Fourier transform

"Problem": Express $f(x)$ as an infinite linear combination of the sin, cos, exp functions!

Motivation: Heat conduction: $\partial_t \varphi(t,x) = \partial_{xx}\varphi(t,x)$, $\varphi(t,x+2\pi) = \varphi(t,x)$, $\varphi(0,x) = f(x)$

Solution: If for example $f(x) = \ldots + 13\cdot\sin(17\cdot x) + \ldots$

then $\varphi(t,x) = \ldots + 13\cdot e^{-17^2 t}\cdot\sin(17\cdot x) + \ldots$

Finite dim: $\vec{v}_1 \ldots \vec{v}_n$ basis, $\vec{x} = c_1\vec{v}_1 + \ldots + c_n\vec{v}_n = \underbrace{\begin{bmatrix} | & & | \\ \vec{v}_1 & \ldots & \vec{v}_n \\ | & & | \end{bmatrix}}_{S} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$

$$\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \underbrace{\begin{bmatrix} | & & | \\ \vec{v}_1 & \ldots & \vec{v}_n \\ | & & | \end{bmatrix}^{-1}}_{S^{-1}} \vec{x}$$

S^{-1}, might be hard to compute

If $\vec{v}_1, \ldots \vec{v}_n$ is an <u>orthonormed</u> basis: $(\vec{v}_i, \vec{v}_j) = \delta_{ij}$, then

$\vec{x} = c_1\vec{v}_1 + \ldots + c_n\vec{v}_n$,

$(\vec{v}_1, \vec{x}) = c_1\underbrace{(\vec{v}_1,\vec{v}_1)}_{1} + c_2\underbrace{(\vec{v}_1,\vec{v}_2)}_{0} + \ldots = c_1$, so

$$\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} (\vec{v}_1,\vec{x}) \\ (\vec{v}_2,\vec{x}) \\ \vdots \\ (\vec{v}_n,\vec{x}) \end{bmatrix} = \underbrace{\begin{bmatrix} — \vec{v}_1^{\,T} — \\ \vdots \\ — \vec{v}_n^{\,T} — \end{bmatrix}}_{S^{-1}} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$\vec{v}_k$ real: $S^{-1} = S^T$

$\vec{v}_k$ complex: $S^{-1} = S^* = \overline{S}^T$

Discrete Fourier transform:

$$P = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 1 & & & 0 \end{bmatrix}, \quad P\vec{v}_k = \varepsilon^k v_k, \quad \varepsilon^N = 1$$

Strategy: We try to treat the infinite dimensional Hilbert space $\mathcal{H}$ of the square integrable functions $L^2([-\pi,\pi], dx)$ as if it was the finite dimensional $\mathbb{C}^n$ with inner product $(\vec{f}, \vec{g}) = \sum_k \overline{f_k} g_k$.

① $f \in \mathcal{H} \longleftrightarrow (f,f) = \|f\|^2 = \int_{-\pi}^{\pi} |f(x)|^2\, dx < \infty$

Potential problem: Let $f(x)$ be nonzero only at finitely many points. Then $\|f\| = 0$, even though $f(x) \not\equiv 0$. So if we desire that $\|f-g\| = 0 \iff f = g$, than we are forced to regard the elements of $\mathcal{H}$ as certain equivalence classes of functions.

Another problem: We would like to have $\lim_{n\to\infty} \int f_n(x)\,dx = \int \lim_{n\to\infty} f_n(x)\,dx$

This requires a new definition of integration.

Mostly we ignore these sorts of problems.

Hilbert space: $\mathcal{H} = L^2([-\pi,\pi], dx)$: $f \in \mathcal{H}$, if $\int_{-\pi}^{\pi} \overline{f(x)} f(x)\,dx < \infty$

inner product: $(f,g) = \int_{-\pi}^{\pi} \overline{f(x)}\, g(x)\,dx$

norm: $\|f\| = (f,f)^{1/2} = \left(\int_{-\pi}^{\pi} |f(x)|^2 dx\right)^{1/2}$, distance: $d(f,g) = \|f-g\|$

orthonormed basis: $e_n(X) = \frac{1}{\sqrt{2\pi}} e^{inx}$, $n \in \mathbb{Z}$, $n = \dots -1, 0, 1, 2, \dots$

Remark: orthogonality is easy:

$$(e_n, e_m) = \int_{-\pi}^{\pi} \overline{\frac{1}{\sqrt{2\pi}} e^{inx}} \cdot \frac{1}{\sqrt{2\pi}} e^{imx} dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(m-n)x} dx = \begin{cases} \frac{1}{2\pi}\int_{-\pi}^{\pi} 1\,dx = 1, & \text{if } n=m \\ \frac{1}{2\pi i(m-n)} e^{i(m-n)x}\Big|_{-\pi}^{\pi} = 0 & \text{if } n \neq m \end{cases}$$

e_n is a basis $\iff \big((e_n, f) = 0 \ \ \forall n \in \mathbb{Z} \Rightarrow f = 0\big)$,
this is nontrivial.

Fourier transform:

$$f(x) = \sum_{n\in\mathbb{Z}} \hat{f}_n e_n(X) \longrightarrow \hat{f}_n = (e_n, f) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} e^{-inx} f(x)\,dx$$

$\mathcal{F}: L^2([-\pi,\pi], dx) \longrightarrow l_2$ $\qquad l_2: \hat{f} \in l_2 \iff \sum_n |\hat{f}_n|^2 < \infty$

$$(f,f) = \left(\sum_n \hat{f}_n e_n, \sum_m \hat{f}_m e_m\right) = \sum_n |\hat{f}_n|^2 = \int_{-\pi}^{\pi} |f(x)|^2 dx$$

So $\mathcal{F}$ gives an isometry between the square integrable functions and the square summable sequences. The realize the same Hilbert space $\mathcal{H}$.

Summary: $\mathcal{F}: f(X) \to \hat{f}_n$, $\qquad \hat{f}_n = (e_n, f) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} e^{-inx} f(x)\,dx$

$\mathcal{F}^{-1}: \hat{f}_n \to f(x)$, $\qquad f(x) = \sum_{n\in\mathbb{Z}} \hat{f}_n e_n(x) = \frac{1}{\sqrt{2\pi}} \sum_{n\in\mathbb{Z}} \hat{f}_n e^{inx}$

How good is the $f(x) \approx f_N(x) = \sum_{|n|\le N} \hat{f}_n e_n(x)$ approximation?

