Inhomogeneous linear systems

$$\frac{d}{dt}\vec{y}(t) = A\vec{y}(t) + \vec{f}(t)$$

hom. lin: $\left.\begin{aligned} \frac{d}{dt}\vec{y}_1(t) &= A\vec{y}_1(t) \\ \frac{d}{dt}\vec{y}_2(t) &= A\vec{y}_2(t) \end{aligned}\right\} \Longrightarrow \frac{d}{dt}\left(\alpha_1\vec{y}_1(t) + \alpha_2\vec{y}_2(t)\right) = A\left(\alpha_1\vec{y}_1(t) + \alpha_2\vec{y}_2(t)\right)$

Solutions form a linear supspace

inhom. lin: $\left.\begin{aligned} \frac{d}{dt}\vec{y}_p &= A\vec{y}_p + \vec{f}(t) \\ \frac{d}{dt}\vec{y}_{hom} &= A\vec{y}_{hom} \end{aligned}\right\} \Longrightarrow \frac{d}{dt}\left(\vec{y}_p + \vec{y}_{hom}\right) = A\left(\vec{y}_p + \vec{y}_{hom}\right) + \vec{f}(t)$

General solution of inhom. lin. DE = (General sol. of hom. lin) + + (a particulars sol. of inhom. lin.)

inhom. lin: Linear input-output relation

$\left.\begin{aligned} \frac{d}{dt}\vec{y}_1 &= A\vec{y}_1 + \vec{f}_1 \\ \frac{d}{dt}\vec{y}_2 &= A\vec{y}_2 + \vec{f}_2 \end{aligned}\right\} \Longrightarrow \frac{d}{dt}\left(\alpha_1\vec{y}_1 + \alpha_2\vec{y}_2\right) = A\left(\alpha_1\vec{y}_1 + \alpha_2\vec{y}_2\right) + \left(\alpha_1\vec{f}_1 + \alpha_2\vec{f}_2\right)$

input	output
$\vec{f}_1$	$\vec{y}_1$
$\vec{f}_2$	$\vec{y}_2$
$\alpha_1\vec{f}_1 + \alpha_2\vec{f}_2$	$\alpha_1\vec{y}_1 + \alpha_2\vec{y}_2$

Example (radioactive decay):

$$\frac{d}{dt}y(t) = -2y(t) + f(t), \quad y(0) = y_0$$

Solution: $$y(t) = e^{-2t}y_0 + \int_0^t e^{-2(t-s)} f(s)\,ds$$

Proof: ① $y(0) = e^{-2\cdot 0}\cdot y_0 + \int_0^0 e^{-2(0-s)} f(s)\,ds = 1\cdot y_0 + 0 = y_0$ o.k.

② $\frac{d}{dt}y(t) = \frac{d}{dt}(e^{-2t})\cdot y_0 + \frac{d}{dt}\left[\int_0^t e^{-2(t-s)} f(s)\,ds\right]$

$= -2e^{-2t}y_0 + e^{-2(t-t)}f(t) + \int_0^t \frac{d}{dt}\left(e^{-2(t-s)}\right)\cdot f(s)\,ds$

$= -2e^{-2t}y_0 + f(t) + \int_0^t -2\cdot e^{-2(t-s)} f(s)\,ds =$

$= -2\left(e^{-2t}y_0 + \int_0^t e^{-2(t-s)} f(s)\,ds\right) + f(t) = -2y(t) + f(t)$ o.k.

More generally:

$$\boxed{\begin{array}{ll}\text{Let} & \frac{d}{dt}\vec{y}(t) = A\vec{y}(t) + \vec{f}(t), \quad \vec{y}(0) = \vec{y}_0 \\ \text{Then} & \vec{y}(t) = e^{tA}\vec{y}_0 + \int_0^t e^{(t-s)A}\vec{f}(s)\,ds\end{array}}$$

$$\frac{d}{dt}\vec{y}(t) = \left(\frac{d}{dt}e^{tA}\right)\vec{y}_0 + \frac{d}{dt}\int_0^t e^{(t-s)A}\vec{f}(s)\,ds$$

$$= A e^{tA}\vec{y}_0 + e^{(t-t)A}\vec{f}(t) + \int_0^t \frac{d}{dt}\left(e^{(t-s)A}\right)\vec{f}(s)\,ds$$

$$= A e^{tA}\vec{y}_0 + \vec{f}(t) + \int_0^t A e^{(t-s)A}\vec{f}(s)\,ds$$

$$= A\left(e^{tA}\vec{y}_0 + \int_0^t e^{(t-s)A}\vec{f}(s)\,ds\right) + \vec{f}(t) = A\vec{y}(t) + \vec{f}(t) \qquad \text{O.K.}$$

A variant:

$\vec{y}(t) = \vec{f}(t) = 0$, if $t \ll 0$,

$(t-s)+s=t$

$$\vec{y}(t) = \int_{-\infty}^t e^{(t-s)A}\vec{f}(s)\,ds = \int_{-\infty}^{\infty} G(t-s)\vec{f}(s)\,ds, \quad \text{where } G(t) = \begin{cases} e^{tA}, & \text{if } t>0 \\ 0, & \text{if } t<0\end{cases}$$

So: Hom. DE $\longrightarrow e^{tA} \longrightarrow$ Inhom. DE solution

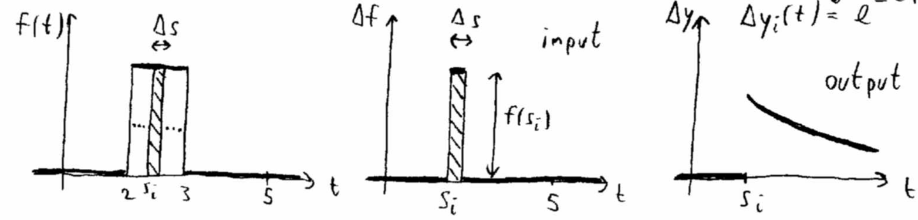
Exercise: $\frac{d}{dt}y = -2y + f(t)$, $y(0) = 0$, $f(t) = \begin{cases} 1, & \text{if } t\in[2,3] \\ 0 & \text{otherwise}\end{cases}$. How much is $y(5)$?

