Unit step, impulse, frequency responses

1^{*}_{VIII}

Example: $y' + 3y = f(t)$

① Impulse response, Green function

$G(t) = 0$ if $t < 0$, $\quad G'(t) + 3G(t) = \delta(t) \longrightarrow G(t) = \begin{cases} 0 & t<0 \\ e^{-3t} & t>0 \end{cases}$

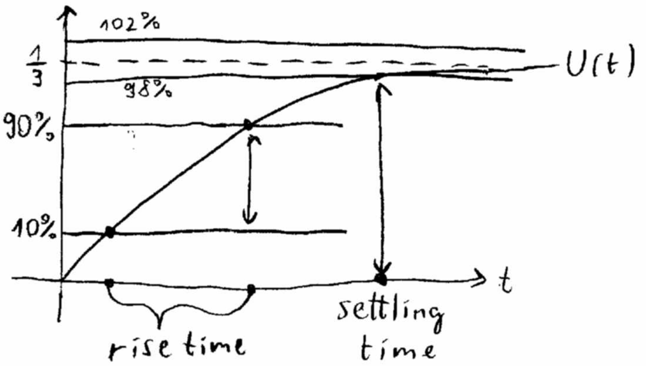
② Unit step response

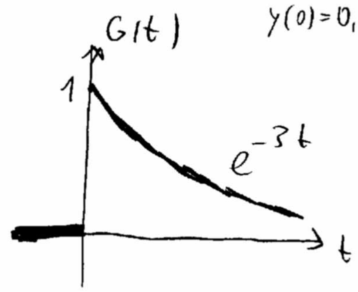
$U(t) = 0$ if $t < 0$, $\quad U'(t) + 3U(t) = \theta(t) \longrightarrow U(t) = \begin{cases} 0 & t<0 \\ \frac{1}{3}(1-e^{-3t}) & t>0 \end{cases}$

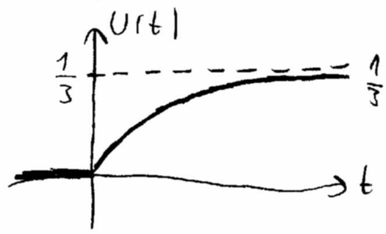
solution of the DE: $y(0)=0$, $y'+3y=1$

Then $\boxed{U'(t) = G(t)}$,

since $(U')' + 3U' = \theta' = \delta = G' + 3G$







$\frac{1}{3}(1-e^{-3t}) = \int_0^t e^{-3\tau}\,d\tau$

③ Frequency response

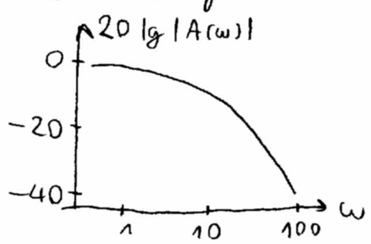
$$y' + 3y = f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(p)\, e^{ipt}\, dp$$

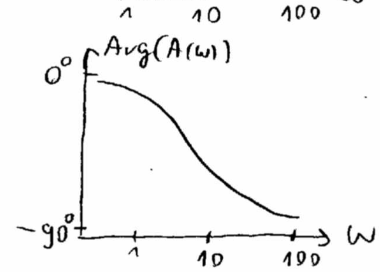
$$\left(A(\omega)\cdot e^{-i\omega t}\right)' + 3\left(A(\omega)\cdot e^{-i\omega t}\right) = e^{-i\omega t} \longrightarrow A(\omega) = \frac{1}{-i\omega + 3} \qquad \omega > 0$$

Transfer function $H(s) = \frac{1}{s+3} = A(is)$

$\mathrm{Arg}(5 \cdot e^{2i}) = 2$

Bode diagram: $\lg(\omega) \longleftrightarrow 20 \lg |A(\omega)|$, $\quad \lg(\omega) \longleftrightarrow \mathrm{Arg}(A(\omega))$





Formally: $y' + 3y = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(p)\, e^{ipt}\, dp$

$$\longrightarrow y(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(p)\, \frac{1}{ip+3}\, e^{ipt}\, dp$$

Can not be well defined as the solution of the DE is determined only up to the general solution of the hom. lin. DE.

Laplace transform

$$\left(\mathcal{L}(f)\right)(s) = \int_0^\infty e^{-st} f(t)\,dt = F(s), \quad f: [0,\infty) \to \mathbb{C}$$

$\mathcal{L}$ linear: $\mathcal{L}(\alpha_1 f_1 + \alpha_2 f_2) = \alpha_1 F_1 + \alpha_2 F_2$

If $f(t)$ continuous and ($f(t) < Me^{\alpha t}$, if $t > k$), then $F(s)$ is defined if $Re(s) > \alpha$

Inverse Laplace tr. $t > 0$

$$\left(\mathcal{L}^{-1}(F)\right)(t) = \frac{1}{2\pi i} \int_{-i\infty+a}^{+i\infty+a} F(s) e^{st}\,ds = f(t),$$

where a is large enough, so that $F(s)$ is defined on the $-i\infty+a \ldots i\infty+a$ countour

Formal proof, $a = 0$.

$$\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} F(s) e^{st}\,ds \underset{s=ip}{=} \frac{1}{2\pi i} \int_{-\infty}^{\infty} F(ip) e^{ipt} \cdot i\,dp, \quad F(ip) = \int_0^\infty e^{-ipt} f(t)\,dt = \int_{-\infty}^{\infty} e^{-ipt}\,\Theta(t) f(t)\,dt$$

So $\tilde{F}(p) = F(ip)$ and $\Theta(t) f(t)$ are related by the Fourier transform with normalization $\hat{g}(p) = \int_{-\infty}^{\infty} e^{-ipt} g(t)\,dt$, $g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{g}(p) e^{ipt}\,dp$.