Without proof:

1. If f' exist everywhere, then $f(x) = \lim_{N\to\infty} f_N(x)$
2. Let f be given piecewise on finitely many intervals, and assume that f and f' has left and right limits on the intervals. Then inside the intervals $f(x) = \lim_{N\to\infty} f_N(x)$, while at the meeting point of two intervals we have $\lim_{N\to\infty} f_N(x) = \frac{1}{2}\left(\lim_{z\to x^+} f(z) + \lim_{z\to x^-} f(z)\right)$
3. It is true, that

$$\|f_N - f\| = \sum_{|n|>N} |\hat{f}_n|^2 \to 0 \quad \text{as } N\to\infty$$

④ Let $f(x)$ be continuous. Then $f(x)$ can be obtained from $f_N(x)$:

$N=1 \quad S_1(x) = f_1(x)$

$N=2 \quad S_2(x) = \frac{1}{2}(f_1(x)+f_2(x))$

$N=3 \quad S_3(x) = \frac{1}{3}(f_1(x)+f_2(x)+f_3(x))$

$\vdots$

$S_N(x) = \frac{1}{N}(f_1(x)+\dots\dots+f_N(x))$

Then $\lim\limits_{N\to\infty} S_N(x) = f(x)$ (Theorem of Lipót Fejér) Ludwig

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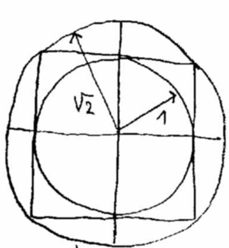
$f(x) \approx f_N(x)$ error. How to measure the "size" of $|f(x)-f_N(x)|$?

Frequently used norms:

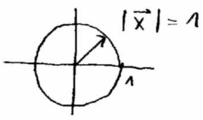
$$\|f\|_2 = \left(\int_{-\pi}^{\pi} |f(x)|^2 dx\right)^{1/2}, \quad \|f\|_1 = \int_{-\pi}^{\pi} |f(x)|dx, \quad \|f\|_\infty = \max_{x\in[-\pi,\pi]} |f(x)|$$

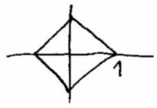
for cont. $f(x)$

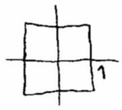


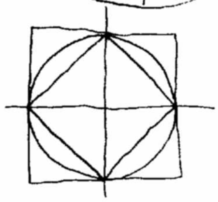
2dim variants:

$$\|\vec{x}\|_2 = \left|\begin{pmatrix}x_1\\x_2\end{pmatrix}\right| = \sqrt{x_1^2+x_2^2}, \quad \left\|\begin{pmatrix}x_1\\x_2\end{pmatrix}\right\|_1 = |x_1|+|x_2|, \quad \left\|\begin{pmatrix}x_1\\x_2\end{pmatrix}\right\|_\infty = \max(|x_1|,|x_2|)$$









Finite dim: $\left(\|\vec{x}_N - x\|_2 \to 0\right) \iff \left(\|\vec{x}_N-\vec{x}\|_1 \to 0\right) \iff \left(\|\vec{x}_N-\vec{x}\| \to 0\right)$

Same is $\underline{\text{FALSE}}$ in ∞ dimension

$$f(x)-f_N(x) = \sum_{|n|>N} \hat{f}_n \frac{e^{inx}}{\sqrt{2\pi}} \approx 0$$ if N is large enough and $|\hat{f}_n|$ has fast decrease as $n\to\infty$

How to ensure the fast decrease of $|\hat{f}_n|$? Assume that $|\hat{f}_n| \approx \frac{1}{n^k}$ if $|n|>>1$.

Then: $f\in\mathcal{H} \implies \sum_n |\hat{f}_n|^2 < \infty \longrightarrow k > 1/2$

$f'\in\mathcal{H} \implies \sum_n n^2|\hat{f}_n|^2 < \infty \longrightarrow k > 3/2 = 1+1/2$

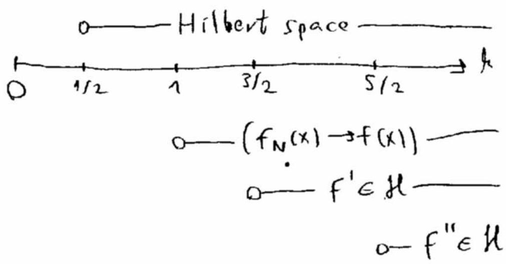
$f''\in\mathcal{H} \implies \sum_n n^4|\hat{f}_n|^2 < \infty \longrightarrow k > 5/2 = 2+1/2$

More derivative exist $\to$ faster decrease for $|\hat{f}_n|$

$$\|f(x)-f_N(x)\|_2 = \sum_{|n|>N} |\hat{f}_n|^2 \sim \frac{1}{N^{2k-1}}$$

$$\|f(x)-f_N(x)\|_\infty \sim \sum_{|n|>N} |\hat{f}_n| \sim \frac{1}{N^{k-1}}$$

worst case



Remarks:

① $\mathcal{H} = L^2([0,1], dx)$: Hilbert space of square integrable functions, $\|f\| = (f,f)^{1/2} = \left[\int_0^1 |f(x)|^2 dx\right]^{1/2}$. That can be zero, for example when $f(x)$ is nonzero only at finitely many points. So it is better to regard $\mathcal{H}$ as equivalenc classes of functions, $f \sim g$ if $\|f-g\| = 0$. It still makes sense to say that a continuous $f(x)$ is in $\mathcal{H}$, since if $f \neq g$ are continuous, then $\|f-g\| \neq 0$.

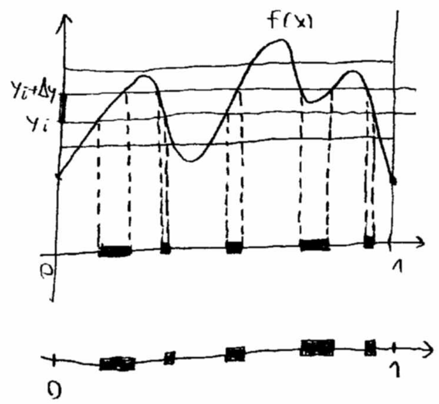
② How can we imagine $\mathcal{H}$, if we do not even know the precise meaning of the $\int_0^1 \ldots dx$ integration sign? As a start, take a simple class of functions, for example the functions consisting of finite number of constant pieces (step functions). Here it is clear what $\|f\|$ is. Then the elements of $\mathcal{H}$ are going to be the equivalence classes of Cauche sequences of these step functions:

f_n is Cauchy if $\lim\limits_{n,m\to\infty} \|f_n - f_m\| = 0$

$f_n \sim g_n$ if $\lim\limits_{n\to\infty} \|f_n - g_n\| = 0$

③ L^2 — 2 ← square integrable, L ← Lebesgue integral,

ℓ_2 ← sequences such that $\sum_n |x_n|^2 < \infty$



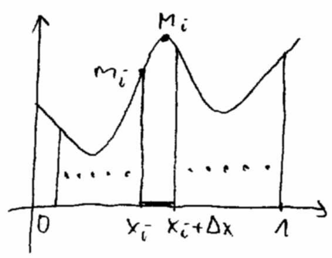
$A_i = \{x \in [0,1] \mid y_i \leq f(x) < y_i + \Delta y\}$

Then define the integral as

$\int_0^1 f(x)dx = \lim\limits_{\Delta y\to 0} \sum_i y_i \cdot \mu(A_i)$,

where $\mu(A_i)$ is the measure of the set A_i. How to define $\mu(A)$?
Cover A with disjunct open intervals and take the infinum (roughly the minimum) of the total length of such coverings.
Do the same for $[0,1]\setminus A$, and if $\mu(A) + \mu([0,1]\setminus A) = 1$, then we defined $\mu(A)$.