Solution:
$$y(5) = \int_0^5 e^{-2(5-s)} f(s)\,ds = \int_2^3 e^{-2(5-s)}\cdot 1\,ds = e^{-10}\cdot\int_2^3 e^{2s}\,ds = \frac{1}{2}\left(e^{-4} - e^{-6}\right)$$

Interpretation:

For $t>s_i$

$$\Delta y_i(t) = e^{-2(t-s_i)}\cdot f(s_i)\Delta s = G(t-s_i)f(s_i)\Delta s$$



if $\Delta s \to 0$: $y(t) = \sum_i \Delta y_i(t) = \sum_i e^{-2(t-s_i)} f(s_i)\Delta s = \int_0^t e^{-2(t-s)} f(s)\,ds = \int_{-\infty}^{\infty} G(t-s)f(s)\,ds$

$= (G*f)(t)$. $*$ convolution. $G(t)$: retarded Green function, impulse response, fundamental solution

Properties of $G(t)$: $t \neq 0 \longrightarrow \frac{d}{dt}G(t) = -2G(t)$

$t<0 \longrightarrow G(t) = 0$

$t \approx 0 \longrightarrow G(0^+) - G(0^-) = G(0^+) = 1$.

What does $G(t)$ solve?

$f(s_i)\Delta s$ input $\longrightarrow$ output: $G(t-s_i)f(s_i)\Delta s$

$F(0)\,\Delta s$ input $\longrightarrow$ output: $G(t)f(0)\Delta s$

$f(0)\Delta s=1$ impulse input $\longrightarrow$ output: $G(t)$

So $G(t)$ is the response of the system for an input $f(t)$, such that $F(t)$ is nonzero only at $t=0$, but $f(0)\Delta s=\int_{-\infty}^{\infty} f(s)ds=1$. Of course there is no such $f(t)$ function, but the desired properties can be approximated by the (for example) functions: $\delta_\varepsilon(t)=\begin{cases}\frac{1}{\varepsilon}, & \text{if } t\in[0,\varepsilon]\\ 0 & \text{otherwise}\end{cases}$, $\varepsilon\longrightarrow 0$.

Then solve $\frac{d}{dt}G_\varepsilon(t)=\delta_\varepsilon(t)$, $G_\varepsilon(-\infty)$, and get

$$\lim_{\varepsilon\to 0} G_\varepsilon(t)=\begin{cases}0 & \text{if } t<0\\ e^{-2t} & \text{if } t>0\end{cases},\quad \text{so } \lim_{\varepsilon\to 0} G_\varepsilon(t)=G(t), \text{ at least for } t\neq 0$$

As G is used in expressions like $\int G(t-s)f(s)ds$, it does not matter that it is undefined at the single point $t=0$.

So $G(t)$ is the solution of the imaginary

$$\frac{d}{dt}G(t)=-2G(t)+\delta(t),\quad G(-\infty)=0 \qquad \text{DE},$$

where the nonexistent $\delta(t)$ Dirac-delta function properties are

$$\delta(t)=0,\ \text{if } t\neq 0,\quad \int_{-\infty}^{\infty}\delta(t)\,dt=1.$$

We desire that $\delta_\varepsilon\to\delta$ as $\varepsilon\to 0$, but the limit should be independent of the actual forms of the approximating δ_ε functions.

Solution (Laurent Schwarz): distributions, generalized functions

Let $\mathcal{D}=\{\varphi(x)\mid \varphi \text{ is smooth and is zero outside of a finite interval}\}$ (test functions)

Bilinear pairing on $\mathcal{D}$: $\langle\psi,\varphi\rangle=\int_{-\infty}^{\infty}\psi(x)\varphi(x)dx=L_\psi(\varphi)$.

So $L_\psi:\mathcal{D}\to\mathbb{C}$ is a linear mapping from $\mathcal{D}$ to $\mathbb{C}$ (linear functional). But not all linear mapping can be written in this form:

$\delta:\mathcal{D}\to\mathbb{C}$, $\delta(\varphi)=\varphi(0)$ is linear, but $\delta\neq L_\psi$ for any $\psi\in\mathcal{D}$.

But $\langle\delta_\varepsilon,\varphi\rangle=\int_{-\infty}^{\infty}\delta_\varepsilon(t)\varphi(t)dt=\int_0^{\varepsilon}\frac{1}{\varepsilon}\varphi(t)dt\longrightarrow\varphi(0)$, so $\delta_\varepsilon\to\delta$, if $\varepsilon\to 0$.

cheating, $\delta_\varepsilon\notin\mathcal{D}$

So formally $\langle\delta_\varepsilon,\varphi\rangle\longrightarrow\varphi(0)=\langle\delta,\varphi\rangle=\int_{-\infty}^{\infty}\delta(t)\cdot\varphi(t)dt$,

where the properties of $\delta(t)$ are

$$\delta(t)=0 \text{ if } t\neq 0,\qquad \int_{-\infty}^{\infty}\delta(t)dt=1$$

Distributions ($\mathcal{D}'$):

Linear mappings $\mathcal{D} \to \mathbb{C}$ (linear functionals on $\mathcal{D}$)

Dirac delta: $\delta(\varphi) = \varphi(0)$, formally $\delta(\varphi) = \langle \delta, \varphi \rangle = \int_{-\infty}^{\infty} \delta(t)\varphi(t)\,dt = \varphi(0)$

Derivation: $\varphi, \psi \in \mathcal{D}$: $\langle \varphi, \psi' \rangle = \int \varphi(t)\psi'(t)\,dt = -\int \varphi'(t)\psi(t)\,dt = -\langle \varphi', \psi \rangle$

Let $f \in \mathcal{D}'$, then f' has definition:

$$\langle f', \varphi \rangle = -\langle f, \varphi' \rangle \quad \text{for all } \varphi \in \mathcal{D}.$$

Examples: $\langle \delta', \varphi \rangle = -\langle \delta, \varphi' \rangle = -\varphi'(0)$, $\langle \delta'', \varphi \rangle = -\langle \delta', \varphi' \rangle = \langle \delta, \varphi'' \rangle = \varphi''(0)$.