So if $\hat{g}(p) = \tilde{F}(p)$, then $\frac{1}{2\pi i} \int_{-\infty}^{\infty} F(ip) e^{ipt} \cdot i\,dp = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{F}(p) e^{ipt}\,dp = \Theta(t) f(t)$

$F(s)$ is analitic (i.e. has convergent Taylor series with nonzero radius of convergence.)

Examples:

① $$\mathcal{L}(1) = (\mathcal{L}(1))(s) = \int_0^\infty e^{-st} \cdot 1\,dt = \frac{1}{-s} e^{-st}\Big|_0^\infty = -\frac{1}{s}\big(\underbrace{e^{-s\cdot\infty}}_{\to 0,\text{ if } Re(s)>0} - e^{-s\cdot 0}\big) = \frac{1}{s}$$

② $$\mathcal{L}(t) = \int_0^\infty e^{-st} \cdot t\,dt = \int_0^\infty \left(\frac{e^{-st}}{-s}\right)' \cdot t\,dt = \frac{e^{-st}}{-s} \cdot t\Big|_0^\infty - \int_0^\infty \frac{e^{-st}}{-s} \cdot 1\,dt = \frac{1}{s^2}$$

③ $$\mathcal{L}(e^{at}) = \int_0^\infty e^{-st} e^{at}\,dt = \int_0^\infty e^{-(s-a)t}\,dt = \frac{1}{s-a}$$

④ $$\mathcal{L}(\cos(at)) = \int_0^\infty \frac{e^{iat}+e^{-iat}}{2} \cdot e^{-st}\,dt = \frac{1}{2}\left(\frac{1}{s-ia} + \frac{1}{s+ia}\right) = \frac{s}{s^2+a^2}.$$

⑤ $$\mathcal{L}(\sin(at)) = \int_0^\infty \frac{e^{iat}-e^{-iat}}{2i} \cdot e^{-st}\,dt = \frac{1}{2i}\left(\frac{1}{s-ia} - \frac{1}{s+ia}\right) = \frac{a}{s^2+a^2}$$

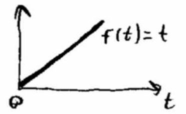
(6) $\mathcal{L}(t^n) = \dfrac{n!}{t^{n+1}}$, can be proved by repeated partial integration , $n = 0, 1, 2, 3 \ldots$

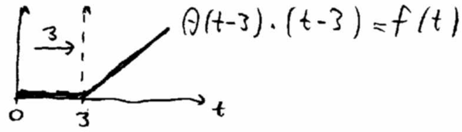
(7) $$\mathcal{L}(\Theta(t-a) f(t-a)) = \int_0^\infty \Theta(\underbrace{t-a}_{\tau})\, e^{-st} f(t-a)\,dt = \int_0^\infty e^{-s(\tau+a)} f(\tau)\,d\tau$$

$$= e^{-sa} \int_0^\infty e^{-s\tau} f(\tau)\,d\tau = e^{-sa} F(s) , \qquad a > 0$$

example: $\mathcal{L}(t) = \frac{1}{s^2}$. let $f(t) = \begin{cases} 0, & \text{if } t \in [0,3] \\ t-3, & \text{if } t \geq 3 \end{cases} = \Theta(t-3)\cdot(t-3)$

Then $F(s) = e^{-s\cdot 3} \cdot \dfrac{1}{s^2}$





(8) Laplace tr: derivation $\longrightarrow$ multiplication

$$\mathcal{L}(f') = \int_0^\infty e^{-st} f'(t)\,dt = e^{-st} f(t)\Big|_0^\infty - \int_0^\infty (-s\cdot e^{-st})\cdot f(t)\,dt$$

$$= \left(e^{-s\cdot\infty} f(\infty) - e^{-s\cdot 0} f(0)\right) + s\int_0^\infty e^{-st} f(t)\,dt = sF(s) - f(0)$$

$$\boxed{\mathcal{L}(f') = s\,F(s) - f(0)}$$

(9) $$\mathcal{L}(f'') = s\,\mathcal{L}(f') - f'(0) = s(sF(s) - f(0)) - f'(0)$$
$$= s^2 F(s) - s f(0) - f'(0)$$

(10) $\mathcal{L}(f''') = s^3 F(s) - s^2 f(0) - s f'(0) - f''(0)$, etc.

