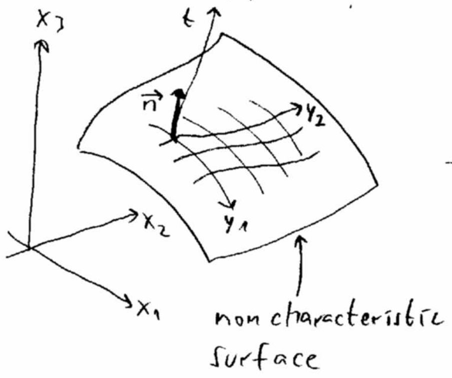
$$\chi_L(\vec{x}, \vec{p}) = \sum_{|\alpha| = k} a_\alpha(\vec{x}) \vec{p}^{\,\alpha}$$

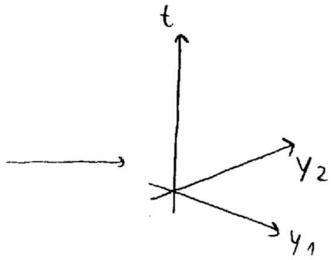
(multiindex notation: $\partial^{(1,2,5)} = \partial_{x_1}^1 \partial_{x_2}^2 \partial_{x_3}^5$, $\underbrace{(7,6,8)}_{\vec{p}}{}^{(1,2,5)} = 7^1 \cdot 6^2 \cdot 8^5$)

$\vec{p} \neq \vec{0}$ is a characteristic vector of L at $\vec{x}$, if $\chi_L(\vec{x}, \vec{p}) = 0$

A (hyper) surface is characteristic, if its normal vector is characteristic at any point. If that never happens, then the surface is non characteristic

Motivation: $\sum_{|\alpha| = k} a_\alpha(\vec{x}) \partial^\alpha \varphi + a(\vec{x}, \partial^\beta \varphi) = 0$, $|\beta| < k$





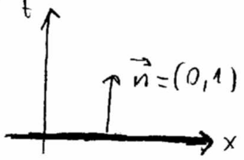
$$\partial_t^k \psi = b(t, \vec{y}, \partial_{\vec{y}}^\beta \partial_t^\ell \psi)$$

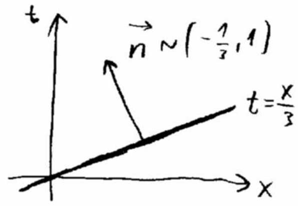
where $\ell < k$
$|\beta| + \ell \leq k$

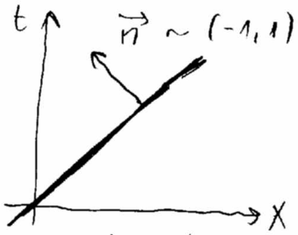
Cauchy-Kowalevskii:
b, $\psi^\ell(0, \vec{y})$, $\ell < k$ real analytic
$\longrightarrow$ locally exists analytic solution

Examples:

① $(\partial_t + \partial_x)\varphi = 0$, $L = 1\cdot\partial_t + 1\cdot\partial_x$, $\chi_L(\underbrace{(x,t)}_{\vec{x}}, \underbrace{(p_1,p_2)}_{\vec{p}}) = 1\cdot p_1 + 1\cdot p_2$







$\chi_L((x,t),(0,1)) = 1\cdot 0 + 1\cdot 1 \neq 0$

non char. surface

line ~ codim 1 surface on the plane

$\chi_L((x,t),(-\frac{1}{3},1)) = 1\cdot(-\frac{1}{3}) + 1\cdot 1 \neq 0$

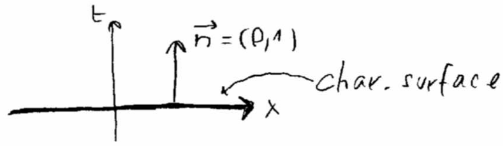
$\chi_L((x,t),(-1,1)) = 1\cdot(-1) + 1\cdot 1 = 0$

characteristic surface, not sure that we can specify the initial condition on it

② $\partial_t\varphi = \partial_{xx}\varphi$, $\partial_{xx}\varphi = \partial_t\varphi$, $L = [\partial_x\ \partial_t]\begin{bmatrix}1&0\\0&0\end{bmatrix}\begin{bmatrix}\partial_x\\\partial_t\end{bmatrix}$, $\chi_L((x,t),(p_1,p_2)) = [p_1\ p_2]\begin{bmatrix}1&0\\0&0\end{bmatrix}\begin{bmatrix}p_1\\p_2\end{bmatrix} = p_1^2$

(p_1,p_2) characteristic $\longleftrightarrow p_1^2 = 0$, $\vec{p} = (0,p_2)$

characteristic surface: $\vec{n}\sim(0,1)$

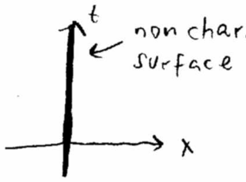


We run into problems, if we try to specify φ and φ_t on the $t=0$ char. surface

① if $\varphi(x,0)$ is given, $\varphi_t(x,0) = \partial_{xx}\varphi(x,0)$ is not arbitrary

② usually $\varphi(x,0)$ is given, solution exist only for $t \geqslant 0 \longrightarrow$ not analytic in t.

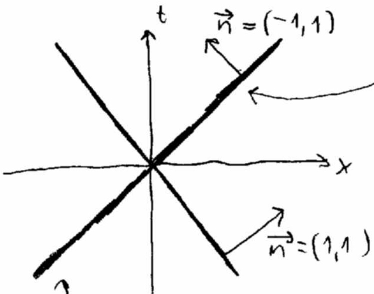
③



$\varphi_t = \varphi_{xx}$, $\varphi(0,t) = f(t)$, $\varphi_x(0,t) = g(t)$ has local analytic solution, but that sort of problem does not occur in real life.