$f \in \mathcal{D}' \longrightarrow f', f''$, etc, $\varphi \cdot f = \varphi(t) f(t)$ is sensible,

but $f \cdot g$ might not exist.

Example: $d_\varepsilon(t) = \begin{cases} 1/\varepsilon & \text{if } t \in [0,\varepsilon] \\ 0 & \text{otherwise} \end{cases}$, $\tilde{d}_\varepsilon(t) = \begin{cases} 1/\varepsilon & \text{if } t \in [-\varepsilon, 0) \\ 0 & \text{otherwise} \end{cases}$.

Then $d_\varepsilon, \tilde{d}_\varepsilon \to \delta$, but $d_\varepsilon \cdot \tilde{d}_\varepsilon = 0$, however $\langle d_\varepsilon \cdot d_\varepsilon, \varphi \rangle = \int_0^\varepsilon \frac{1}{\varepsilon^2}\varphi(t)\,dt \to \varphi(0)\cdot\infty$

However $\delta_2(x,y) = \delta_2(\vec{r}) = \delta(x)\delta(y)$ makes sense:

$$\langle \delta_2, \varphi(x)\psi(y) \rangle = \iint \delta(x)\delta(y)\,\varphi(x)\psi(y)\,dx\,dy = \int \delta(x)\varphi(x)\,dx \cdot \int \delta(y)\psi(y)\,dy$$
$$= \varphi(0)\psi(0) = (\varphi\cdot\psi)(0,0)$$

Now we can derive nondifferentiable functions:

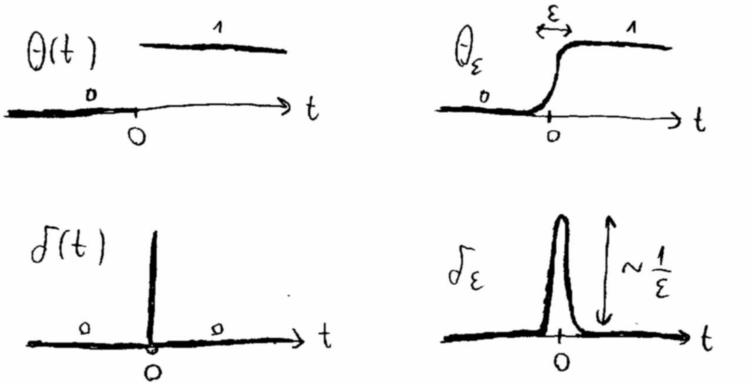
Heaviside theta, unit step function: $\theta(t) = \begin{cases} 0, & t<0 \\ 1, & t>0 \end{cases}$

Statement: $\theta'(t) = \delta(t)$

Proof: $\langle \theta', \varphi \rangle = -\langle \theta, \varphi' \rangle = -\int_{-\infty}^{\infty} \theta(t)\varphi'(t)\,dt = -\int_0^\infty \varphi'(t)\,dt = -\varphi(t)\Big|_0^\infty$

$$= \left(\varphi(\infty) - \varphi(0)\right) = \varphi(0) = \langle \delta, \varphi \rangle$$

So $\theta' = \delta$



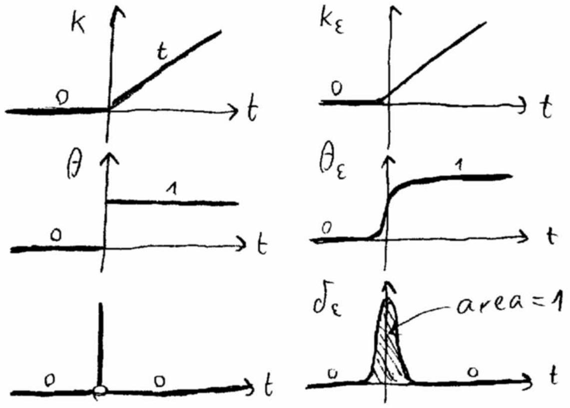
pictures of θ, δ, and their smooth $\theta_\varepsilon, \delta_\varepsilon$ approximations.

$$k(t)=\begin{cases} t & t>0 \\ 0 & t<0 \end{cases}$$

$$\theta(t)=\begin{cases} 1 & t>0 \\ 0 & t<0 \end{cases}$$

$$\delta(t): \quad t\neq 0 \;\; \delta(t)=0, \quad \int_{-\infty}^{\infty}\delta(t)\,dt=1$$

$$\boxed{k'=\theta,\ \theta'=\delta,\ k''=\delta}$$



5 $_{\overline{VII}}$ *

$\varepsilon \to 0$

Exercise: $\frac{d}{dt}y=-3y+f(t)$, $y(-\infty)=0=f(-\infty)$ $\quad (f(t)=y(t)=0$, if $t\ll 0)$

Solution:

① $y(t)=\int_{-\infty}^{t}e^{-3(t-s)}f(s)\,ds=\int_{-\infty}^{\infty}G(t-s)f(s)\,ds=(G*f)(t)$, where $G(t)=\begin{cases} e^{-3t} & t>0 \\ 0 & t<0 \end{cases}$

② ⓐ Solve $\frac{d}{dt}G(t)=-3G(t)+\delta(t)$, $G(-\infty)=0$

① $t\neq 0$: $\frac{d}{dt}G(t)=-3G(t) \longrightarrow G(t)=C\cdot e^{-3t}$ (with different C for $t<0$ and $t>0$)