(11) Let $g(t) = \int_0^t f(t)\,dt$, so $g(0) = 0$, $g'(t) = f(t)$

Then $\mathcal{L}(g') = \mathcal{L}(f) = s\,G(s) - g(0) = s\,G(s)$,

so $G(s) = \frac{1}{s} F(s)$

Consequently $$\mathcal{L}\left(\int_0^t f(\tau)\,d\tau\right) = \frac{1}{s} F(s)$$

(12) $$\mathcal{L}(t\cdot f(t)) = \int_0^\infty e^{-st}\cdot t\cdot f(t)\,dt = -\frac{d}{ds}\int_0^\infty e^{-st} f(t)\,dt = -\frac{d}{ds} F(s) = -F'(s)$$

Constant coefficent linear DE

Example: $y'(t)+3y(t)=5, \quad y(0)=6.$

$$\Big\downarrow \mathcal{L}$$

Solution:

$$\underbrace{(sY(s)-y(0))}_{\mathcal{L}(y')} + 3\underbrace{Y(s)}_{\mathcal{L}(y)} = 5\cdot\underbrace{\frac{1}{s}}_{\mathcal{L}(1)}, \qquad (s+3)Y(s) = \frac{5}{s}+6$$

$$(s+3)Y(s)=\frac{5}{s}+6 \quad \longrightarrow \quad Y(s) = \underbrace{\frac{1}{s+3}}_{\substack{\text{transfer}\\ \text{function}}}\Big(\underbrace{\frac{5}{s}}_{\text{input}} + \underbrace{6}_{\text{initial condition}}\Big)$$

Remark: $y'+3y=5, \; y(0)=6, \; t>0$ can be traded for:

$y(t)=0$ if $t<0$, $\quad y'(t)+3y(t) = \underbrace{\Theta(t)\cdot 5}_{\substack{\text{input for}\\ t\geq 0}} + \underbrace{6\cdot\delta(t)}$

$$\hookrightarrow \; y(0^+)-\underbrace{y(0^-)}_{=0} = y(0^+) = \underbrace{6\cdot 1}_{\substack{y(0)=6\\ \text{initial cond.}}}$$

So formally $\frac{5}{s}+6 = \mathcal{L}(5+6\cdot\delta(t))$,

since $\mathcal{L}(\delta(t)) = \int_0^\infty e^{-st}\delta(t)\,dt = \lim_{a\to 0^+}\int_0^\infty e^{-st}\delta(t-a)\,dt =$

$$= \lim_{a\to 0^+} e^{-sa} = 1$$

not well defined

$$Y(s) = \frac{1}{s+3}\Big(\frac{5}{s}+6\Big) = \frac{5+6s}{s(s+3)} = \frac{A}{s} + \frac{B}{s+3} = \frac{5/3}{s} + \frac{13/3}{\underbrace{s+3}_{s+3=s-(-3)}}$$

partial fraction decomposition

$$\Big\downarrow \mathcal{L}^{-1}$$

$$y(t) = \frac{5}{3}\cdot 1 + \frac{13}{3}e^{-3t}$$

Example:

$$y''+4y'+5y=3+t,\quad y(0)=2,\quad y'(0)=7$$

Solution: $Y(s)=\mathcal{L}(y(t))$

$$(s^2Y(s)-s\cdot 2-7)+4(sY(s)-2)+5Y(s)=\frac{3}{s}+\frac{1}{s^2}$$

$$Y(s)=\frac{1}{s^2+4s+5}\left(\frac{3}{s}+\frac{1}{s^2}+[2s+15]\right)=\frac{A}{s}+\frac{B}{s^2}+\frac{C}{s-(-2+i)}+\frac{D}{s-(-2-i)}$$

$$=-\frac{4}{25}\cdot\frac{1}{s}+\frac{1}{5}\cdot\frac{1}{s^2}+\left(\frac{27}{25}-\frac{631}{100}i\right)\cdot\frac{1}{s-(-2+i)}+\left(\frac{27}{25}+\frac{631}{100}i\right)\cdot\frac{1}{s-(-2-i)}$$

$$y(t)=-\frac{4}{25}\cdot 1+\frac{1}{5}t+\left(\frac{27}{25}-\frac{631}{100}i\right)e^{(-2+i)t}+\left(\frac{27}{25}+\frac{631}{109}i\right)e^{(-2-i)t}$$

Remark:

DE: $y''+4y'+5y=(3+t)\Theta(t)+2\delta'(t)+7\delta(t)$, $\quad y(t)=0$, if $t<0$

$$\mathcal{L}(\delta(t))=1,\quad \mathcal{L}(\delta'(t))=\int_0^\infty e^{-st}\delta'(t)\,dt=-\frac{d}{dt}e^{-st}\Big|_{t=0^+}=s$$

Remark:

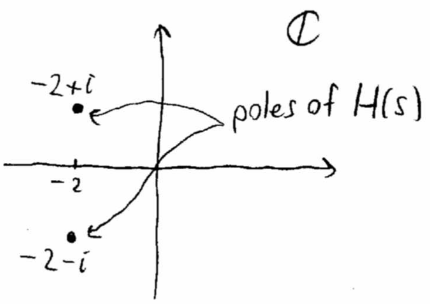
$y(t)=0$, if $t<0$, $\quad y''(t)=\delta'(t)\longrightarrow y(t)=\Theta(t)$, since $[[\Theta(t)]']'=[\delta(t)]'=\delta'(t)$