③ $\partial_{tt}\varphi = \partial_{xx}\varphi$, $(\partial_{xx} - \partial_{tt})\varphi = 0$, $L = [\partial_x, \partial_t]\begin{bmatrix}1&0\\0&-1\end{bmatrix}\begin{bmatrix}\partial_x\\\partial_t\end{bmatrix}$, $\chi_L((x,t),(p_1,p_2)) = [p_1\ p_2]\begin{bmatrix}1&0\\0&-1\end{bmatrix}\begin{bmatrix}p_1\\p_2\end{bmatrix}$

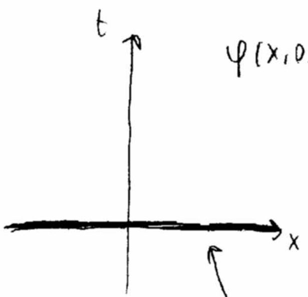
(p_1,p_2) char.: $\longleftrightarrow p_1^2 - p_2^2 = 0 \longleftrightarrow |p_1| = |p_2|$



$\varphi(x,x) = f(x)$, $-1\cdot\varphi_x'(x,x) + 1\cdot\varphi_t'(x,x) = g(x)$

illegitim initial condition

char. surface



$\varphi(x,0) = F(x)$, $\varphi_t'(x,0) = G(x)$

legitim init. cond.

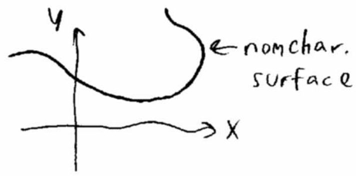
non char. surface

(4) $(\partial_{xx}+\partial_{yy})\varphi(x,y)=0$

$$L=[\partial_x\ \partial_y]\begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{bmatrix}\partial_x\\\partial_y\end{bmatrix}\qquad \chi_L\big((x,y),(p_1,p_2)\big)=[p_1\ p_2]\begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{bmatrix}p_1\\p_2\end{bmatrix}=p_1^2+p_2^2$$

$p_1^2+p_2^2>0$ if $\vec{p}\neq\vec{0}$, no nontrivial characteristic vector.

$\longrightarrow$ all surfaces are non characteristic.



Remark:

$\partial_t-\partial_{xx}$	parabolic	
$\partial_t^2-\partial_x^2$	hyperbolic	diff. operators
$\partial_x^2+\partial_y^2$	elliptic	

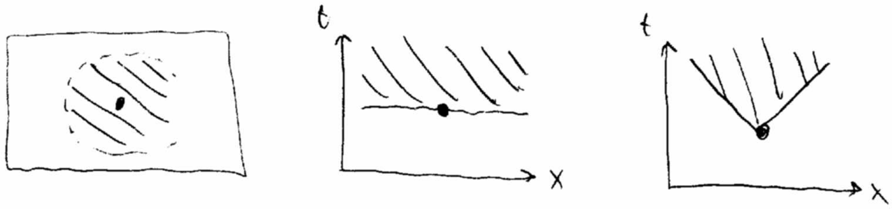
Their typical occurences:

parabolic, hyperbolic $\Big\}\longrightarrow$ evolution equations — ∞ / finite speed of propagation

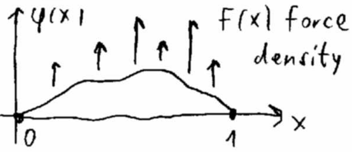
elliptic: static problems

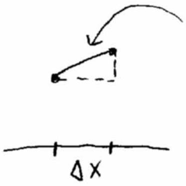
Regions of influence:

elliptic: — parabolic — hyperbolic



Calculus of variations

Example: 



length's increase: $\sqrt{\Delta x^2 + (\varphi' \Delta x)^2} - \sqrt{\Delta x^2}$

$\approx \Delta x \cdot \frac{1}{2} (\varphi')^2$ if $\varphi' \approx 0$,

since $\sqrt{1+x} \approx 1 + \frac{1}{2} x + \dots$

balance of forces:

$\varphi''(x) = -f(x)$, $\varphi(0) = \varphi(1) = 0$

Energy increase:

$$E[\varphi] = \int_0^1 \frac{1}{2} \left(\frac{d\varphi}{dx}\right)^2 - f(x)\varphi(x)\, dx$$

Two equivalent problems:

1. Solve $\varphi''(x) = -f(x)$, $\varphi(0) = \varphi(1) = 0$
2. Minimize $E[\varphi] = \int_0^1 \frac{1}{2} \left(\frac{d\varphi}{dx} \right)^2 - f(x)\varphi(x)\, dx$ with $\varphi(0) = \varphi(1) = 0$.

Minimalizaton: 1 dim

1. find critical point $h'(x_{crit}) = 0$
2. type of crit. point $h''(x_{crit}) < 0$ (max), >0 (min)

∞ dim:

$$S[\varphi] = \int_0^1 L(\varphi(x), \varphi'(x), x)\, dx, \quad \varphi(0) = a, \quad \varphi(1) = b$$

$\varphi_c(x)$ critical point of th $S[\varphi]$ functional:

$$S[\varphi_c + \delta\varphi] \approx S[\varphi_c] \quad \text{up to first order of } \delta\varphi$$

$$S[\varphi_c + \delta\varphi] = \int_0^1 L(\varphi_c + \delta\varphi, \varphi_c' + \delta\varphi', x)\, dx \approx S[\varphi_c] + \int_0^1 \delta\varphi(x) \cdot \frac{\partial L}{\partial \varphi}\Big|_{\varphi = \varphi_c} + \delta\varphi'(x) \frac{\partial L}{\partial \varphi'}\Big|_{\varphi=\varphi_c} dx$$

$$= S[\varphi_c] + \int_0^1 \left(\frac{\partial L}{\partial \varphi} - \frac{d}{dx} \frac{\partial L}{\partial \varphi'} \right)\Big|_{\varphi = \varphi_c} dx + \underbrace{\underbrace{\delta\varphi(x)}_{=0, \text{ if } x = 0 \text{ or } 1} \cdot \frac{\partial L}{\partial \varphi'} \Big|_0^1}_{=0}$$