$G(-\infty)=0 \longrightarrow \boxed{G(t)=0,\ \text{if } t<0}$

② $t\approx 0$: $\frac{d}{dt}G(t)=-3G(t)+\delta(t)\approx\delta(t) \longrightarrow \left.\begin{array}{l} G(t)=\theta(t)+C \\ G(-\infty)=0 \end{array}\right\} \to G(0^+)=1$

③ $t>0$: $\frac{d}{dt}G(t)=-3G(t)$, $G(0^+)=1 \longrightarrow G(t)=e^{-3t}$

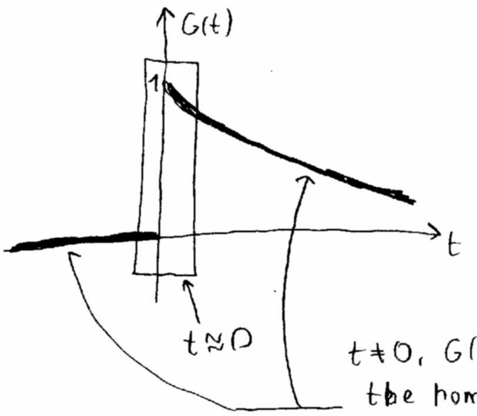
So $G(t)=\begin{cases} 0 & t<0 \\ e^{-3t} & t>0 \end{cases}$

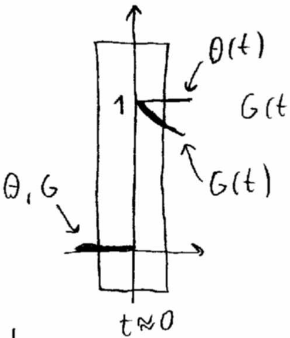
Remark: $G'=-3G+\delta \iff \underbrace{\langle G',\varphi\rangle}_{-\langle G,\varphi'\rangle}=-3\langle G,\varphi\rangle+\langle\delta,\varphi\rangle \quad \forall\varphi\in D$

$$-\int_0^{\infty}e^{-3t}\cdot\varphi'(t)\,dt=-3\int_0^{\infty}e^{-3t}\cdot\varphi(t)\,dt+\varphi(0)$$

ⓑ Consequently

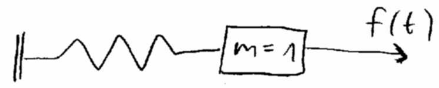
$$y(t)=\int_{-\infty}^{\infty}G(t-s)f(s)\,ds=\int_{-\infty}^{t}e^{-3(t-s)}f(s)\,ds$$





$G(t)\approx\theta(t)$, since $\delta(t)\gg -3G(t)$,

so $\frac{d}{dt}G(t)=-3G(t)+\delta(t)\approx\delta(t)$



$$\frac{d^2}{dt^2} y(t) = -4y(t) + f(t), \quad y(t), f(t) = 0, \text{ if } t \ll 0$$

Solution:

(1) $\frac{d^2}{dt^2} G(t) = -4G(t) + \delta(t), \quad G(t) = 0, \text{ if } t<0$

(a) $t \neq 0$: $G''(t) = -4G(t) \longrightarrow G(t) = C_1 \cos(2t) + C_2 \sin(2t)$

C_i different for $t<0, t>0$

$(G(t)=0, \text{ if } t<0) \longrightarrow G(0^-) = G'(0^-) = 0$

(b) $t \approx 0$: $G''(t) = -4G(t) + \delta(t) \approx \delta(t)$

$K''(t) = \delta(t) \longrightarrow K(t) = (a+bt) + \begin{cases} t & t>0 \\ 0 & t<0 \end{cases}$

$G \approx K, \; G(0^-) = G'(0^-) = 0 \longrightarrow K(t) = \begin{cases} t & t>0 \\ 0 & t<0 \end{cases}$

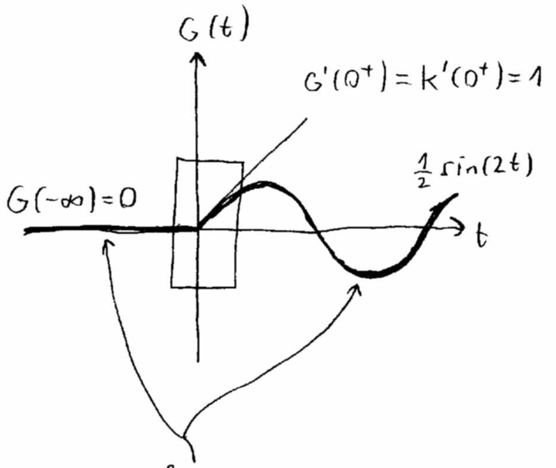
So $G(0^+) = K(0^+) = 0, \; G'(0^+) = K'(0^+) = 1$

(c) $t>0$:

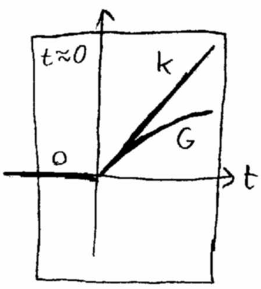
$G(0^+) = 0, \; G'(0^+) = 1, \; G''(t) = -4G(t) \longrightarrow$

$$\longrightarrow G(t) = \begin{cases} 0, & \text{if } t<0 \\ \frac{1}{2}\sin(2t), & \text{if } t>0 \end{cases}$$

(2) Consequently $y(t) = \int_{-\infty}^{t} \frac{1}{2} \sin(2(t-s)) f(s)\, ds = \int_{-\infty}^{\infty} G(t-s) f(s)\, ds$