$y(t)=0$, if $t<0$, $\quad y''(t)=\delta(t)\longrightarrow y(t)=k(t)$, since $K''=\Theta'=\delta$

$y''=\delta\longrightarrow y(0^-)=y(0^+)$, $y'(0^+)-y'(0^-)=1$

$y''=\delta'\longrightarrow y(0^+)-y(0^-)=1$

Transfer function: $$H(s)=\frac{1}{s^2+4s+5}=\frac{1}{(s-[-2+i])(s-[-2-i])}$$



$Re(-2+i)=Re(-2-i)=-2<0$

stabil rendszer,

$$\lim_{t\to\infty}e^{(-2\pm i)t}=0$$

angular frequency of the oscillation: $|Im(-2\pm i)|=1$,

Frequency: $1\cdot\frac{1}{2\pi}\approx 0.16$ Hertz

Green function Transfer function $\boxed{6_{VIII}}$

$$y''(t)+4y'(t)+5y(t)=f(t)\cdot\Theta(t) \text{ "="} f(t)$$

$y(t)=f(t)=0$, if $t<0$ $\qquad$ $y(0)=y'(0)=0$

Retarded Green function

$$G(t)=\begin{cases} 0, & \text{if } t<0 \\ e^{-2t}\sin t, & \text{if } t>0 \end{cases}$$

Transfer function

$$H(s)=\frac{1}{s^2+4s+5}=\mathcal{L}(G(t))$$

Solution:

$$y(t)=\int_{-\infty}^{\infty} G(t-\tau)f(\tau)d\tau$$
$$=(G*f)(t)$$

$$y(t)=\mathcal{L}^{-1}(H(s)\cdot \underbrace{F(s)}_{\mathcal{L}(f(t))})$$

$G(t)\longleftrightarrow$ frequency response $\longleftrightarrow$ Transfer function

$$\left(\frac{d^2}{dt^2}+4\frac{d}{dt}+5\right)G(t)=\delta(t)$$

$$\delta(t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ipt}\cdot 1\,dp$$

$$G(t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{1}{-p^2+4ip+5}\cdot e^{ipt}dp \longrightarrow p=-is \longrightarrow \frac{1}{-p^2+4ip+5}=\frac{1}{s^2+4s+5}$$

formal calculations!

$$\left(\frac{d^2}{dt^2}+4\frac{d}{dt}+5\right)y(t)=f(t)$$

$$\hat{y}(p)=\frac{1}{-p^2+4ip+5}\hat{f}(p) \quad ???$$

$$\left(\frac{d^2}{dt^2}+4\frac{d}{dt}+5\right)y(t)=f(t)$$

$$Y(s)=\frac{1}{s^2+4s+5}F(s)$$

Convolution in the "t" time domain $\longleftrightarrow$ Multiplication in the

Fourier: p frequency domain
Laplace s imaginary frequency

Convolution

Fourier transform

$$\hat{f}(p) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ipt} f(t)\,dt \qquad f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(p)\, e^{ipt}\,dp$$

Convolution: $(f*g)(t) = \int_{-\infty}^{\infty} f(t-\tau)\, g(\tau)\, d\tau$

Statement: $\widehat{f*g} = \hat{f} \cdot \hat{g} \cdot \sqrt{2\pi}$

Proof:

$$(\widehat{f*g})(p) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ipt} \left(\int_{-\infty}^{\infty} f(t-\tau)\, g(\tau)\, d\tau \right) dt \qquad \begin{pmatrix} t \\ \tau \end{pmatrix} \longleftrightarrow \begin{pmatrix} t_1 \\ \tau \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}}_{\det = 1} \begin{pmatrix} t \\ \tau \end{pmatrix}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t_1)\, e^{-ip_1 t_1} \cdot g(\tau)\, e^{-ip\tau}\, d\tau\, dt_1$$

$$= \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t_1)\, e^{-ipt_1}\, dt_1 \right) \cdot \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(\tau)\, e^{-ip\tau} \right) \cdot \sqrt{2\pi} = \hat{f}(p) \cdot \hat{g}(p) \cdot \sqrt{2\pi}$$

Laplace transform

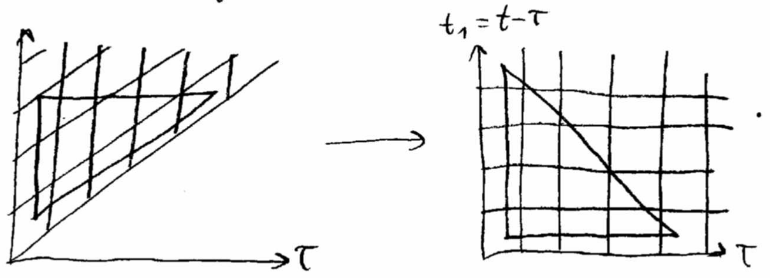
$$(f*g)(t) = \int_0^t f(t-\tau)\, g(\tau)\, d\tau$$