So φ_c critical point $\longrightarrow$ φ_c satisfies the $\frac{d}{dx} \frac{\partial L}{\partial \varphi'} - \frac{\partial L}{\partial \varphi} = 0$

Euler-Lagrange equation

$\vec{\varphi}(x)$: $\boxed{\frac{d}{dx} \frac{\partial L}{\partial \varphi_i'} - \frac{\partial L}{\partial \varphi_i}}$ $\quad i = 1 \dots \dim \vec{\varphi}$

Field theory: $S = \int_{\mathbb{R}^n} L(\varphi, \partial_i \varphi, \vec{x})\, d\vec{x}$

$$\frac{\partial}{\partial x_i} \frac{\partial L}{\partial(\partial_i \varphi)} - \frac{\partial L}{\partial \varphi} = 0$$

Examples:

① $L(\varphi, \varphi', x) = \frac{1}{2}\varphi'^2 - f(x)\varphi(x)$

$$\frac{\partial L}{\partial \varphi} = -f(x), \quad \frac{\partial L}{\partial \varphi'} = \frac{1}{2}\cdot 2\varphi' = \varphi'$$

EL: $\frac{d}{dx}\varphi' - (-f) = 0, \quad \frac{d^2}{dx^2}\varphi(x) + f(x) = 0$ Poisson equation

② $L(\varphi, \varphi', x) = (\varphi')^3 \cdot e^x - \varphi' \cdot \varphi^2 + \varphi^3 + \sin x$ (nonsens example)

$$\frac{\partial L}{\partial \varphi'} = 3(\varphi')^2 \cdot e^x - \varphi^2 \qquad \frac{\partial L}{\partial \varphi} = -\varphi' \cdot 2\varphi + 3\varphi^2$$

EL: $\frac{d}{dx}\left[3(\varphi')^2 \cdot e^x - \varphi^2\right] - \left[-\varphi' \cdot 2\varphi + 3\varphi^2\right] = 0$

$$\left[6\varphi'\varphi'' \cdot e^x + 3\varphi'^2 \cdot \underbrace{e^x}_{\frac{d}{dx}e^x} - 2\varphi \cdot \varphi'\right] - \left[-\varphi' \cdot 2\varphi + 3\varphi^2\right] = 0$$

③ $L(x(t), \dot{x}(t), t) = \frac{1}{2}\dot{x}^2 - V(x)$ = kinetic − potential energy

$$\frac{\partial L}{\partial x} = -V'(x) \qquad \frac{\partial L}{\partial \dot{x}} = \dot{x}$$

EL: $\frac{d}{dt}\dot{x} - (-V'(x)) = 0 \qquad \frac{d^2}{dt^2}x(t) = -\frac{\partial V(x(t))}{\partial(x(t))}$

gradient of the potential energy

④ $L(x_1, x_2, \dot{x}_1, \dot{x}_2, t) = \sqrt{\dot{x}_1^2 + \dot{x}_2^2}$ geodetic curve (straight line) between $\vec{x}(0) = \vec{a}$ and $\vec{x}(1) = \vec{b}$

$$\frac{\partial L}{\partial x_i} = 0, \quad \frac{\partial L}{\partial \dot{x}_i} = \frac{2\dot{x}_i}{2\sqrt{\dot{x}_1^2 + \dot{x}_2^2}} \qquad i = 1, 2$$

EL:
$$\left.\begin{aligned} \frac{d}{dt}\left(\frac{\dot{x}_1}{\sqrt{\dot{x}_1^2 + \dot{x}_2^2}}\right) - 0 = 0 &= \frac{1}{(\dot{x}_1^2 + \dot{x}_2^2)^{3/2}}\dot{x}_2\left(-\dot{x}_1\ddot{x}_2 + \dot{x}_2\ddot{x}_1\right) \\ \frac{d}{dt}\left(\frac{\dot{x}_2}{\sqrt{\dot{x}_1^2 + \dot{x}_2^2}}\right) - 0 = 0 &= \frac{1}{(\dot{x}_1^2 + \dot{x}_2^2)^{3/2}}\dot{x}_1\left(\dot{x}_1\ddot{x}_2 - \dot{x}_2\ddot{x}_1\right) \end{aligned}\right\} \rightarrow \dot{\vec{x}} \sim \ddot{\vec{x}}$$

solution: $\vec{x}(t)$ is a straight line, arbitrary parametrization.

Remark: $S[\vec{r}(t)] = \int |\dot{\vec{r}}(t)|\,dt$ arclength is invariant with respect to a $\vec{r}(t) \rightarrow \vec{r}(f(t))$ reparametrization. So there is no unique solution of the EL eq., but all solutions give the same geodesic.

(5) $L(x_1, x_2, \dot{x}_1, \dot{x}_2, t) = \frac{1}{2}(\dot{x}_1^2 + \dot{x}_2^2)$

$$\frac{\partial L}{\partial x_1} = \frac{\partial L}{\partial x_2} = 0 , \quad \frac{\partial L}{\partial \dot{x}_1} = \dot{x}_1 , \quad \frac{\partial L}{\partial \dot{x}_2} = \dot{x}_2$$

$$EL: \frac{d}{dt}\dot{x}_1 - 0 = 0 , \quad \frac{d}{dt}\dot{x}_2 - 0 = 0 , \qquad \ddot{x}_1 = 0 , \quad \ddot{x}_2 = 0$$

constant velocity motion

Remark: generates the same curves as (4) did, but now the parametrization is proportional to the arc length.