$\frac{d^2}{dt^2} G(t) = -4G(t)$ if $t \neq 0$



$k(t) = G(t) = 0$, if $t<0$

$k(t) \approx G(t)$, if $t \approx 0$

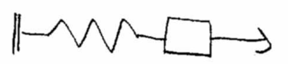
$k(0^-) = 0, \; k(0^+) - k(0^-) = 0$

$k'(0^-) = 0, \; k'(0^+) - k'(0^-) = 1$

$\longrightarrow k(0^+) = 0, \; k'(0^+) = 1$

$\longrightarrow G(0^+) = 0, \; G'(0^+) = 1$

$\longrightarrow G(t) = \frac{1}{2} \sin(2t)$, if $t>0$

 $y''+4y=f(t)$

$$\frac{d}{dt}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix} \qquad (\text{here } f_1(t)=0,\ f_2(t)=f(t))$$

$$A=\begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix},\ \det(A-\lambda E)=\lambda^2+4,\ \lambda_{1,2}=\pm 2i,\quad \lambda_1=2i,\ \vec{V}_1=\begin{bmatrix} 1 \\ 2i \end{bmatrix},\quad \lambda_2=-2i,\ \vec{V}_2=\begin{bmatrix} 1 \\ -2i \end{bmatrix}$$

$$e^{tA}=\begin{bmatrix} 1 & 1 \\ 2i & -2i \end{bmatrix}\begin{bmatrix} e^{2it} & 0 \\ 0 & e^{-2it} \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 2i & -2i \end{bmatrix}^{-1}=\begin{bmatrix} \cos(2t) & \frac{1}{2}\sin(2t) \\ -2\sin(2t) & \cos(2t) \end{bmatrix}$$

(1) $\vec{y}(t)=\vec{f}(t)=0,\ \text{if } t<<0$

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}=\int_{-\infty}^{t} e^{(t-s)A}\vec{f}(s)\,ds=\int_{-\infty}^{t}\begin{bmatrix} \cos(2(t-s)) & \frac{1}{2}\sin(2(t-s)) \\ -2\sin(2(t-s)) & \cos(2(t-s)) \end{bmatrix}\begin{bmatrix} f_1(s) \\ f_2(s) \end{bmatrix}ds$$

$f_1(s) \leftarrow 0$, $f_2(s) \leftarrow f(t)$

$$y(t)=y_1(t)=\int_{-\infty}^{t}\cos(2(t-s))\cdot 0+\frac{1}{2}\sin(2(t-s))\cdot f(t)\,dt$$

$$\vec{y}(t)=\int_{-\infty}^{t}e^{(t-s)A}\vec{f}(s)\,ds=\int_{-\infty}^{\infty}G(t-s)\vec{f}(s)\,ds,\qquad G(t)=\begin{cases} 0, & \text{if } t<0 \\ e^{tA}, & \text{if } t>0 \end{cases}$$

$$G(t)=\begin{bmatrix} \boxed{\begin{matrix} G_{11}(t) \\ G_{21}(t) \end{matrix}} & \boxed{\begin{matrix} G_{12}(t) \\ G_{22}(t) \end{matrix}} \end{bmatrix}$$

$$\frac{d}{dt}G(t)=\begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix}G(t)+\delta(t)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\quad \text{since } G(0^+)-G(0^-)=E \longrightarrow \frac{d}{dt}G(t)\Big|_{t=0}=\delta(t)\cdot E$$

$$\frac{d}{dt}\begin{bmatrix} G_{11}(t) \\ G_{21}(t) \end{bmatrix}=\begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix}\begin{bmatrix} G_{11}(t) \\ G_{21}(t) \end{bmatrix}+\begin{bmatrix} \delta(t) \\ 0 \end{bmatrix},\qquad \begin{bmatrix} G_{11}(0^+) \\ G_{21}(0^+) \end{bmatrix}=\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\frac{d}{dt}\begin{bmatrix} G_{12}(t) \\ G_{22}(t) \end{bmatrix}=\begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix}\begin{bmatrix} G_{12}(t) \\ G_{22}(t) \end{bmatrix}+\begin{bmatrix} 0 \\ \delta(t) \end{bmatrix},\qquad \begin{bmatrix} G_{12}(0^+) \\ G_{22}(0^+) \end{bmatrix}=\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

(2) $\vec{y}(0)=\vec{y}_0$

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}=e^{tA}\vec{y}_0+\int_0^t e^{(t-s)A}\vec{f}(s)\,ds=G(t)\vec{y}_0+\int_0^{\infty}G(t-s)f(s)\,ds$$

Remark: $[t>0,\ \vec{y}(0)=\vec{y}_0,\ f(t)]$ can be traded for $[\vec{y}(t)=\vec{f}(t)=0 \text{ for } t<0,\ \tilde{f}(t)=\Theta(t)\vec{f}(t)+\vec{y}_0\delta(t)]$

Remark: If $y''+4y=f$, then f has direct influence only on $y'=y_2$, while it might be possible, that we can observe only the $y=y_1$ coordinate of the state space $(y,y')=(y_1,y_2)$.

In matrix form:

$$\frac{d}{dt}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}=\begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}+\begin{bmatrix} 0 \\ 1 \end{bmatrix}[f(t)],\qquad [z_1(t)]=[1,0]\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}=[y(t)]$$

$$\frac{d}{dt}\vec{y}=A\vec{y}+B\vec{u}(t),\qquad \vec{z}(t)=C\vec{y}(t)$$

B: input matrix

C: output matrix

$\frac{d}{dt}\vec{y} = A\vec{y} + \vec{f}(t), \quad \vec{f}(t) = \vec{y}(t) = \vec{0}, \text{ if } t << 0$ $\boxed{8_{\mathrm{VII}}}$

Then $\vec{y}(t) = \int_{-\infty}^{t} e^{(t-s)A} \vec{f}(s)\, ds = \int_{-\infty}^{\infty} G(t-s)\vec{f}(s)\, ds$

What is G for?