Riemann integral



$\sum_i m_i \Delta x \leq \int_0^1 f(x)dx \leq \sum_i M_i \Delta x$

If $\lim\limits_{\Delta x\to 0} \sum_i m_i \Delta x = \lim\limits_{\Delta x\to\infty} \sum_i M_i \Delta x$, then this limit is $\int_0^1 f(x)dx$

Advantage of Lebesgue over Riemann:

$\lim\limits_{n\to\infty} \int f_n(x)dx = \int \lim\limits_{n\to\infty} f_n(x)dx$

holds under weaker conditions.

Heat equation, periodic, 1+1 dim case

$\boxed{4^{*}_{VI}}$

$\partial_t \varphi = \partial_{xx}\varphi$, $\varphi(t,x) = \varphi(t,x+2\pi)$, $\varphi(0,x) = f(x)$

Motivation: heatcurrent $\sim$ $-$(temperature gradient)

temperature change rate $\sim$ $-$ divergence(heatcurrent) $\sim$ div(grad(temperature))

Strategy: (Fourier)

1. Find special, fatorized solutions: $\varphi(t,x) = T(t)\cdot X(x)$
2. As the PDE hom. lin., take the superpositions of them
3. Find a superposition such that the $\varphi(0,x) = f(x)$ initial condition is satisfied.

(1) $\partial_t\varphi = \partial_{xx}\varphi$, $\varphi = T(t)\cdot X(x) \Rightarrow T'(t)\cdot X(x) = T(t)\cdot X''(x) \Rightarrow \frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} = \text{const} = k$

if $g(t)=h(x)$, then g,h constant

if $X'' = kX$ and $X(x) = X(x+2\pi)$, then $k = -n^2$, $n = 0,1,2,\dots$

$X(x) = \cos(nx)$, $n = 0,1,2..$ or $X(x) = \sin(nx)$, $n = 1,2,3\dots$

$T'(t) = -n^2 T(t) \Rightarrow T(t) = e^{-n^2 t}$

Special solutions: $n=0$, $\varphi(t,x) = 1$

$n \geq 1$, $\varphi(t,x) = e^{-n^2 t}\cos(nx)$, $\varphi(t,x) = e^{-n^2 t}\sin(nx)$

(2) Produce an orthonormed basis from the cos, sin functions

(a) orthogonality: $\int_{-\pi}^{\pi} 1\cdot\cos(nx)dx = \int_{-\pi}^{\pi} 1\cdot\sin(nx)dx = \int_{-\pi}^{\pi}\sin(nx)\cdot\sin(mx)dx = \dots = 0$

sin, cos
cos, cos

$n \neq m$
$n,m \geq 1$

We are lucky

(b) normalization:

orthonormed system: $\frac{1}{\sqrt{2\pi}}, \frac{\cos(nx)}{\sqrt{\pi}}, \frac{\sin(nx)}{\sqrt{\pi}}$, $n = 1,2,3\dots$

(3) $f(x) = a_0\cdot\frac{1}{\sqrt{2\pi}} + \sum_{n=1}^{\infty} a_n\frac{\cos(nx)}{\sqrt{\pi}} + b_n\frac{\sin(nx)}{\sqrt{\pi}}$, where

$a_0 = \left(\frac{1}{\sqrt{2\pi}}, f(x)\right) = \int_{-\pi}^{\pi}\frac{1}{\sqrt{2\pi}}\cdot f(x)dx$, $a_n = \left(\frac{\cos(nx)}{\sqrt{\pi}}, f(x)\right) = \int_{-\pi}^{\pi}\frac{\cos(nx)}{\sqrt{\pi}}\cdot f(x)dx$

$b_n = \left(\frac{\sin(nx)}{\sqrt{\pi}}, f(x)\right) = \int_{-\pi}^{\pi}\frac{\sin(nx)}{\sqrt{\pi}}\cdot f(x)dx$

Solution:

$$\varphi(t,x) = \underbrace{e^{-0^2\cdot t}}_{=1}\cdot a_0\frac{1}{\sqrt{2\pi}} + \sum_{n=1}^{\infty} e^{-n^2 t}\left(a_n\frac{\cos(nx)}{\sqrt{\pi}} + b_n\frac{\sin(nx)}{\sqrt{\pi}}\right)$$

Strategy II:

$$\frac{d}{dt}\vec{\vec{\varphi}}(t) = \partial_{xx}\vec{\vec{\varphi}}(t), \quad \vec{\vec{\varphi}}(0) = \vec{\vec{f}}, \qquad \vec{\vec{f}}, \vec{\vec{\varphi}}(t) \in \mathcal{H} = L^2([-\pi,\pi], dx)$$

$\boxed{5_{\text{VI}}}$

$\partial_{xx} = (\partial_x)^2$, so search for the eigenvalues and eigenvectors of ∂_x !

$$\partial_x \vec{v} = \lambda \vec{v} \longleftrightarrow v'(x) = \lambda \cdot v(x), \quad v(x+2\pi) = v(x)$$

$$\longrightarrow \lambda_n = in, \quad v_n(x) = \frac{e^{inx}}{\sqrt{2\pi}}$$

so $\int_{-\pi}^{\pi} |v_n(x)|^2 dx = \|v_n\|^2 = 1^2$

So $\partial_x \vec{\vec{v}}_n = in \cdot \vec{\vec{v}}_n, \qquad \frac{d}{dx}\left(\frac{e^{inx}}{\sqrt{2\pi}}\right) = i \cdot n \left(\frac{e^{inx}}{\sqrt{2\pi}}\right), \qquad n \in \mathbb{Z}$

$v_n(x)$, $n \in \mathbb{Z}$ is an orthonormed basis:

$$(v_n, v_m) = \int_{-\pi}^{\pi} \overline{\frac{e^{inx}}{\sqrt{2\pi}}} \cdot \frac{e^{imx}}{\sqrt{2\pi}} dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(m-n)x} dx = \begin{cases} 1, & \text{if } m = n \\ \frac{1}{2\pi}\cdot\frac{1}{i(m-n)} e^{i(m-n)x}\Big|_{-\pi}^{\pi} = 0, & \text{if } m \neq n \end{cases}$$

Why? ∂_x is anti self-adjoint:

$$(f, \partial_x g) = \int_{-\pi}^{\pi} \overline{f(x)} \frac{d}{dx} g(x) dx = \underbrace{\overline{f(x)} \cdot g(x)\Big|_{-\pi}^{\pi}}_{=0 \text{ if } f, g \text{ periodic}} - \int_{-\pi}^{\pi} \overline{\frac{d}{dx} f(x)} \cdot g(x) dx = -(\partial_x f, g)$$

Consequently

$$(f, \partial_x g) = ((\partial_x)^* f, g) = -(\partial_x f, g) \implies (\partial_x)^* = -\partial_x$$

anti self-adjoint $\longrightarrow$ purely imaginary eigenvalues, orthogonal eigenvectors

Remark: These are formal calculations, we discard the question of the domain of the unbounded operator $\frac{d}{dx}$. We also believe that result from finite dimensional lin. alg. are applicable in infinite dimensional spaces.