Statement: $\mathcal{L}(f*g) = \mathcal{L}(f) \cdot \mathcal{L}(g)$

Proof:

$$\mathcal{L}(f*g) = \int_0^\infty e^{-st} \left(\int_0^t f(t-\tau)\, g(\tau)\, d\tau \right) dt = \int_0^\infty \left(\int_0^t e^{-s(t-\tau)} f(t-\tau) \cdot e^{-s\tau} g(\tau)\, d\tau \right) dt$$

$$= \int_0^\infty e^{-st_1} f(t_1)\, dt_1 \cdot \int_0^\infty e^{-s\tau} g(\tau)\, d\tau = \mathcal{L}(f) \cdot \mathcal{L}(g)$$

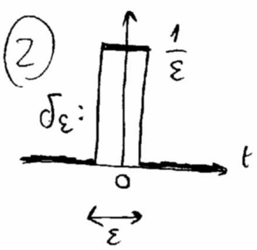


Remarks

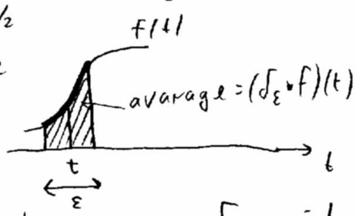
① $(f*g)(t)=\int_{-\infty}^{\infty} f(t-\tau)g(\tau)d\tau = \int_{-\infty}^{\infty} f(t_1)g(t-t_1)dt_1 = (g*f)(t)$ $\quad t_1 = t-\tau$

② $(\delta_\varepsilon * f)(t) = (f*\delta_\varepsilon)(t) = \int_{-\infty}^{\infty} f(t-\tau)\delta_\varepsilon(\tau)\,d\tau = \int_{-\varepsilon/2}^{\varepsilon/2} f(t-\tau)\cdot\frac{1}{\varepsilon}\,d\tau$

δ_ε:



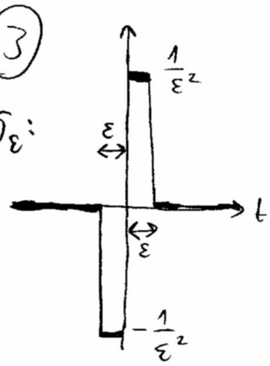
$(\delta_\varepsilon * f)(t)$ = avarage value of f on the $[t-\frac{\varepsilon}{2}, t+\frac{\varepsilon}{2}]$ interval



$\varepsilon\to 0,\ \delta_\varepsilon \to \delta \qquad (\delta * f)(t) = (f*\delta)(t) = \int_{-\infty}^{\infty} f(t-\tau)\delta(\tau)d\tau = f(t)$, so $\delta * = id.$

③ $(\sigma_\varepsilon * f)(0) = (f*\sigma_\varepsilon)(0) = \frac{1}{\varepsilon}\left[\left(\int_0^\varepsilon f(t)\cdot\frac{1}{\varepsilon}dt\right) - \left(\int_{-\varepsilon}^{0} f(t)\cdot\frac{1}{\varepsilon}dt\right)\right]$

σ_ε:



$\approx \frac{1}{\varepsilon}\left(f\left(\frac{\varepsilon}{2}\right) - f\left(-\frac{\varepsilon}{2}\right)\right) \approx f'(0)$

$(\delta' * f)(t) = (f*\delta')(t) = \int_{-\infty}^{\infty} f(t-\tau)\delta'(\tau)d\tau = -f'(t)$

so $\lim_{\varepsilon\to 0^+} \sigma_\varepsilon = -\delta'$

④ f and g play different roles in $(f*g)(t) = \int_{-\infty}^{\infty} f(t-\tau)g(\tau)d\tau$.

A more symmetric definition: Let $\{T_a \mid a\in\mathbb{R}\}$ be the transformation group of translations of $\mathbb{R}$: Let $T_a:\mathbb{R}\to\mathbb{R}$, $T_a(x)=x+a$. Then $T_aT_b = T_{a+b}$.

Let $T(f) = \int_{-\infty}^{\infty} f(t)T_t\,dt$. Then

$$T(f)T(g) = \left(\int_{-\infty}^{\infty} f(t_1)T_{t_1}dt_1\right)\left(\int_{-\infty}^{\infty} g(t_2)T_{t_2}dt_2\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t_1)g(t_2)T_{t_1+t_2}\,dt_1\,dt_2$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t-t_2)g(t_2)\,dt_2\cdot T_t\,dt = \int_{-\infty}^{\infty}\left[(f*g)(t)\right]\cdot T_t\,dt$$

Same can be done for: n dim, discrete n dim, circle,

$T_{\vec a}:\mathbb{R}^n\to\mathbb{R}^n,\quad T_{\vec a}(\vec x) = \vec x + \vec a$

$T_{\vec a}:\mathbb{Z}^n\to\mathbb{Z}^n,\quad T_{\vec a}(\vec n) = \vec n + \vec a$

$T_\alpha: S^1\to S^1,\quad T_\alpha(\beta) = \beta+\alpha$ modulo 2π $\qquad$ groups, too.