Let L be a translation invariant differential operator, $(LG)(\vec{x}) = \delta(\vec{x})$

Then the solution of $L\varphi(\vec{x}) = f(\vec{x})$ is $\int d\vec{s}\, G(\vec{x}-\vec{s}) f(\vec{s}) = \varphi(\vec{x})$

Examples:

Operator	Green function, impulse response
$\partial_t + a$	$\Theta(t)\, e^{-at}$
$\partial_t^2 + k^2$	$\Theta(t)\, \frac{1}{k}\sin(kt)$
$\Delta = \partial_x^2 + \partial_y^2 + \partial_z^2$	$-\frac{1}{4}(x^2+y^2+z^2)^{-1/2} = -\frac{1}{4\pi r}$
$\partial_t - \partial_x^2$	$\Theta(t)\, \frac{1}{\sqrt{4\pi t}}\, e^{-x^2/4t}$

Exercise: $(\Delta\varphi)(x_1, x_2, x_3) = \exp\left(-(x_1^2+x_2^2+x_3^2)\right) = e^{-r^2}$. Find a solution φ!

Solution:

$$\varphi(x_1,x_2,x_3) = \iiint_{\mathbb{R}^3} ds_1\, ds_2\, ds_3 \; -\frac{1}{4\pi}\left((x_1-s_1)^2 + (x_2-s_2)^2 + (x_3-s_3)^2\right)^{-1/2} \cdot \exp\left(-(s_1^2+s_2^2+s_3^2)\right)$$

Exercise: $\partial_t \varphi = \partial_{xx}\varphi + f(t,x)$. Find a solution φ!

Solution:

$$(\partial_t - \partial_{xx})\left[\Theta(t)\frac{1}{\sqrt{4\pi t}}\, e^{-x^2/4t}\right] = \delta(t)\,\delta(x)$$

$$\varphi(t,x) = \iint ds\, dy\; \Theta(t-s)\, \frac{1}{\sqrt{4\pi(t-s)}} \exp\left[-\frac{(x-y)^2}{4(t-s)}\right] f(s,y)$$

Exercise: $\partial_t\varphi = \partial_{xx}\varphi$, $\varphi(0,x) = \frac{1}{1+x^2}$. How much is $\varphi(t,x)$, $t>0$?

Solution:

$$\varphi(t,x) = \int dy\; \Theta(t-0)\frac{1}{\sqrt{4\pi(t-0)}} \exp\left[-\frac{(x-y)^2}{4(t-0)}\right] = \int dy\, \frac{1}{\sqrt{4\pi t}}\, e^{-\frac{(x-y)^2}{4t}}$$

Time dependent linear systems

$$\frac{d}{dt}\vec{y}(t) = A(t)\,\vec{y}(t) + \vec{f}(t), \quad \vec{y}(0) = \vec{y}_0$$

① Compute the evolution operator of the hom. lin. system!

$$\vec{y}(t) = \Phi_{t,0}\,\vec{y}(0) \longrightarrow \frac{d}{dt}\vec{y}(t) = \left(\frac{d}{dt}\Phi_{t,0}\right)\cdot\vec{y}(0) = A(t)\,\Phi_{t,0}\,\vec{y}(0) \longrightarrow$$

$$\longrightarrow \frac{d}{dt}\Phi_{t,0} = A(t)\,\Phi_{t,0}, \qquad \Phi_{0,0} = E$$

Remark: Usually it is hard to get an explicit expression for $\Phi_{t,0}$.
However if $[A(t), A(t')] = 0$ for all t, t', then $\Phi_{t,0} = \exp\left(\int_0^t A(\tilde{t})\,d\tilde{t}\right)$

② The $t_2 \leftarrow t_1$ evolution operator:

$$\vec{y}(t_2) = \Phi_{t_2,t_1}\,\vec{y}(t_1), \qquad \Phi_{t_2,t_1} = \Phi_{t_2,0}\cdot\Phi_{0,t_1} = \Phi_{t_2,0}\,\Phi_{t_1,0}^{-1}$$

③ Solution of the inhom. DE:

$$\frac{d}{dt}\vec{y}(t) = A(t)\,\vec{y}(t) + \vec{f}(t), \quad \vec{y}(0) = \vec{y}_0$$

$$\vec{y}(t) = \Phi_{t,0}\,\vec{y}_0 + \int_0^t \Phi_{t,s}\,\vec{f}(s)\,ds = \Phi_{t,0}\,\vec{y}_0 + \int_0^t \Phi_{t,0}\,\Phi_{s,0}^{-1}\,\vec{f}(s)\,ds$$

④ Green függvény:

$$\frac{d}{dt}\vec{y}(t) = A(t)\,\vec{y}(t) + \vec{f}(t), \quad \vec{y}(t) = \vec{f}(t) = \vec{0}, \text{ if } t \ll 0$$

$$\vec{y}(t) = \int_{-\infty}^{t} \Phi_{t,s}\,\vec{f}(s)\,ds = \int_{-\infty}^{\infty} G(t,s)\,\vec{f}(s)\,ds$$

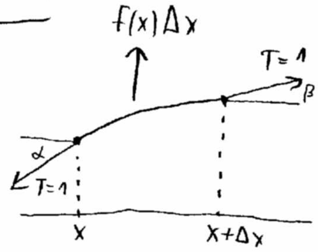
$G(t,s) = 0$ if $t < s$

$G(t,s) = \Phi_{t,s}$ if $t > s$

$$\left(\frac{\partial}{\partial t} - A(t)\right) G(t,s) = \delta(s-t)\,E$$

1 dim Poisson equation

Streched string



$\sin\alpha \approx tg\,\alpha = \varphi' x$

$\sin\beta \approx tg\,\beta = \varphi'(x+\Delta x$

$\approx \varphi'(x) + \varphi''(x)\,\Delta x$

$f(x)\Delta x + T\cdot(\sin\beta - \sin\alpha) \approx 0$

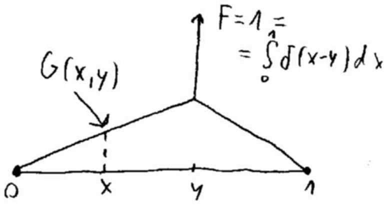
$\approx (f(x) + 1\cdot\varphi''(x))\,\Delta x \approx 0$