So the eigensystem of ∂_x^2: $\quad \partial_x^2\left(\frac{e^{inx}}{\sqrt{2\pi}}\right) = -n^2\left(\frac{e^{inx}}{\sqrt{2\pi}}\right), \quad n \in \mathbb{Z}$

Remark: 1 On the periodic functions of $L^2([-\pi,\pi], dx)$ things are pretty much the same as in finite dim. lin. alg.

2 Try to repeat this on $L^2((-\infty,\infty), dx)$: $\partial_x e^{ipx} = ip\, e^{ipx}$, $p \in \mathbb{R}$, but $e^{ipx} \notin \mathcal{H}$, since $\int_{-\infty}^{\infty} \overline{e^{ipx}} \cdot e^{ipx} dx = \int_{-\infty}^{\infty} 1\, dx = \infty = \|e^{ipx}\|$, so we are out of $\mathcal{H}$.
Nevertheless this case is still close to 1.

3 What happens on $L^2([0,\infty), dx)$? $\quad \partial_x e^{px} = p \cdot e^{px}$, $\quad e^{px} \in \mathcal{H}$ if $\mathrm{Re}(p) < 0$.
But usually e^{px} not orthogonal to e^{qx}, $p \neq q$. Finite dim. lin. alg does not work.

1 $\longrightarrow$ 2 $\longrightarrow$ 3

Good $\qquad$ Not bad $\qquad$ Bad

Exercise: $\partial_t \varphi = \partial_x^2 \varphi$, $\varphi(t, x+2\pi) = \varphi(t,x)$,
$\varphi(0,x) = f(x) = \mathrm{sgn}(x)$ if $x \in [-\pi, \pi)$

Solution:

(1) Compute $\hat{f}$:

$$\hat{f}_n = \left(\frac{e^{inx}}{\sqrt{2\pi}}, \mathrm{sgn}(x) \right) = \int_{-\pi}^{\pi} \frac{e^{-inx}}{\sqrt{2\pi}} \mathrm{sgn}(x)\,dx = \int_{-\pi}^{0} \frac{e^{-inx}}{\sqrt{2\pi}} \cdot (-1)\,dx + \int_{0}^{\pi} \frac{e^{-inx}}{\sqrt{2\pi}} \cdot 1\,dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{-inx}}{-in} \cdot (-1) \Big|_{-\pi}^{0} + \frac{e^{-inx}}{-in} \cdot 1 \Big|_{0}^{\pi} \right] = \frac{1}{\sqrt{2\pi}} \cdot \frac{i}{n} \cdot \begin{cases} 0, & \text{if } n \text{ even} \\ -4, & \text{if } n \text{ odd} \end{cases}$$

So $$\mathrm{sgn}(x) = \frac{-4}{\sqrt{2\pi}} \cdot \sum_{\substack{n=2k+1 \\ k \in \mathbb{Z}}} \frac{i}{n} \frac{e^{inx}}{\sqrt{2\pi}} = \sum_{k=0}^{\infty} \frac{8}{2\pi} \cdot \frac{1}{2k+1} \cdot \sin((2k+1)x)$$

This is not quite an equality of functions, here it works for $x \in (-\pi, \pi)$

Remarks: (1) $\mathrm{sgn}(x)$ is real $\longrightarrow \hat{f}_n = \overline{\hat{f}_{-n}}$

(2) $|\hat{f}_n| \sim \frac{1}{|n|}$, $\mathrm{sgn}(x) \in \mathcal{H}$, but $\mathrm{sgn}'(x) \notin \mathcal{H}$ $\longleftarrow \sum_n n^2 |\hat{f}_n|^2 = \infty$

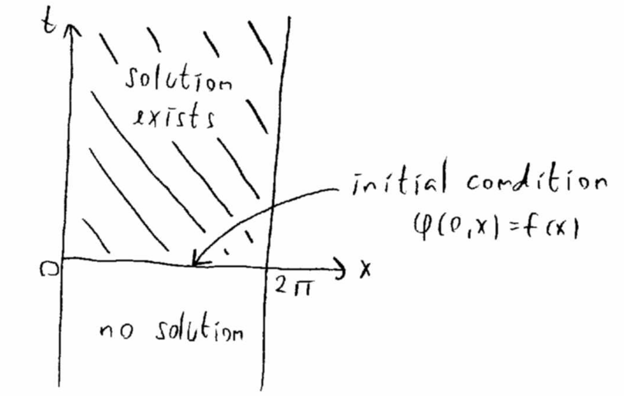
(3) $\mathrm{sgn}(x) = -\mathrm{sgn}(-x) \longrightarrow \mathrm{Re}(\hat{f}_n) = 0$ $\longleftarrow$ sine series

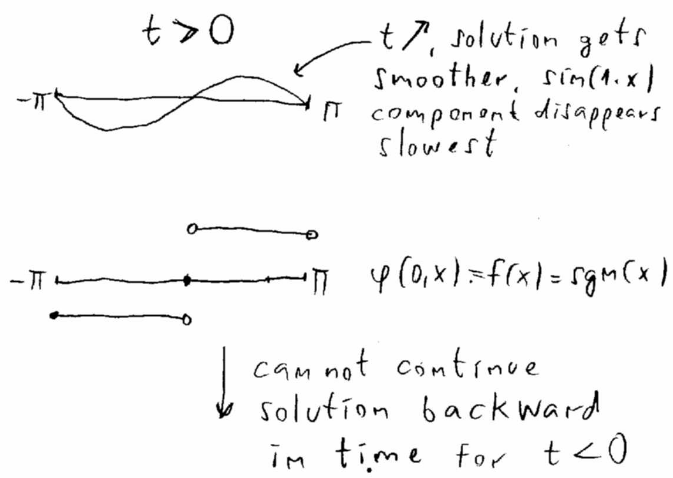
(2) $\varphi(t,x)$:

$$\varphi(t,x) = -\frac{4}{\sqrt{2\pi}} \cdot \sum_{\substack{n=2k+1 \\ k \in \mathbb{Z}}} \frac{i}{n} e^{-n^2 t} \cdot \frac{e^{inx}}{\sqrt{2\pi}} = \sum_{k=0}^{\infty} \frac{1}{2k+1} \frac{4}{\pi} e^{-n^2 t} \sin([2k+1]x)$$

Remark: $t > 0 \longrightarrow$ fast convergence, for example $t = 0.1$, $n = 20$, $e^{-n^2 t} = e^{-40} \sim 10^{-17}$

$t < 0 \longrightarrow e^{-n^2 t} = e^{n^2 |t|}$ has rapid increase, $\varphi(t,x)$ almost never exists.