(6) $L(\vec{x}, \dot{\vec{x}}, t) = \frac{1}{2}\sum_{ij} g_{ij}(\vec{x})\, \dot{x}_i \dot{x}_j = \frac{1}{2} \dot{\vec{x}}^T g \dot{\vec{x}}$

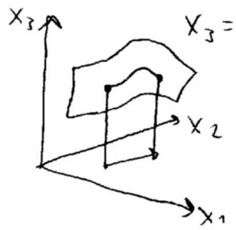
g_{ij}: metric tensor
Euclidean space: $g_{ij} = \delta_{ij}$

$$\frac{\partial L}{\partial \dot{x}_i} = \sum_j g_{ij}(\vec{x}) \cdot \dot{x}_j \qquad \frac{\partial L}{\partial x_i} = \frac{1}{2}\sum_{kl} \frac{\partial g_{kl}}{\partial x_i} \dot{x}_k \dot{x}_l$$

$$EL: \quad \frac{d}{dt}\left(\sum_j g_{ij}\dot{x}_j\right) - \frac{1}{2}\sum_{kl}\frac{\partial g_{kl}}{\partial x_i}\dot{x}_k\dot{x}_l = 0$$

$$\sum_j g_{ij}\ddot{x}_j + \sum_{kj}\frac{\partial g_{ij}}{\partial x_k} \cdot \dot{x}_k \dot{x}_j - \frac{1}{2}\sum_{kl}\frac{\partial g_{kl}}{\partial x_i}\dot{x}_k\dot{x}_l = 0$$

Remark: metric tensor of a 2dim surface



$x_3 = f(x_1, x_2)$ $\quad \Delta x_1^2 + \Delta x_2^2 + \Delta x_3^2 \approx \Delta x_1^2 + \Delta x_2^2 + \left[(f'_{x_1} \cdot \Delta x_1) + (f'_{x_2} \cdot \Delta x_2)\right]^2$

$$= [\Delta x_1, \Delta x_2] \underbrace{\begin{bmatrix} 1 + (f'_{x_1})^2 & f'_{x_1} \cdot f'_{x_2} \\ f'_{x_2} \cdot f'_{x_1} & 1 + (f'_{x_2})^2 \end{bmatrix}}_{g} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix}$$

(7) Monocycle (one wheel bicycle) $\qquad$ State space: (x, y, α)

$$S = \int_0^1 \frac{1}{2}\dot{x}^2 + \frac{1}{2}\dot{y}^2 + \frac{1}{2}\dot{\alpha}^2\, dt, \quad \text{constraint: } (\dot{x}, \dot{y}) \sim (\cos\alpha, \sin\alpha) \longleftrightarrow -\sin\alpha \cdot \dot{x} + \cos\alpha \cdot \dot{y} = 0$$

$$S = \int_0^1 \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{\alpha}^2) + \lambda(t)(-\sin\alpha \cdot \dot{x} + \cos\alpha \cdot \dot{y})\, dt$$

EL: "λ": $\quad -\sin\alpha \cdot \dot{x} + \cos\alpha \cdot \dot{y} = 0 \qquad$ constraint, λ: Lagrange multiplier

"x": $\quad \frac{d}{dt}(\dot{x} - \lambda\sin\alpha) - 0 = 0$

"y": $\quad \frac{d}{dt}(\dot{y} + \lambda\cos\alpha) - 0 = 0$

"α": $\quad \frac{d}{dt}\dot{\alpha} - \lambda(-\cos\alpha \cdot \dot{x} - \sin\alpha \cdot \dot{y}) = 0$

(8) Magnetic field

$$L = \frac{1}{2}(\dot x_1^2 + \dot x_2^2) + A_1 \dot x_1 + A_2 \dot x_2, \qquad A_i = A_i(x_1, x_2)$$

$$\frac{\partial L}{\partial \dot x_1} = \dot x_1 + A_1, \quad \frac{\partial L}{\partial \dot x_2} = \dot x_2 + A_2, \quad \frac{\partial L}{\partial x_1} = \frac{\partial A_1}{\partial x_1}\dot x_1 + \frac{\partial A_2}{\partial x_1}\cdot \dot x_2, \quad \frac{\partial L}{\partial x_2} = \frac{\partial A_1}{\partial x_2}\dot x_1 + \frac{\partial A_2}{\partial x_2}\cdot \dot x_2$$

EL: $$\frac{d}{dt}(\dot x_1 + A_1) - \left(\frac{\partial A_1}{\partial x_1}\dot x_1 + \frac{\partial A_2}{\partial x_1}\dot x_2\right) = 0 = \left(\ddot x_1 + \frac{\partial A_1}{\partial x_1}\dot x_1 + \frac{\partial A_1}{\partial x_2}\dot x_2\right) - \left(\frac{\partial A_1}{\partial x_1}\dot x_1 + \frac{\partial A_2}{\partial x_1}\dot x_2\right)$$

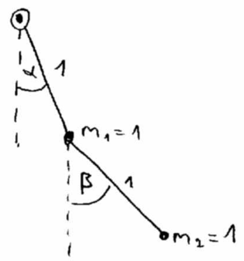
$$\ddot x_1 = \left(\frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2}\right)\cdot \dot x_2$$

$$\ldots \quad \ddot x_2 = -\left(\frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2}\right)\cdot \dot x_1$$

3d: $$\ddot{\vec X} = \underbrace{\vec B \times \dot{\vec X}}_{\text{Lorentz force}}, \quad \vec B = \mathrm{rot}\, \vec A$$

(9) Double pendulum

State space: $S^1 \times S^1 \times \mathbb{R} \times \mathbb{R}$ $(\alpha, \beta, \dot\alpha, \dot\beta)$



Potential energy:

$$V(\alpha, \beta) = -\left[\cos(\alpha) + (\cos(\alpha) + \cos(\beta))\right]$$

Kinetic energy:

$$T = \frac{1}{2}\vec V_1^2 + \frac{1}{2}\vec V_2^2$$

$$\vec V_1 = \frac{d}{dt}[\cos\alpha, \sin\alpha], \quad \vec V_2 = \frac{d}{dt}[\cos\alpha + \cos\beta, \sin\alpha + \sin\beta]$$

$$L(\alpha, \beta, \dot\alpha, \dot\beta) = T - V = \dot\alpha^2 + \frac{1}{2}\dot\beta^2 + \cos(\alpha - \beta)\dot\alpha\dot\beta + 2\cos\alpha + \cos\beta$$

$$EL_\alpha: \ 2\ddot\alpha + \cos(\alpha-\beta)\ddot\beta + \sin(\alpha-\beta)\dot\beta^2 + 2\sin\alpha = 0$$

$$EL_\beta: \ \ddot\beta + \cos(\alpha-\beta)\ddot\alpha - \sin(\alpha-\beta)\dot\alpha^2 + \sin\beta = 0$$