Problem: $\varphi''(x) = -f(x)$, $\varphi(0) = \varphi(1) = 0$. Compute φ! $\longrightarrow$ $\boxed{\varphi''(x) = -f(x)}$

$\Delta\varphi = \partial_x^2\varphi$

Solution:

① Green function $G(x,y)$:



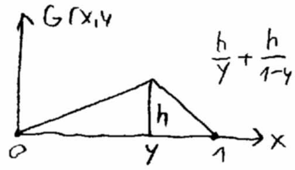
$G(x,y)$: Response of the system at x for a unit force concentrated at y.

ⓐ $G(0,y) = G(1,y) = 0$

ⓑ $\frac{d^2}{dx^2} G(x,y) = -\delta(x,y) \longrightarrow \frac{d^2}{dx^2} G(x,y) = 0$ if $x \in [0,y)$ or $x \in (y,1]$

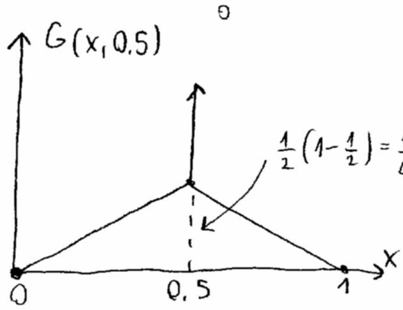
$\searrow G'(y+0,y) - G'(y-0,y) = -1$

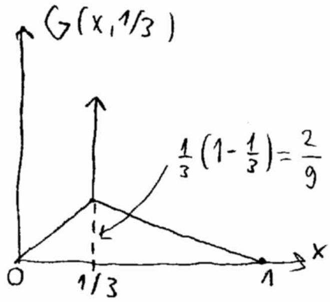
$$\longrightarrow G(x,y) = \begin{cases} (1-y)x & \text{if } x<y \\ (1-x)y & \text{if } x>0 \end{cases}$$

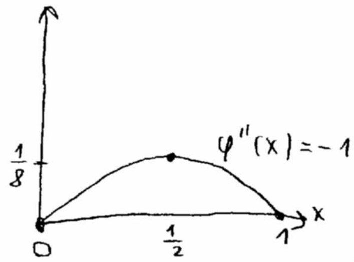


$\frac{h}{y} + \frac{h}{1-y} = 1 \rightarrow h = y(1-y)$

ⓒ $\varphi(x) = \int_0^1 G(x,y)\, f(y)\,dy$







ⓓ If $f(x) = 1$,

$$\varphi(x) = \int_0^1 G(x,y)\cdot 1\,dy = \int_0^x (1-x)y\,dy + \int_x^1 (1-y)x\,dy = \frac{1}{2}x(1-x)$$

Summary

① $\frac{d}{dt}\vec{y}(t) = A\vec{y}(t) + \vec{f}(t)$

ⓐ $\vec{y}(t) = \vec{f}(t) = \vec{0}$, if $t \ll 0$

ⓑ $\vec{y}(0) = \vec{y}_0$, $t \geq 0$

Solution: ⓐ $\vec{y}(t) = \int_{-\infty}^{t} e^{(t-s)A}\vec{f}(s)\,ds = \int_{-\infty}^{\infty} G(t-s)\vec{f}(s)\,ds$

ⓑ $\vec{y}(t) = e^{tA}\vec{y}_0 + \int_0^t e^{(t-s)A}\vec{f}(s)\,ds = G(t)\vec{y}_0 + \int_{-\infty}^{\infty} G(t-s)\cdot\theta(s)f(s)\,ds$

Green function: $G(t) = \begin{cases} 0, & \text{if } t<0 \\ e^{tA}, & \text{if } t>0 \end{cases}$ $\quad \left(\frac{d}{dt} - A\right)G(t) = \delta(t)\cdot E$

② Distributions, generalized functions

ⓐ Space $\mathcal{D}$ of test functions: $\varphi \in \mathcal{D} \longrightarrow \varphi$ smooth, zero outside of a finite interval

ⓑ $\mathcal{D}'$ distributions: Linear mappings: $\mathcal{D} \to \mathbb{C}$.

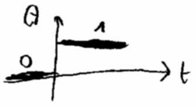
ⓒ Dirac delta δ: $\delta(\varphi) = \varphi(0) = \langle\delta,\varphi\rangle = \int_{-\infty}^{\infty}\delta(t)\varphi(t)\,dt$

ⓓ $f \in \mathcal{D}' \to f' \in \mathcal{D}'$, $\quad \langle f',\varphi\rangle = -\langle f,\varphi'\rangle = \int_{-\infty}^{\infty} f'(t)\varphi(t)\,dt = -\int_{-\infty}^{\infty} f(t)\varphi'(t)\,dt$

ⓔ $K(t) = \begin{cases} t, & t>0 \\ 0, & t<0 \end{cases}$

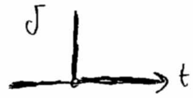
$K' = \theta$

$\theta(t) = \begin{cases} 1, & t>0 \\ 0, & t<0 \end{cases}$



$\theta' = \delta$ Heaviside theta

$\delta(t)$: $\delta(t) = 0$ if $t \neq 0$
$\int_{-\infty}^{\infty}\delta(t)\,dt = 1$



$K'' = \theta' = \delta$

③ Examples:

ⓐ $y' + 5y = f(t) \longrightarrow G'(t) + 5G(t) = \delta(t)$

$t<0 \longrightarrow G(t) = 0$, $\quad t\approx 0$: $\quad G(t) \approx \theta(t)$, since $G'(t) = -5G(t) + \delta(t) \approx \delta(t)$