Trigonometric Fourier series:

sin, cos are eigenvectors of ∂_x^2:

$$\partial_x^2 \sin(nx) = -n^2 \sin(nx), \quad \partial_x^2 \cos(nx) = -n^2 \cos(nx)$$

Advantage of exponential Fourier series:

① easyer to find eigenvectors for ∂_x, ② 1dim eigensupspaces

Where to go?

① 1dim: $[-\pi, \pi] \to [0,1]$ signalprocessing

$[-\pi, \pi] \to [-L\pi, L\pi] \xrightarrow{L \to \infty} (-\infty, \infty)$

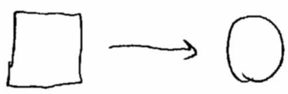
$[-\pi, \pi] \to (-\infty, \infty)$

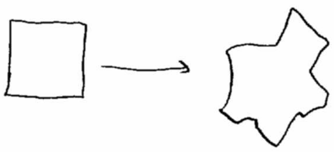
② Boundary conditions:

$\varphi(t,x) = \varphi(t, x+2\pi) \longrightarrow \varphi(0) = \varphi(\pi) = 0, \ \varphi'(0) = \varphi'(-\pi) = 0,$ etc.

③ 1dim $\longrightarrow$ 2, 3.. dim

$[-\pi, \pi] \longrightarrow [-\pi, \pi]^2 \longrightarrow [-L_1\pi, L_1\pi] \times [-L_2\pi, L_2\pi]$, etc.

④ $[0,1]^2 =$  , special functions, orthogonal series

⑤  Helmholtz equation $(\Delta f)(\vec{x}) = -k^2 f(\vec{x})$

①, ③ are relatively simple,

special cases of ② is simple, too.

④ has a well developed theory

⑤ numerical methods

$\boxed{8^{*}_{VI}}$

1. one dimension

(a) $[-\pi,\pi]$: orthonormed basis: $\frac{e^{inx}}{\sqrt{2\pi}}$, $n\in\mathbb{Z}$

$$\partial_x \frac{e^{inx}}{\sqrt{2\pi}} = (-in)\cdot\frac{e^{inx}}{\sqrt{2\pi}}$$

(b) $[-L\pi, L\pi]$: $\frac{e^{in\frac{x}{L}}}{\sqrt{2\pi L}}$, $n\in\mathbb{Z}$

$$\partial_x \frac{e^{in\frac{x}{L}}}{\sqrt{2\pi L}} = \left(-i\frac{n}{L}\right)\frac{e^{in\frac{x}{L}}}{\sqrt{2\pi L}}$$

(c) $L\to 0$, $\frac{n}{L}=p_n$, $\Delta p\to\frac{1}{L}$, $(-\infty,\infty)=\mathbb{R}$

orthonormed basis: $\frac{e^{ip_nx}}{\sqrt{2\pi L}}$, $\partial_x\frac{e^{ip_nx}}{\sqrt{2\pi L}} = (-ip_n)\frac{e^{ip_nx}}{\sqrt{2\pi L}}$, $n\in\mathbb{Z}$

Fourier-tr:

$$f(x)\longrightarrow\int_{-L\pi}^{L\pi}\frac{e^{-ip_nx}}{\sqrt{2\pi L}}\cdot f(x)\,dx = \hat{f}_n$$

$$\hat{f}\longrightarrow f(x) = \sum_{p_n=n\Delta p}\hat{f}_n\frac{e^{ip_nx}}{\sqrt{2\pi L}}$$

Modification:

"definition" of $\hat{f}(p)$

$$f(x)\longrightarrow\frac{1}{\sqrt{2\pi}}\int_{-L\pi}^{L\pi}e^{-ip_nx}f(x)\,dx = \hat{f}_n = \hat{f}(p_n)$$

$$\hat{f}(p)\longrightarrow\frac{1}{\sqrt{2\pi}}\sum_{p_n=\frac{n}{L}}\hat{f}(p_n)\,e^{ip_nx}\cdot\frac{1}{L}$$

$p_{n+1}-p_n=\frac{1}{L}=\Delta p$

$L\to\infty$ limit:

$$f(x)\longrightarrow\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-ipx}f(x)\,dx = \hat{f}(p)$$

$$\hat{f}(p)\longrightarrow\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(p)\,e^{ipx}\,dp = f(x)$$

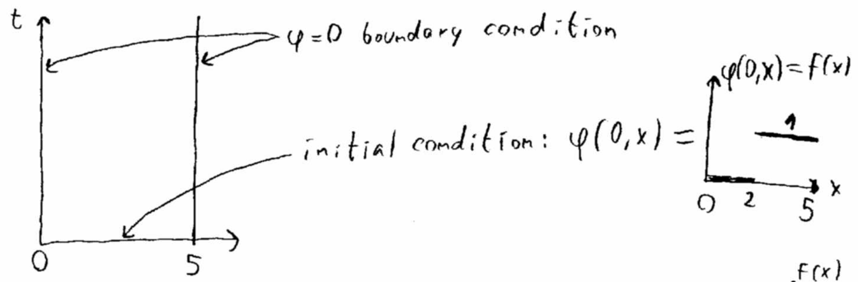
Fourier integral transform: $L^2((-\infty,\infty)\,dx)\longrightarrow L^2((-\infty,\infty),dx)$

nonperiodic signal $\longrightarrow$ continuous fequency spectrum

Sine transform

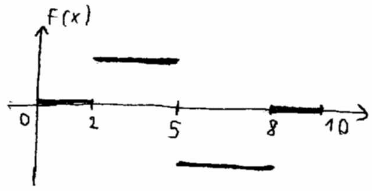
$\boxed{9^{*}_{\underline{VI}}}$

Problem: $\partial_t \varphi = \partial_{xx}\varphi$, $\varphi(t,0)=0$, $\varphi(t,5)=0$, $\varphi(0,x)=\begin{cases} 0, & \text{if } x\in[0,2] \\ 1, & \text{if } x\in(2,5) \end{cases}$



Strategy I: Reduce to the periodic case:

period $= 2\cdot 5 = 10$, initial condition:



orthonormed basis:

$$V_n(x) = \frac{1}{\sqrt{10}} \exp\left(i\frac{2\pi n}{10}x\right), \quad \partial_x V_n(x) = i\cdot\frac{2\pi n}{10} V_n(x), \quad n\in\mathbb{Z}$$

Fourier-tr.: $$\hat{f}_n = (V_n, f) = \int_0^{10} \frac{1}{\sqrt{10}} e^{-i\frac{2\pi n}{10}x} f(x)dx = \int_2^5 \frac{1}{\sqrt{10}} e^{-i\frac{2\pi n}{10}x}\cdot 1 dx + \int_5^8 \frac{1}{\sqrt{10}} e^{-i\frac{2\pi n}{10}x}\cdot(-1)dx$$