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Finite elements method FEM

Problem:

(1) Solve it: $\varphi''(x) = -f(x)$, $\varphi(0)=\varphi(1)=0$

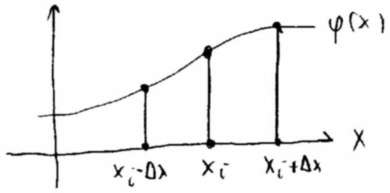
(2) Equivalent problem: Calculus of variations, minimalization

Find the minimum of $\int_0^1 \frac{1}{2}[\varphi'(x)]^2 - f(x)\varphi(x)\,dx$, $\varphi(0)=\varphi(1)=0$

Euler-Lagrange: $S[\varphi] = \int_0^1 \frac{1}{2}\varphi'^2 - f\varphi\,dx \longleftrightarrow \frac{d}{dx}\varphi' + f = 0$

Numerical, approximate solution:

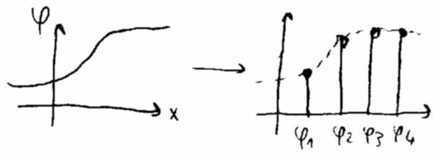
(1) Finite differences:



$\varphi''(x_i) \approx \frac{1}{\Delta x^2}\left(\varphi(x_i+\Delta x) - 2\varphi(x_i) + \varphi(x_i-\Delta x_i)\right) \approx -f(x_i)$

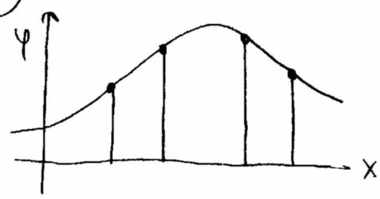
Problem: $\varphi \in \mathcal{F}un([0,1]) \longleftrightarrow \vec{\varphi} \in \mathbb{R}^n$

∞ dim $\longleftrightarrow$ n dim

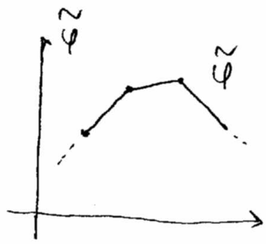


$\varphi(x) \longleftrightarrow \vec{\varphi} = [\varphi_1, \varphi_2, \varphi_3, \varphi_4]$

(2) FEM



$\longleftrightarrow$



$\varphi \in \mathcal{F}un([0,1])$

∞ dim

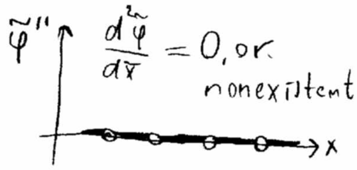
$\tilde{\varphi} \in V \subset \mathcal{F}un([0,1])$

finitedim $\qquad$ ∞ dim

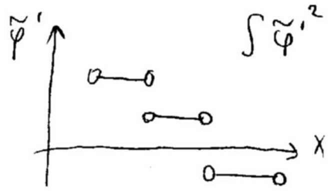
PDE: $\varphi'' = -f$ $\qquad$ $\tilde{\varphi}'' = -\tilde{f}$ nonsense

Minimalization:

$\underset{\varphi \in \mathcal{F}un}{\text{Min}} \int \frac{1}{2}\varphi'^2 - f\varphi\,dx$ $\qquad$ $\underset{\tilde{\varphi} \in V}{\text{Min}} \int \frac{1}{2}\tilde{\varphi}'^2 - \tilde{f}\tilde{\varphi}\,dx$ makes sense



$\frac{d^2\tilde{\varphi}}{d\tilde{x}} = 0$, or nonexistent



$\int \tilde{\varphi}'^2$ exists

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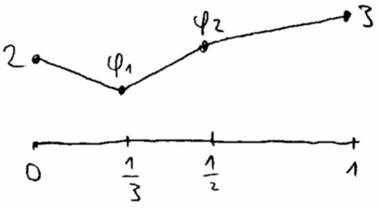
"Problem": Let $S[\varphi]=\int_0^1 [\varphi'(x)]^2 + x\varphi(x)\,dx$, $\varphi(0)=2$, $\varphi(1)=3$

Minimize S if $\varphi \in V$, where φ is continuous an piecewise affine on the $[0,1/3]$, $[1/3,1/2]$, $[1/2,1]$ intervals and $\varphi(0)=2$, $\varphi(1)=3$

"Solution":



φ': $\quad \dfrac{\varphi_1-2}{1/3} \quad \dfrac{\varphi_2-\varphi_1}{1/2-1/3} \quad \dfrac{3-\varphi_2}{1-1/2}$

$\dim V = 2$, $\quad \varphi \sim (\varphi_1,\varphi_2)\in\mathbb{R}^2$

$\varphi'^2 \quad \Delta x$

$$S[(\varphi_1,\varphi_2)] = \left(\frac{\varphi_1-2}{1/3}\right)^2\cdot\frac{1}{3} + \left(\frac{\varphi_2-\varphi_1}{1/6}\right)^2\cdot\frac{1}{6} + \left(\frac{3-\varphi_2}{1/2}\right)^2\cdot\frac{1}{2}$$

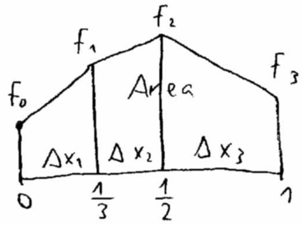
$$+\int_0^{1/3} x\cdot\left(2+\frac{\varphi_1-2}{1/3}x\right)dx + \int_{1/3}^{1/2} x\left(\varphi_1+\frac{\varphi_2-\varphi_1}{1/6}\cdot\left(x-\frac{1}{3}\right)\right)dx \;+$$

$$+\int_{1/2}^{1} x\left(\varphi_2+\frac{3-\varphi_2}{1/2}\left(x-\frac{1}{2}\right)\right)dx$$