$G(0^+) - G(0^-) = 1 \longrightarrow G(0^+) = 1$

$t>0$: $\quad G(0^+) = 1$, $G'(t) + 5G(t) = 0 \longrightarrow G(t) = \begin{cases} e^{-5t}, & t>0 \\ 0, & t<0 \end{cases}$

ⓑ $y'' + 4y = f(t) \longrightarrow G''(t) + 4G(t) = \delta(t)$

$t<0 \longrightarrow G(t) = 0$, $\quad t\approx 0$: $\quad G(t) = K(t)$, since $G''(t) = -4G(t) + \delta(t) \approx \delta(t)$

$G(0^+) - G(0^-) = 0$, $G'(0^+) - G'(0^-) = 1 \longrightarrow$

$\left.\begin{array}{l} \longrightarrow G(0^+) = 0,\ G'(0^+) = 1, \\ t>0:\ G''(t) + 4G(t) = 0 \end{array}\right\} \longrightarrow G(t) = \begin{cases} \frac{1}{2}\sin t & t>0 \\ 0 & t<0. \end{cases}$

Sample problems

(1) $y'(t)=3y(t)+f(t)$, $f(t)=\begin{cases}4, & \text{if } t\in[1,2] \\ 0 & \text{otherwise}\end{cases}$, $y(0)=7$.

(a) What is the retarded solution of $G'(t)=3G(t)+\delta(t)$? (b) How much is $y(8)$?

Solution:

(a) G retarded $\longrightarrow G(t)=0$, if $t<0 \longrightarrow G(0^-)=0$

$t\approx 0$ $\quad G'(t)=3G(t)+\delta(t)\approx\delta(t) \longrightarrow G(0^+)-G(0^-)=\theta(0^+)-\theta(0^-)=1$ $\Big\} \longrightarrow G(0^+)=1$

$t>0$ $\quad G(0^+)=1,\ G'(t)=3G(t) \longrightarrow G(t)=\begin{cases}0 & \text{if } t<0 \\ e^{3t} & \text{if } t>0\end{cases}$

G can be used to express solutions of the DE, for example:

$(y(t)=f(t)=0, \text{ if } t\ll 0) \longrightarrow y(t)=\int_{-\infty}^{t} e^{3(t-s)}f(s)\,ds=\int_{-\infty}^{\infty} G(t-s)f(s)\,ds=(G*f)(t)$

(b) $y(t)=e^{3t}\cdot 7+\int_0^t e^{3(t-s)}f(s)\,ds$

$y(8)=e^{3\cdot 8}\cdot 7+\int_0^8 e^{3(8-s)}f(s)\,ds=e^{3\cdot 8}\cdot 7+\int_1^2 e^{3\cdot(8-s)}\cdot 4\,ds$

(2) $y''(t)=-9y(t)+f(t)$, $y(t)=f(t)=0$, if $t\ll 0$. Compute $y(t)$!

What is the retarded solution of $G''(t)=-9G(t)+\delta(t)$?

Solution:

(a) G retarded $\longrightarrow G(t)=0$ if $t<0 \longrightarrow G(0^-)=G'(0^-)=0$

$t\approx 0$ $\quad G''=-9G+\delta\approx\delta \longrightarrow G(t)\approx k(t) \longrightarrow G(0^+)=0,\ G'(0^+)=1 \longrightarrow$

$t>0$ $\quad G''=-9G,\ G(0^+)=0,\ G'(0^+)=1 \longrightarrow G(t)=\begin{cases}0, & \text{if } t<0 \\ \frac{1}{3}\sin(3t), & \text{if } t>0\end{cases}$

(b) $y(t)=\int_{-\infty}^{t}\frac{1}{3}\sin(3(t-s))f(s)\,ds=\int_{-\infty}^{\infty}G(t-s)f(s)\,ds$

(3) $\frac{d}{dt}\begin{bmatrix}y_1\\ y_2\end{bmatrix}=\begin{bmatrix}-2 & 0\\ 2 & -3\end{bmatrix}\begin{bmatrix}y_1\\ y_2\end{bmatrix}+\begin{bmatrix}f_1(t)\\ f_2(t)\end{bmatrix}$. Solve the $\frac{d}{dt}G(t)=\begin{bmatrix}-2 & 0\\ 2 & -3\end{bmatrix}+\delta(t)\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$ DE, then use G to express the solution of the initial value problem $\vec{y}(0)=\begin{bmatrix}4\\ 5\end{bmatrix}$!

Solution:

$$e^{tA}=\begin{bmatrix}1 & 0\\ 2 & 1\end{bmatrix}\begin{bmatrix}e^{-2t} & 0\\ 0 & e^{-3t}\end{bmatrix}\begin{bmatrix}1 & 0\\ 2 & 1\end{bmatrix}^{-1}=\begin{bmatrix}e^{-2t} & 0\\ 2e^{-2t}-2e^{-3t} & e^{-3t}\end{bmatrix}$$

$$G(t)=\begin{cases}0, & \text{if } t<0\\ e^{tA}, & \text{if } t>0\end{cases}$$

$$\vec{y}(t)=G(t)\begin{bmatrix}4\\ 5\end{bmatrix}+\int_0^t G(t-s)\vec{f}(s)\,ds=$$

$$=\begin{bmatrix}e^{-2t} & 0\\ 2e^{-2t}-2e^{-3t} & e^{-3t}\end{bmatrix}\begin{bmatrix}4\\ 5\end{bmatrix}+\int_0^t\begin{bmatrix}e^{-2(t-s)} & 0\\ 2e^{-2(t-s)}-2e^{-3(t-s)} & e^{-3(t-s)}\end{bmatrix}\begin{bmatrix}f_1(s)\\ f_2(s)\end{bmatrix}ds.$$