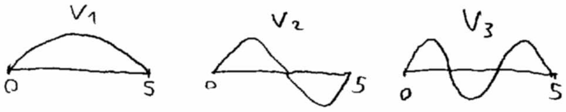
solution: $$\varphi(t,x) = \sum_{n\in\mathbb{Z}} \hat{f}_n \cdot \exp\left[-\left(\frac{2\pi n}{10}\right)^2 t\right]\cdot\frac{1}{\sqrt{10}}\exp\left[i\frac{2\pi n}{10}x\right], \quad x\in[0,5]$$

Strategy II:

Helmholtz equation: $\frac{d^2}{dx^2}V_n(x) = \lambda_n V_n(x)$, $V_n(0) = V_n(5) = 0$

$$\longrightarrow V_n(x) = \sin\left(\frac{\pi}{5}\cdot nx\right) \xrightarrow{\text{normalization}} V_n(x) = \sqrt{\frac{2}{5}}\cdot\sin\left(\frac{\pi}{5}\cdot nx\right), \quad n = 1,2,3,\dots$$

$$\frac{d^2}{dx^2}V_n(x) = -\left(\frac{\pi}{5}\right)^2\cdot n^2 V_n(x)$$



Sine-tr: $$\hat{f}_n = (V_n, f(x)) = \int_0^5 \sqrt{\frac{2}{5}}\cdot\sin\left(\frac{\pi}{5}\cdot nx\right)\cdot f(x)dx = \int_2^5 \sqrt{\frac{2}{5}}\sin\left(\frac{\pi}{5}\cdot nx\right)\cdot 1\,dx$$

Solution: $$\varphi(t,x) = \sum_{n=1}^{\infty} \hat{f}_n\cdot\exp\left[-\left(\frac{\pi}{5}\right)^2\cdot n^2\cdot t^2\right]\cdot\sqrt{\frac{2}{5}}\sin\left(\frac{\pi}{5}\cdot nx\right)$$

$\partial_x^2 = (\partial_x^2)^* \longrightarrow V_n(x)$ orthogonal system

$$(f,g'') = \int_0^5 \overline{f(x)}\, g''(x)dx = \underbrace{\overline{f(x)}\, g'(x)\Big|_0^5}_{=0} - \int_0^5 \overline{f'(x)}\, g'(x)dx$$

$$= \underbrace{-\overline{f'(x)}\, g(x)\Big|_0^5}_{=0} + \int_0^5 \overline{f''(x)}\, g(x)dx = (f'', g)$$

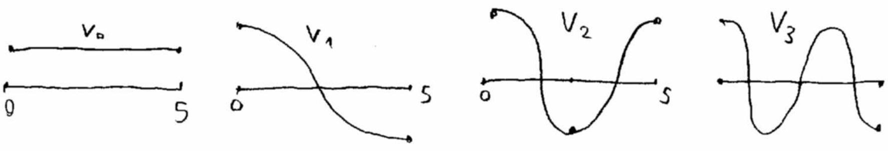
Cosine transform

Problem: $\partial_t \varphi = \partial_{xx}\varphi$, $\varphi'(t,0) = \varphi'(t,5)$, $\varphi(0,x) = f(x)$

Remark: Heat current $\sim -\text{grad}(\text{temperature}) = -\varphi'$. Zero at $x=0$, $x=5$, so no heat loss.

Orthonormed basis:

$$V_0(x) = \frac{1}{\sqrt{5}}, \qquad V_n(x) = \sqrt{\frac{2}{5}}\cos\left(\frac{\pi}{5}\cdot nx\right), \quad n = 1, 2, 3 \dots$$



$$\frac{d^2}{dx^2} V_n(x) = -\left(\frac{\pi}{5}\right)^2 n^2 V_n(x)$$

2 dimension, orthogonal periods

(a) $f(x,y) = f(x+2\pi, y) = f(x, y+2\pi)$, $\quad L^2([-\pi,\pi]^2, dx\,dy) = L^2(\square, dx\,dy)$

inner product: $(f,g) = \iint_{(x,y)\in\square} \overline{f(x,y)}\, g(x,y)\, dx\, dy$

orthonormed basis: $V_{n,m}(x,y) = V_n(x)\cdot V_m(y) = \frac{e^{inx}}{\sqrt{2\pi}}\cdot\frac{e^{imx}}{\sqrt{2\pi}} = \frac{1}{2\pi} e^{i(nx+my)} =$

$= \frac{1}{2\pi} \exp(i\underbrace{(n,m)\cdot(x,y)}_{\text{scalar product}})$

$\Delta = \partial_x^2 + \partial_y^2$, $\quad \Delta V_{n,m} = -(n^2+m^2) V_{n,m}$, $\quad n,m \in \mathbb{Z}$

(b) $f(x,y) = f(x+2\pi L_x, y) = f(x, y+2\pi L_y)$, $\quad L^2([-\pi L_x, \pi L_x]\times[-\pi L_y, \pi L_y], dx\,dy) = L^2(\square, dx\,dy)$

inner product $(f,g) = \iint_{(x,y)\in\square} \overline{f(x,y)}\, g(x,y)\, dx\, dy$

orthonormed basis:

$$V_{n,m}(x,y) = \frac{e^{in\frac{x}{L_x}}}{\sqrt{2\pi L_x}}\cdot\frac{e^{im\frac{x}{L_y}}}{\sqrt{2\pi L_y}} = \frac{1}{2\pi\sqrt{L_x L_y}}\exp\left[i\left(n\frac{x}{L_x} + m\frac{y}{L_y}\right)\right]$$

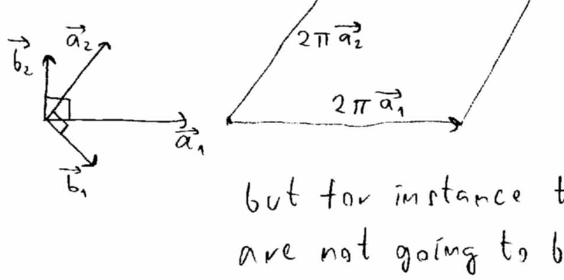
$n, m \in \mathbb{Z}$

$$\Delta V_{n,m} = -\left(n^2/L_x^2 + m^2/L_y^2\right) V_{n,m}$$

(c) non orthogonal periods:

$f(\vec{x}) = f(\vec{x} + 2\pi\vec{a}_1) = f(\vec{x} + 2\pi\vec{a}_2)$. $\qquad \begin{pmatrix} | & | \\ \vec{a}_1 & \vec{a}_2 \\ | & | \end{pmatrix}^{-1} = \begin{pmatrix} -\vec{b}_1- \\ -\vec{b}_2- \end{pmatrix}$, $\quad \vec{b}_i\vec{a}_j = \delta_{ij}$

$$V_{n,m}(\vec{x}) = \frac{1}{2\pi}\cdot\frac{1}{\sqrt{|\det(\vec{a}_1,\vec{a}_2)|}}\cdot\exp\left[i\cdot(n_1\vec{b}_1 + n_2\vec{b}_2)\vec{x}\right], \quad n_1, n_2 \in \mathbb{Z}$$



Remark: $V_{n,m}$ is eigenvector of the Δ operator. The sine and cosine transformations can be generalized for this paraleograma shaped domain, but for instance the $\sin[(n_1\vec{b}_1 + n_2\vec{b}_2)\vec{x}]$ functions are not going to be eigenvalues of Δ.