As we are searching for approximate solutions, it does not make too much sense to compute the integrals exactly, we use (for example) the trapesoid method:

$$S[(\varphi_1,\varphi_2)] \approx \left(\frac{\varphi_1-2}{1/3}\right)^2\cdot\frac{1}{3} + \left(\frac{\varphi_2-\varphi_1}{1/6}\right)^2\cdot\frac{1}{6} + \left(\frac{3-\varphi_2}{1/2}\right)\cdot\frac{1}{2} +$$

$$+\frac{1/3}{2}\cdot 0\cdot 2 + \underbrace{\frac{1/3+1/6}{2}}_{\frac{\Delta x_1+\Delta x_2}{2}}\cdot\underset{x}{\frac{1}{3}}\cdot\underset{\varphi(x)}{\varphi_1} + \frac{1/6+1/2}{2}\cdot\frac{1}{2}\cdot\varphi_2 + \frac{1/2}{2}\cdot 1\cdot 3 =$$



$$\text{Area} = \Delta x_1\frac{f_0+f_1}{2} + \Delta x_2\frac{f_1+f_2}{2} + \Delta x_3\frac{f_2+f_3}{2}$$
$$= \frac{\Delta x_1}{2}f_0 + \frac{\Delta x_1+\Delta x_2}{2}f_1 + \frac{\Delta x_2+\Delta x_3}{2}f_2 + \frac{\Delta x_3}{2}f_3$$

$$= [\varphi_1\ \varphi_2]\underbrace{\begin{bmatrix} l_{11} & l_{12} \\ l_{21} & l_{22}\end{bmatrix}}_{\substack{l_{12}=l_{21} \\ L=L^T}}\begin{bmatrix}\varphi_1\\ \varphi_2\end{bmatrix} + [m_1\ m_2]\begin{bmatrix}\varphi_1\\ \varphi_2\end{bmatrix} + c = \vec{\varphi}^{\,T}L\vec{\varphi} + \vec{m}^{\,T}\vec{\varphi} + c$$

$$\tilde{S} = \vec{\varphi}^{\,T}L\vec{\varphi} + \vec{m}^{\,T}\vec{\varphi} + c$$

critical point of $\tilde{S}$: $\operatorname{grad}\tilde{S}=0=2L\vec{\varphi}+\vec{m}$, $\quad \dfrac{\partial\tilde{S}}{\partial\varphi_i} = \sum_j 2L_{ij}\varphi_j + m_i = 0$

$$\vec{\varphi}_{crit} = -\frac{1}{2}L^{-1}\vec{m}$$

$\vec{\varphi}_{crit}$ is the global minimum if L is positive definite, i.e. its eigen values are all positive (since $L=L^T$)

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What to do, when there is no (or we do not know) Lagrange function?

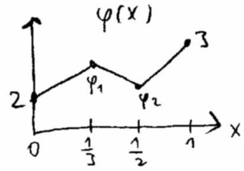
"Problem": find an approximate solution of the $\varphi'' = -f(x)$, $\varphi(0)=2$, $\varphi(1)=3$ DE!

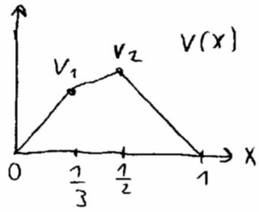
"Solution":

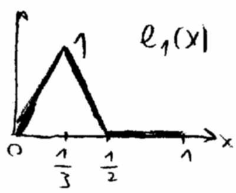
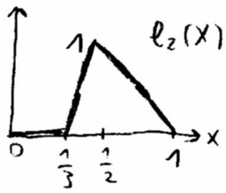
$$\varphi''(x) = -f(x) \longleftrightarrow \int^1 \overbrace{(\varphi''(x)+f(x))}^{=0} v(x)\,dx \qquad \text{"for all"} \quad v(x)$$

$$0 = \int_0^1 \varphi'' v + f v\,dx = \varphi' v\Big|_0^1 - \int_0^1 \varphi' v' - f v\,dx$$

if $v(0)=v(1)=0$, then $\int_0^1 -\varphi' v' + f v\,dx = 0$

Search for the approx. solution here:  $\varphi(x) \in \Phi$

get $v(x)$ from: 

a basis of the functions $v(x)$:   $v(x) = v_1 e_1(x) + v_2 e_2(x)$

We require for $\varphi \in \Phi$ that $\int_0^1 -\varphi' v' - f v = 0$ for all v_1, v_2, which means that

$$\int_0^1 -\varphi' e_1' + f e_1\,dx = 0, \qquad \int_0^1 -\varphi' e_2' + f e_2\,dx = 0.$$

Moreover we hope that the approximate computation of the integrals (for example with the trapesoid method) will do no harm.

$$e_1: 0 = \int_0^{1/2} -\varphi' e_1' + f e_1\,dx \approx \frac{1}{3}\cdot\left(-\frac{\varphi_1 - 2}{1/3}\right)\cdot\frac{1}{1/3} + \frac{1}{6}\left(-\frac{\varphi_2-\varphi_1}{1/6}\right)\cdot\left(-\frac{1}{1/6}\right) +$$

$$+ \frac{1/3}{2} f(0)\cdot 0 + \frac{1/3+1/6}{2} f\left(\tfrac{1}{3}\right)\cdot 1 + \frac{1/6}{2} f\left(\tfrac{1}{2}\right)\cdot 0 \quad \longleftarrow \text{trapesoid method}$$

$$e_2: 0 = \int_{1/3}^{1} -\varphi' e_2' + f e_2\,dx \approx \frac{1}{6}\left(-\frac{\varphi_2-\varphi_1}{1/6}\right)\cdot\frac{1}{1/6} + \underbrace{\frac{1}{2}}_{\Delta x}\underbrace{\left(-\frac{3-\varphi_2}{1/2}\right)}_{-\varphi'}\cdot\underbrace{\left(-\frac{1}{1/2}\right)}_{e_2'} +$$