$T_{\vec\alpha}:(S^1)^n\to(S^1)^n,\quad T_{\vec\alpha}(\vec\beta) = \vec\beta+\vec\alpha$ mod $2\pi\vec e_1 \ldots 2\pi\vec e_n$

Example: convolution for $\mathbb{Z}\to\mathbb{C}$ sequences:

$$(a*b)_n = \sum_k a_{n-k}b_k$$

Linear systems

9^{*} $\underline{\text{VIII}}$

$$\frac{d}{dt}\vec{y} = A\vec{y} + \vec{f}(t), \quad \vec{y}(0) = \vec{y}_0$$

$$s\vec{Y}(s) - \vec{y}_0 = A\vec{Y}(s) + \vec{F}(s)$$

$$\vec{Y}(s) = (sE - A)^{-1}\left(\vec{F}(s) + \vec{y}_0\right)$$

How to compute $(sE-A)^{-1}$?

$$A\vec{v}_i = \lambda_i \vec{v}_i, \quad U = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix}, \quad A = UDU^{-1}, \quad D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

$$(sE-A)^{-1} = \left(UsEU^{-1} - UDU^{-1}\right)^{-1} = \left(U[sE-D]U^{-1}\right)^{-1} = U[sE-D]^{-1}U^{-1}$$

$$= U\begin{bmatrix} \frac{1}{s-\lambda_1} & & 0 \\ & \ddots & \\ 0 & & \frac{1}{s-\lambda_n} \end{bmatrix} U^{-1}$$

If $\vec{f}(t) = \vec{F}(s) = 0$, then

$$\vec{y}(t) = e^{tA}\vec{y}_0 = U e^{tD} U^{-1}\vec{y}_0 = U\begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{bmatrix} U^{-1}\vec{y}_0$$

$$\vec{Y}(s) = U\begin{bmatrix} \frac{1}{s-\lambda_1} & & 0 \\ & \ddots & \\ 0 & & \frac{1}{s-\lambda_n} \end{bmatrix} U^{-1}\vec{y}_0 \qquad \mathcal{L}\left(e^{\lambda_n t}\right) = \frac{1}{s-\lambda_n}$$

Example: $\frac{d}{dt}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}[f(t)]$, $f(t) = \begin{cases} 4, & \text{if } t \in [1,2] \\ 0 & \text{otherwise} \end{cases}$, $\vec{y}(0) = \begin{bmatrix} 7 \\ 8 \end{bmatrix}$

Solution: $\mathcal{L}(f) = \int_1^2 e^{-st} \cdot 4\,dt = \frac{4}{-s}\left(e^{-2s} - e^{-1\cdot s}\right) = \frac{4}{s}\left(e^{-s} - e^{-2s}\right)$

$$\begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix} = \begin{bmatrix} s+2 & 0 \\ -2 & s+3 \end{bmatrix}^{-1}\begin{bmatrix} \frac{4}{s}(e^{-s}-e^{-2s})+7 \\ 8 \end{bmatrix} = \frac{1}{(s+2)(s+3)}\begin{bmatrix} s+3 & 0 \\ +2 & s+2 \end{bmatrix}\begin{bmatrix} \frac{4}{s}(e^{-s}-e^{-2s})+7 \\ 8 \end{bmatrix}$$

$$= \begin{bmatrix} \boxed{\begin{matrix} 1 \\ 2 \end{matrix}} & \boxed{\begin{matrix} 0 \\ 1 \end{matrix}} \end{bmatrix}\begin{bmatrix} (s+2)^{-1} & 0 \\ 0 & (s+3)^{-1} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}^{-1}\begin{bmatrix} \frac{4}{s}(e^{-s}-e^{-2s})+7 \\ 8 \end{bmatrix}$$

eigenvectors

$-2, -3$: eigenvalues

Summary

(1) Unit step response: $LG = \delta, \quad LU = \theta, \quad U' = G.$

Example: $\left(\frac{d^2}{dt^2} + 4\right) G(t) = \delta(t) \longrightarrow G(t) = \begin{cases} 0, & \text{if } t<0 \\ \frac{1}{2}\sin(2t), & \text{if } t>0 \end{cases}$

$$\left(\frac{d^2}{dt^2} + 4\right) U(t) = \theta(t) \longrightarrow U(t) = \begin{cases} 0, & \text{if } t<0 \\ \frac{1}{4}(1-\cos(2t)) = \int_0^t \frac{1}{2}\sin(2t)\,dt, & \text{if } t>0 \end{cases}$$

(2) Frequency response:

$$\left(\frac{d^2}{dt^2} + 4\right)\left(A(\omega)e^{-i\omega t}\right) = e^{-i\omega t} \longrightarrow A(\omega) = \frac{1}{-\omega^2+4}$$

(3) Laplace transform:

$$[\mathcal{L}(f(t))](s) = \int_0^\infty e^{-st} f(t)\,dt = F(s), \quad [\mathcal{L}^{-1}(F(s))](t) = \frac{1}{2\pi i}\int_{-i\infty+a}^{i\infty+a} F(s)\,e^{s\cdot t}\,ds = f(t)$$