Plane wave solutions

Linear, real DE: φ is a complex solution $\longrightarrow$

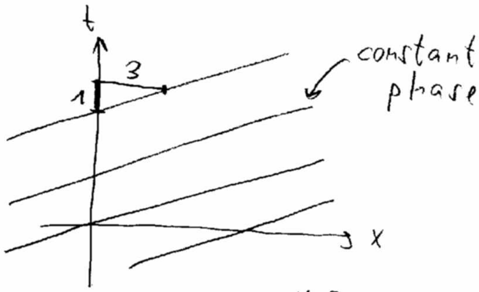
$\longrightarrow Re(\varphi)$, $Im(\varphi)$ are solutions, too.

Example: Wave equation: $\partial_{tt}\varphi(t,x) = 9\,\partial_{xx}\varphi(t,x)$

plane wave solution: $\varphi(t,x) = exp(i(kx-\omega t))$

$$\partial_{tt}\, exp(i(kx-\omega t)) = -\omega^2 exp(i(kx-\omega t)) =$$
$$= 9\,\partial_{xx} exp(i(kx-\omega t)) = -9k^2 exp(i(kx-\omega t))$$

$$\omega^2 = 9k^2$$
$$|\omega| = 3|k|$$

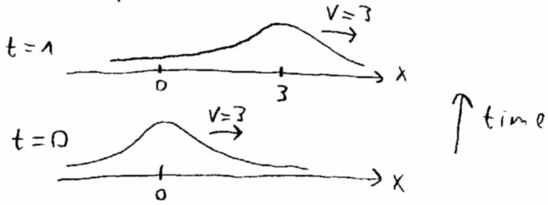


Take $\omega > 0$: $\quad \varphi(x,t) = exp(i(3\cdot|k|\cdot t - kx))$

① $k>0$: $\quad 3|k|t - kx = const \longrightarrow x - 3t = \widetilde{const}$

② $k<0$: $\quad x + 3t = \widetilde{const}$

Velocity $= -3$



velocity = 3

velocity: $\dfrac{d\omega(k)}{dk}$. $\quad \omega(k)$: dispersion relation

Exercise: $(\partial_{tt} + \partial_{tx} - 6\cdot\partial_{xx})\,\varphi(x,t) = 0$

$$\varphi(t,x) = exp(i(kx-\omega t))$$
$$\omega^2 - \omega k - 6k^2 = 0$$

constant phase

$\omega>0,\ k>0$: $\quad \omega = 3\cdot k \quad$ velocity $= 3 \quad$ $x - 3t = const.$

$\omega>0,\ k<0$: $\quad \omega = 2\cdot|k| \quad$ velocity $= -2 \quad$ $x + 2t = const.$

Remark: Does not make too much sense to search for complex solutions of real nonlinear equations.

Special solution (travelling wave):

$$(\partial_{tt} - \partial_{xx})\varphi = -\varphi^3$$

travelling wave: $\varphi(t,x) = f(x-tv) \longrightarrow v^2 f''(x-tv) - f''(x-tv) = -[f(x-tv)]^3$

$\longrightarrow (v^2-1)\, f''(z) = -f^3(z)$. When $f(\pm\infty) = 0$, we call the solution of the ODE travelling wave.

Linear elasticity

$$\vec{\varphi}_{tt} - \mu \Delta \vec{\varphi} - (\lambda+\mu)\nabla(\operatorname{div}\vec{\varphi}) = \vec{0}, \quad \vec{\varphi} = \vec{\varphi}(t, x_1, x_2, x_3)$$

$$\frac{\partial^2}{\partial t^2}\varphi_n - \mu\left(\partial_{x_1}^2 + \partial_{x_2}^2 + \partial_{x_3}^2\right)\varphi_n - (\lambda+\mu)\frac{\partial}{\partial x_n}\left(\partial_{x_1}\varphi_1 + \partial_{x_2}\varphi_2 + \partial_{x_3}\varphi_3\right) = 0, \quad n = 1,2,3$$

Homogeneous, isotropic material, $\vec{k} = (k, 0, 0)$

$$\vec{\varphi}(t, \vec{x}) = \vec{a} \cdot \exp\left(i[\vec{k}\cdot\vec{x} - \omega t]\right) = \vec{a} \cdot \exp\left(i[k x_1 - \omega t]\right)$$

Let $\mu = 9$, $\lambda = 7$. Then

$$a_1(25k^2 - \omega^2) = 0, \quad a_2(9k^2 - \omega^2) = 0, \quad a_3(9k^2 - \omega^2) = 0$$

So a basis of the $\omega > 0$ solutions:

$$\underbrace{\exp\left(i[\pm k x_1 - 5k\cdot t]\right)\begin{bmatrix}1\\0\\0\end{bmatrix}}_{\substack{\text{longitudional polarisation} \\ \text{velocity} = \pm 5 \ \text{(fast)}}}, \quad \underbrace{\exp\left(i[\pm k x_1 - \omega t]\right)\cdot\begin{bmatrix}0\\1\\0\end{bmatrix} \ \text{or} \cdot \begin{bmatrix}0\\0\\1\end{bmatrix}}_{\substack{\text{transversal polarization} \\ \text{velocity} = \pm 3 \ \text{(slower)}}}, \quad k > 0$$

Also solution: $\omega = k = 0$, $\vec{a} = \overrightarrow{\text{const.}}$ (translation)

$\vec{\varphi} = t \cdot \vec{v}$ constant velocity motion, not seen by plane waves

Real world $\rightarrow$ real part of φ:

$$\vec{\varphi}(t, \vec{x}) = \cos\left(\pm k x_1 - 5k\cdot t + \delta\right)\begin{bmatrix}1\\0\\0\end{bmatrix}, \quad \text{etc.}$$

Summary

① $L^2([0,1],dx) \simeq \ell_2 \simeq L^2([-\pi,\pi],dx)$

Fourier transform:

ⓐ orthonormed basis: $v_n(x) = \frac{e^{inx}}{\sqrt{2\pi}}$, $n \in \mathbb{Z}$

ⓑ $\mathcal{F}: f(x) \longrightarrow \hat{f}$, $\hat{f}_n = (v_n, f) = \int_{-\pi}^{\pi} \frac{\overline{e^{inx}}}{\sqrt{2\pi}} \cdot f(x)\, dx$