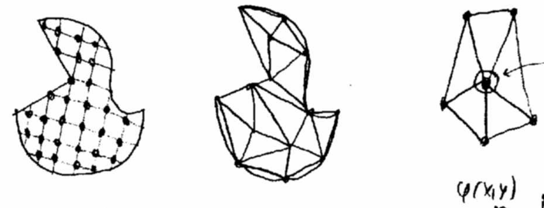
$$+ \frac{1/6}{2} f\left(\tfrac{1}{3}\right)\cdot 0 + \underbrace{\frac{1/6+1/2}{2}}_{\frac{\Delta x_2 + \Delta x_3}{2}} \underbrace{f\left(\tfrac{1}{2}\right)}_{f(\frac{1}{2})}\cdot \underbrace{1}_{e_2(\frac{1}{2})} + \frac{1/2}{2} f(1)\cdot 0$$

Remarks

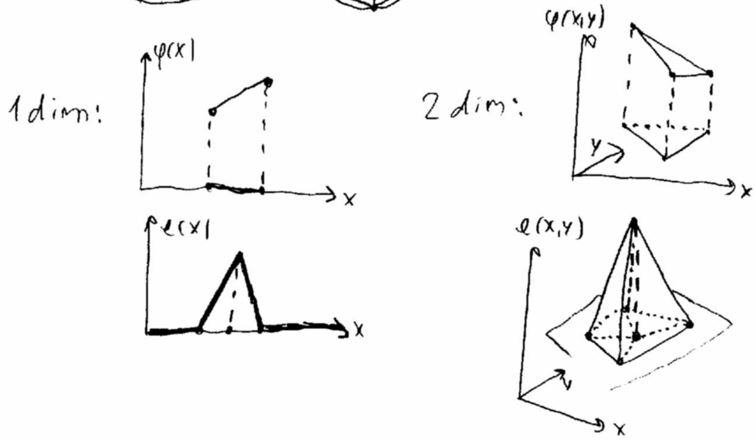
(1)

1 dimension: finite differences is simpler

2,3 dim: complicated shapes are hard to approximate with rectangular lattices

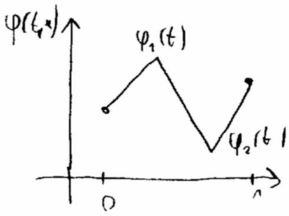


How to compute for example $\Delta\varphi$?



(2) Dynamics

Example: $\varphi_t(t,x) = \varphi_{xx}(t,x) + f(t,x)\varphi(t,x) + g(t,x)$



$$\varphi(t,x) \longrightarrow \vec{\varphi}(t) = \begin{pmatrix} \varphi_1(t) \\ \varphi_2(t) \end{pmatrix}$$

$$\int_0^1 (\varphi_t - \varphi_{xx} - f\varphi - g)\, v(x)\, dx = 0 \qquad \text{"for all" } v(x)$$

$$\int_0^1 (\varphi_t - f\varphi - g)\, v + \varphi_x v_x \, dx = 0 \qquad \text{for all } v(x) = e_i(x)$$

$$\longrightarrow \frac{d}{dt}\vec{\varphi}(t) = A(t)\vec{\varphi}(t) + \vec{b}(t)$$

Computed by FEM

ODE, solved by numerical methods like Runge-Kutta

Summary

① Wave equation (1+1 dim) $\varphi_{tt} - \varphi_{xx} = 0$

Solution: $\varphi(t,x) = f(x-t) + g(x+t) = \frac{1}{2}\left[\varphi(0,x-t) + \varphi(0,x+t)\right] + \frac{1}{2}\int_{x-t}^{x+t} \dot{\varphi}(0,y)\,dy$

Plane wave: $\varphi = e^{i(kx-\omega t)} \longrightarrow |k| = |\omega| \longrightarrow$ speed of propagation $= \pm 1$

Vibrating string: $\varphi(0,x) = F(x)$, $\dot{\varphi}(0,x) = G(x)$, $\varphi(t,0) = \varphi(t,\pi) = 0$

Solution: Sine tr. $F(x) = \sum_{n=1}^{\infty} \hat{F}_n \sqrt{\frac{2}{\pi}} \sin(nx)$, $\hat{F}_n = \int_0^{\pi} \sqrt{\frac{2}{\pi}} \sin(nx)\,dx$, $\hat{G}$ is similar

$$\varphi(t,x) = \sum_{n=1}^{\infty} \left(\hat{F}_n \cos(nt) + \frac{\hat{G}_n}{n}\sin(nt)\right) \sqrt{\frac{2}{\pi}} \sin(nx)$$

② Calculus of Variations: $S[\varphi] = \int_0^1 L(\varphi(x), \varphi'(x), x)\,dx$, $\varphi(0) = a$, $\varphi(1) = b$

φ_c critial point: $S[\varphi_c + \delta\varphi] \approx S[\varphi_c]$

Euler-Lagrange: $\frac{d}{dx}\frac{\partial L}{\partial \varphi'} - \frac{\partial L}{\partial \varphi} = 0$

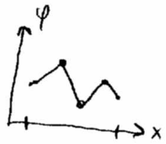
Example: $L(\varphi(x), \varphi'(x), x) = \varphi(x)\cdot[\varphi'(x)]^3 + x\,\varphi(x) + x^2$

EL: $\frac{d}{dx}\left(\varphi(x)\cdot 3[\varphi'(x)]^2\right) - \left([\varphi'(x)]^3 + x\right) = 0$

③ Finite elements method DE $\longleftrightarrow$ ⓐ Calculus of variations, minimalization

DE $\longleftrightarrow$ ⓑ weak solution

ⓐ approximate solution:

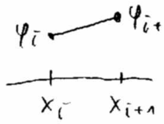


φ'': hard to interpret

$S = \int L(\varphi', \varphi, x)\,dx$ makes sense

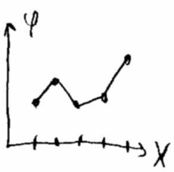
trapesoid method

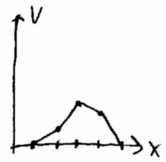
$L = \varphi'^2 - f\varphi$, $S \approx \dots + \underbrace{(x_{i+1} - x_i)}_{\Delta x_i}\left[\left(\frac{\varphi_{i+1} - \varphi_i}{\Delta x_i}\right)^2 - \frac{f(x_i)\varphi(x_i) + f(x_{i+1})\varphi(x_{i+1})}{2}\right] + \dots$

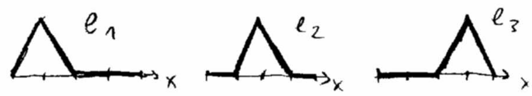


$S \approx \vec{\varphi}^T L \vec{\varphi} + \vec{m}^T \vec{\varphi} + c$, grad $S = 0 \longrightarrow \vec{\varphi}_{crit} = -\frac{1}{2} L^{-1}\vec{m}$

ⓑ $\varphi'' + f = 0 \longleftrightarrow \int_0^1 (\varphi'' + f)\,v\,dx = \int_0^1 -\varphi' v' + f v\,dx = 0$ if $v(1) = v(0) = 0$

φ: 

V: 

$\xrightarrow{\text{basis}}$ 

$\int_0^1 -\varphi' e_1' + f e_i\,dx = 0$ for all i.

Sample problems

15* IX

(1) $\varphi_{tt} = \varphi_{xx}$, $\varphi(0,x) = 3\sin(4x)$, $\dot\varphi(0,x) = 5\sin(6x)$, $\varphi(t,0) = \varphi(t,\pi) = 0$

Compute $\varphi(t,x)$ for $x \in [0,\pi]$!

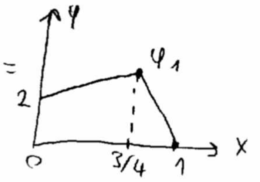
Solution: $\varphi(t,x) = 3\cos(4t)\sin(4x) + \frac{5}{6}\sin(6t)\cdot\sin(6x)$

(2) Let $L(x,\dot x,t) = \dot x^4 + x^2 + t^2$. If $x(1)=2$, $\dot x(1)=3$, then what does the Euler-Lagrange equation predict for $\ddot x(1)$?

Solution: $0 = \frac{d}{dt}\frac{\partial L}{\partial \dot x} - \frac{\partial L}{\partial x} = \frac{d}{dt}(4\dot x^3) - 2x = 12\dot x^2\ddot x - 2x = 0$

So if $x(1)=2$, $\dot x(1)=3$, then $12\cdot 3^2\cdot \ddot x(1) - 2\cdot 2 = 0 \longrightarrow \ddot x(1) = \frac{4}{12\cdot 3^2} = \frac{1}{27}$

(3) Let $\varphi(x) =$

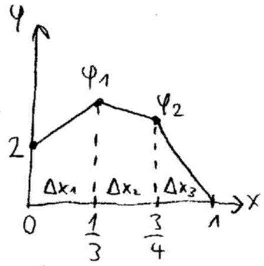


Compute $\int_0^1 \varphi'^2 + (1-x)\varphi\, dx$, using the trapezoid method for the approximate computation of the integral $\int_0^1 (1-x)\varphi\, dx$ with a single division point at $x = 3/4$.

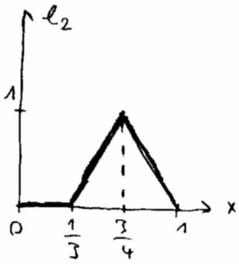
Solution: $\int_0^1 \varphi'^2 + (1-x)\varphi\, dx \approx$

$$\approx \frac{3}{4}\left(\frac{\varphi_1 - 2}{3/4}\right)^2 + \frac{1}{4}\left(\frac{0-\varphi_1}{1/4}\right)^2 + \frac{3}{4}\,\frac{(1-0)\cdot 2 + (1-3/4)\cdot\varphi_1}{2} + \frac{1}{4}\,\frac{(1-\frac{3}{4})\cdot\varphi_1 + (1-1)\cdot 0}{2}$$

(4) Let $\varphi(x) =$



, $e_2(x) =$



How much is $\int_0^1 (\varphi'' - (1-x))\, e_2(x)\, dx$, if we allow the use of partial integration to rewrite $\int_0^1 \varphi'' e_2\, dx$ as $-\int_0^1 \varphi' e_2'\, dx$? (Use the trapesoid method for the computation of the integrals.)

Solution: $\int_0^1 (\varphi'' - (1-x))e_2\, dx = \int_0^1 -\varphi' e_2' - (1-x)e_2\, dx = \int_{1/3}^1 -\varphi' e_2' - (1-x)e_2\, dx$

$$= \underbrace{\left(\frac{3}{4} - \frac{1}{3}\right)}_{\Delta x_2} \underbrace{\left(-\frac{\varphi_2 - \varphi_1}{\frac{3}{4} - \frac{1}{3}}\right)}_{-\varphi' = -\frac{\Delta\varphi}{\Delta x_2}} \cdot \underbrace{\frac{1}{\frac{3}{4} - \frac{1}{3}}}_{e_2' = \frac{1}{\Delta x_2}} + \underbrace{\left(1 - \frac{3}{4}\right)}_{\Delta x_3} \underbrace{\left(-\frac{0-\varphi_2}{1-\frac{3}{4}}\right)}_{-\varphi' = -\frac{\Delta\varphi}{\Delta x_3}} \underbrace{\left(-\frac{1}{1-\frac{3}{4}}\right)}_{e_2' = -\frac{1}{\Delta x_3}} + \underbrace{\frac{\left(\frac{3}{4} - \frac{1}{3}\right) + \left(1 - \frac{3}{4}\right)}{2}}_{\frac{\Delta x_2 + \Delta x_3}{2}} \cdot \underbrace{\varphi_2 \cdot 1}_{(\varphi \cdot e_2)\left(\frac{3}{4}\right)}$$

at $x = \frac{3}{4}$