$\mathcal{L}(1) = \frac{1}{s}$ $\quad \mathcal{L}(e^{at}) = \frac{1}{s-a}$ $\quad \mathcal{L}(t^n) = \frac{n!}{s^{n+1}}$ $\quad \mathcal{L}(\theta(t-a)f(t-a)) = e^{-sa}F(s), \; a>0$

$\mathcal{L}(\delta(t)) = 1$ $\quad \mathcal{L}(\cos(at)) = \frac{s}{s^2+a^2}$ $\quad \mathcal{L}\left(\int_0^t f(\tau)\,d\tau\right) = \frac{1}{s}F(s)$ $\quad \mathcal{L}(f') = sF(s) - f(0)$

$\mathcal{L}(\delta'(t)) = s$ $\quad \mathcal{L}(\sin(at)) = \frac{a}{s^2+a^2}$ $\quad \mathcal{L}(t\cdot f(t)) = -F'(s)$ $\quad \mathcal{L}(f'') = s^2F(s) - sf(0) - f'(0)$

(4) DE solutions

(a) $y' + 2y = \theta(t-3), \; y(0) = 4 \longrightarrow sY(s) - 4 + 2Y(s) = \frac{e^{-3s}}{s} \longrightarrow Y(s) = \frac{1}{s+2}\left(4 + \frac{e^{-3s}}{s}\right) \longrightarrow$

$\longrightarrow \mathcal{L}^{-1}\left(\frac{4}{s+2}\right) = 4e^{-2t}, \; \mathcal{L}^{-1}\left(\frac{1}{s+2}\cdot\frac{e^{-3s}}{s}\right) = \mathcal{L}^{-1}\left(e^{-3s}\left[\frac{1/2}{s} - \frac{1/2}{s+2}\right]\right) = \theta(t-3)\cdot\frac{1}{2}\left(1 - e^{-2(t-3)}\right)$

$\longrightarrow y(t) = 4\cdot e^{-2t} + \frac{1}{2}\theta(t-3)\left[1 - e^{-2(t-3)}\right]$

(b) $y'' + 2y' + y = (t+1)^2, \; y(0) = 4, \; y'(0) = 5 \longrightarrow [s^2Y(s) - 4s - 5] + 2[sY(s) - 4] + Y(s) = \frac{2!}{s^3} + \frac{2}{s^2} + \frac{1}{s} \Longrightarrow$

$\longrightarrow \mathcal{L}^{-1}\left(\frac{1}{(s+1)^2}\right) = \mathcal{L}^{-1}\left(-\left[\frac{1}{s+1}\right]'\right) = t\cdot e^{-1.t} \longrightarrow y(t) = Ate^{-t} + Be^{-t} + \frac{C}{2}t^2 + Dt + E$

$\Longrightarrow Y(s) = \frac{1}{s^2+2s+1}\left[4s + 13 + \frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}\right] = \frac{A}{(s+1)^2} + \frac{B}{(s+1)} + \frac{C}{s^3} + \frac{D}{s^2} + \frac{E}{s} \longrightarrow$

(c) $\frac{d}{dt}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} 1 \\ t \end{bmatrix}, \; \vec{y}(0) = \begin{bmatrix} 6 \\ 8 \end{bmatrix} \longrightarrow s\cdot\begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix} - \begin{bmatrix} 6 \\ 8 \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix} + \begin{bmatrix} 1/s \\ 1/s^2 \end{bmatrix}$

$$\begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix} = \begin{bmatrix} s-5 & -2 \\ -2 & s-5 \end{bmatrix}^{-1}\begin{bmatrix} 6 + 1/s \\ 8 + 1/s^2 \end{bmatrix} = \begin{bmatrix} \boxed{\begin{matrix} 1 \\ 1 \end{matrix}} & \boxed{\begin{matrix} 1 \\ -1 \end{matrix}} \end{bmatrix}\begin{bmatrix} (s-7)^{-1} & 0 \\ 0 & (s-3)^{-1} \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1}\begin{bmatrix} 6 + 1/s \\ 8 + 1/s^2 \end{bmatrix}$$

(5) Convolution:

Fourier: $(f * g)(t) = \int_{-\infty}^{\infty} f(t-\tau)\,g(\tau)\,d\tau$ $\qquad$ Laplace: $(f * g)(t) = \int_0^t f(t-\tau)\,g(\tau)\,d\tau$

Fourier tr. $\mathcal{F}$: $\qquad$ Laplace tr. $\mathcal{L}$:

$\mathcal{F}(f * g) = \sqrt{2\pi}\,\mathcal{F}(f)\cdot\mathcal{F}(g)$ $\qquad$ $\mathcal{L}(f * g) = \mathcal{L}(f)\cdot\mathcal{L}(g)$

Sample problems

(1) Find the impulse and unit step responses of the $y''+9y=f(t)$ DE!