ⓒ $\mathcal{F}^{-1}: \hat{f} \longrightarrow f(x)$, $f(x) = \sum_{n\in\mathbb{Z}} \hat{f}_n \frac{e^{inx}}{\sqrt{2\pi}}$

$\mathcal{F}, \mathcal{F}^{-1}$ isometry between $L^2([-\pi,\pi],dx)$ and ℓ_2

$f \in L^2([-\pi,\pi],dx)$: $\int_{-\pi}^{\pi} |f(x)|^2 dx < \infty$, $\hat{f} \in \ell_2$: $\sum_n |\hat{f}_n|^2 < \infty$, $\int_{-\pi}^{\pi} |f(x)|^2 dx = \sum_n |\hat{f}_n|^2$

② $v_n(x)$ eigenvector of $\frac{d}{dx}$:

$$\frac{d}{dx} v_n(x) = in\, v_n(x)$$

③ Solution of the $\partial_t \varphi = \partial_{xx}\varphi$, $\varphi(t, x+2\pi) = \varphi(t,x)$, $\varphi(0,x) = f(x)$ heat equation:

① $f(x) = \sum_n \hat{f}_n \frac{e^{inx}}{\sqrt{2\pi}}$, where $\hat{f}_n = (v_n, f) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} e^{-inx} f(x)\, dx$

② $\varphi(t,x) = \sum_n e^{-n^2 t} \hat{f}_n \frac{e^{inx}}{\sqrt{2\pi}}$

④ Other orthonormed bases:

$L^2([0,1],dx)$: $e^{i2\pi nx}$, $n \in \mathbb{Z}$

$L^2([-\pi,\pi],dx)$: $\frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}}\cos(nx), \frac{1}{\sqrt{\pi}}\sin(nx)$, $n = 1,2,3\ldots$

$L^2([0,\pi],dx)$: $\left[\frac{1}{\sqrt{\pi}}, \sqrt{\frac{2}{\pi}}\cos(nx)\right]$, or $\left[\sqrt{\frac{2}{\pi}}\sin(nx)\right]$, where $n = 1,2,3\ldots$

$L^2([0,1]^2, dxdy)$: $e^{i2\pi(nx+my)}$, $(n,m) \in \mathbb{Z}^2$

⑤ Fourier integral transform: $L^2(\mathbb{R},dx) \longrightarrow L^2(\mathbb{R},dx)$

$$\hat{f}(p) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ipx} f(x)\, dx, \qquad f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(p) e^{ipx}\, dp$$

⑥ Plane wave solution: $\varphi_{tt} + 3\varphi_{tx} - 10\varphi_{xx} = 0$

$$\varphi(t,x) = \exp(i[kx - \omega t])$$

$$-\omega^2 - 3k\omega + 10k^2 = 0$$

Sample proplems

$\boxed{14^{*}_{\text{VI}}}$

① Solve it! $\partial_t\varphi = \partial_{xx}\varphi$, $\varphi(t,x) = \varphi(t,x+7)$, $\varphi(0,x) = f(x) = \begin{cases} -1, & \text{if } x\in[0,4] \\ 1, & \text{if } x\in(4,7) \end{cases}$

Solution: ⓐ orthonormed basis: $V_n(x) = \frac{1}{\sqrt{7}} e^{in\frac{2\pi x}{7}}$, $n\in\mathbb{Z}$

ⓑ Fourier-tr:

$$\hat{f}_n = (V_n, f) = \int_0^7 \frac{1}{\sqrt{7}} e^{-in\frac{2\pi x}{7}} \cdot f(x)\,dx$$

$$= \frac{1}{\sqrt{7}}\left[\int_0^4 e^{-in\frac{2\pi x}{7}} \cdot (-1)\,dx + \int_4^7 e^{-in\frac{2\pi x}{7}} \cdot 1\,dx\right]$$

ⓒ $$\varphi(t,x) = \sum_{n\in\mathbb{Z}} \exp\left[-t\left(\frac{2\pi}{7}\right)^2 \cdot n^2\right] \cdot \hat{f}_n \cdot \frac{1}{\sqrt{7}} \exp\left(in\frac{2\pi x}{7}\right)$$

② Solve it! $\partial_t\varphi = \partial_{xx}\varphi$, $\varphi(t,0) = \varphi(t,1) = 0$, $\varphi(0,x) = f(x) = \begin{cases} 0, & \text{if } x\in[0,1/2] \\ 1, & \text{if } x\in(\frac{1}{2},1) \end{cases}$

ⓐ orthonormed basis: $\sqrt{2}\sin(\pi k x)$, $k = 1,2,3\ldots$

ⓑ Sine tr.: $\hat{f}_k = (V_k, f) = \int_0^1 \sqrt{2}\sin(\pi k x)\cdot f(x)\,dx = \int_{1/2}^{1} \sqrt{2}\sin(\pi k x)\cdot 1\,dx$

ⓒ $\partial_{xx} V_k(x) = -(\pi^2 k^2) V_k(x)$, so $\varphi(t,x) = \sum_{k=1}^{\infty} e^{-(\pi k)^2 t} \cdot \hat{f}_k \cdot \sqrt{2}\sin(\pi k x)$

③ Find an orthonormed basis for $L^2([0,2]\times[0,3], dx\,dy)$!

Solutions: ⓐ $\exp\left(in\frac{2\pi x}{2}\right)\cdot\exp\left(im\frac{2\pi x}{3}\right)\cdot\frac{1}{\sqrt{2\cdot 3}}$, $n,m\in\mathbb{Z}$

ⓑ $\sqrt{\frac{2}{3}}\cdot\frac{1}{\sqrt{2}}\exp\left(in\frac{2\pi x}{2}\right)\cdot\sin\left(m\cdot\frac{\pi x}{3}\right)$, $n\in\mathbb{Z}$, $m = 1,2,3\ldots$ stb.

④ What are the velocities of the plane wave solutions of

$$\varphi_{tt} - 2\varphi_{tx} - 13\varphi_{xx} = 0 \quad ?$$

Solution: plane wave: $\varphi(t,x) = \exp(i[kx - wt])$

$$-w^2 + 2wk + 13k^2 = 0$$

Let $k = 1$, then the positive root of $-w^2+2w+13=0$ is $1+\sqrt{14}$

$w>0,\ k>0$: $w = k\cdot(1+\sqrt{14}) \longrightarrow$ velocity $= 1+\sqrt{14}$

$w>0,\ k<0$: $w = |k|\cdot(-1+\sqrt{14}) \longrightarrow$ velocity $= -(-1+\sqrt{14})$
$= -\sqrt{14}+1$

Let $k=-1$, then $-w^2-2w+13=0$, whose positive root is $-1+\sqrt{14}$.
Velocity is its opposite, since the $k=-1$ wave travels to the negative, left direction.