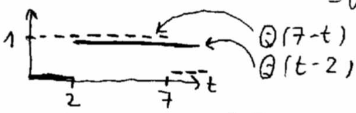
Solution: $G''+9G=\delta \longrightarrow G(t)=\begin{cases} 0, & \text{if } t<0 \\ \frac{1}{3}\sin(3t), & \text{if } t>0 \end{cases}$

$U''+9U=\theta \longrightarrow U'=G \longrightarrow U(t)=\int_{-\infty}^{t} G(\tau)\,d\tau=\begin{cases} 0, & \text{if } t<0 \\ \frac{1}{9}(1-\cos(3t)), & \text{if } t>0 \end{cases}$

(1.b) Find the frequency response, too!

Solution: $[A(\omega)e^{-i\omega t}]''+9[A(\omega)e^{-i\omega t}]=e^{-i\omega t} \longrightarrow A(\omega)=\frac{1}{-\omega^2+9}$

(2) Compute $\mathcal{L}(\theta(t-2)\theta(7-t)\cdot e^{3t})$!



Solution: $=\int_2^7 e^{-st}\cdot e^{3t}\,dt=\frac{1}{3-s}\,e^{(3-s)t}\Big|_{t=2}^{7}=\frac{1}{3-s}\left(e^{(3-s)\cdot 7}-e^{(3-s)\cdot 2}\right)$

(3) $y'-5y=-7-15t$, $y(0)=3$. Compute $Y(s)$ and its partial fraction decomposition! How much is $y(t)$?

Solution: $sY(s)-3-5Y(s)=-\frac{7}{s}-\frac{15}{s^2} \longrightarrow Y(s)=\frac{1}{s-5}\left(3-\frac{7}{s}-\frac{15}{s^2}\right)=\frac{3s^2-7s-15}{(s-5)s^2}=$

$=\frac{A}{s-5}+\frac{B}{s^2}+\frac{C}{s} \longrightarrow Y(s)=\frac{1}{s-5}+\frac{3}{s^2}+\frac{2}{s} \longrightarrow y(t)=1\cdot e^{5t}+3t+2$

(4) $y''-3y'+2y=8$, $y(0)=9$, $y'(0)=8$. Repeat ex.(3)!

Solution: $[s^2Y(s)-9s-8]-3[sY(s)-9]+2Y(s)=\frac{8}{s} \longrightarrow Y(s)=\frac{1}{s^2-3s+2}\left[9s+19+\frac{8}{s}\right]=$

$=\frac{9s^2+19s+8}{(s-1)(s-2)s}=\frac{A}{s-1}+\frac{B}{s-2}+\frac{C}{s} \longrightarrow Y(s)=\frac{2}{s-1}+\frac{3}{s-2}+\frac{4}{s} \longrightarrow y(t)=2\cdot e^t+3e^{2t}+4$

(5) $\frac{d}{dt}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}=\begin{bmatrix} 2 & 0 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}+\begin{bmatrix} 4 \\ 5 \end{bmatrix}$, $\vec{y}(0)=\begin{bmatrix} 6 \\ 7 \end{bmatrix}$. How much is $\vec{Y}(s)$?

Solution: $s\begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix}-\begin{bmatrix} 6 \\ 7 \end{bmatrix}=\begin{bmatrix} 2 & 0 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix}+\begin{bmatrix} 4/s \\ 5/s \end{bmatrix} \longrightarrow \begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix}=\begin{bmatrix} s-2 & 0 \\ 2 & s-3 \end{bmatrix}^{-1}\begin{bmatrix} 6+4/s \\ 7+5/s \end{bmatrix}=$

$=\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} (s-2)^{-1} & 0 \\ 0 & (s-3)^{-1} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 6+4/s \\ 7+5/s \end{bmatrix}$ (Here $\begin{bmatrix} 2 & 0 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix}=2\cdot\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 2 & 0 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}=3\cdot\begin{bmatrix} 0 \\ 1 \end{bmatrix}$)

(6) $f(t)=t$, $g(t)=1-t$. Compute $(f*g)(t)$ and $\mathcal{L}(f*g)$!

Solution: $(f*g)(t)=\int_0^t (t-\tau)(1-\tau)\,d\tau=\int_0^t \tau^2-(1+t)\tau+t\,d\tau=\frac{1}{2}t^2-\frac{1}{6}t^3$, $\mathcal{L}(f*g)=\frac{1}{2}\frac{2!}{s^3}-\frac{1}{6}\cdot\frac{3!}{s^4}=$

$=\left(\frac{1}{s^2}\right)\cdot\left(\frac{1}{s}-\frac{1}{s^2}\right)$

(7) Let $\theta(7-t)\theta(t-2)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(p)e^{ipt}\,dp$. How much is $\hat{f}(2)$?

Solution: $\hat{f}(2)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-i\cdot 2\cdot t}\,\theta(7-t)\,\theta(t-2)\,dt=\frac{1}{\sqrt{2\pi}}\int_2^7 e^{-i\cdot 2\cdot t}\cdot 1\,dt=$

$=\frac{1}{\sqrt{2\pi}\cdot(-i\cdot 2)}\left(e^{-14i}-e^{-4i}\right)$