$\boxed{1^{\delta}\ \overline{\underline{\mathrm{V}}}}$

Problem: Radioactive decay
$\alpha \to \beta \to \emptyset$
$y_1 \quad y_2$

$$\frac{d}{dt}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 2 & -6 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

Eigensystem: $\lambda_1 = -2$, $\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\lambda_2 = -6$, $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$e^{tA} = \exp\left(t\begin{bmatrix} -2 & 0 \\ 2 & 6 \end{bmatrix}\right) = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} e^{-2t} & 0 \\ 0 & e^{-6t} \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} e^{-2t} & 0 \\ \frac{1}{2}e^{-2t} - \frac{1}{2}e^{-6t} & e^{-6t} \end{bmatrix}$$

The meaning of the propagator e^{tA}:

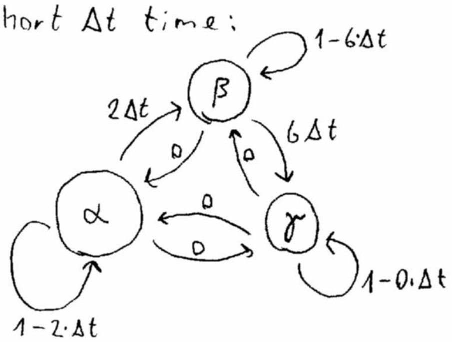
$\vec{y}(t) = e^{tA}\,\vec{y}(0)$

$\vec{y}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{y}(t) = e^{tA}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} e^{-2t} \\ \frac{1}{2}e^{-2t} - \frac{1}{2}e^{-6t} \end{bmatrix}$ ← first column of e^{tA}

$\vec{y}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\vec{y}(t) = e^{tA}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ e^{-6t} \end{bmatrix}$ ← second column of e^{tA}

Probability theoretical interpretation: $\alpha \to \beta \to \gamma$ 3 state system

Transition probabilities in a short Δt time:



The evolution of the $p_\alpha, p_\beta, p_\gamma$ probabilities

$t=0 \qquad\qquad t=\Delta t$

$$\begin{bmatrix} p_\alpha \\ p_\beta \\ p_\gamma \end{bmatrix} \longrightarrow \begin{bmatrix} 1-2\Delta t & 0 & 0 \\ 2\Delta t & 1-6\Delta t & 0 \\ 0 & \boxed{6\Delta t} & 1-0\cdot\Delta t \end{bmatrix}\begin{bmatrix} p_\alpha \\ p_\beta \\ p_\gamma \end{bmatrix}$$

$$= \left(E + \Delta t\begin{bmatrix} -2 & 0 & 0 \\ 2 & -6 & 0 \\ 0 & 6 & 0 \end{bmatrix}\right)\begin{bmatrix} p_\alpha \\ p_\beta \\ p_\gamma \end{bmatrix}$$

Here for example the $\boxed{6\Delta t}$ matrix element at position "$\gamma\beta$" gives the chance of the $\gamma \leftarrow \beta$ transition if the current state is β.

So
$$\frac{d}{dt}\begin{bmatrix} p_\alpha \\ p_\beta \\ p_\gamma \end{bmatrix} = \begin{bmatrix} \boxed{\begin{matrix} -2 \\ 2 \\ 0 \end{matrix}} & \boxed{\begin{matrix} 0 \\ -6 \\ 6 \end{matrix}} & \boxed{\begin{matrix} 0 \\ 0 \\ 0 \end{matrix}} \end{bmatrix}\begin{bmatrix} p_\alpha \\ p_\beta \\ p_\gamma \end{bmatrix}$$

The sum of the elements in each column is zero.

Conservation of probability:

$$\frac{d}{dt}(p_\alpha + p_\beta + p_\gamma) = (-2+2+0)\,p_\alpha + (0-6+6)\,p_\beta + (0+0+0)\,p_\gamma = 0$$

to be cont.

Cont.

$$A = \begin{bmatrix} -2 & 0 & 0 \\ 2 & -6 & 0 \\ 0 & 6 & 0 \end{bmatrix}$$

$$\det(A-\lambda E) = \begin{vmatrix} -2-\lambda & 0 & 0 \\ 2 & -6-\lambda & 0 \\ 0 & 6 & 0-\lambda \end{vmatrix} = (-2-\lambda)(-6-\lambda)(0-\lambda) \longrightarrow \lambda_1=-2,\ \lambda_2=-6,\ \lambda_3=0$$

$$\lambda_1=-2,\ \vec{v}_1 = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}, \quad \lambda_2=-6,\ \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \quad \lambda_3=0,\ \vec{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$e^{tA} = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ -3 & -1 & 1 \end{bmatrix} \begin{bmatrix} e^{-2t} & 0 & 0 \\ 0 & e^{-6t} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ -3 & 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} e^{-2t} & 0 & 0 \\ \frac{1}{2}e^{-2t}-\frac{1}{2}e^{-6t} & e^{-6t} & 0 \\ 1-\frac{3}{2}e^{-2t}+\frac{1}{2}e^{-6t} & \boxed{1-e^{-6t}} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} W_{\alpha\leftarrow\alpha} & W_{\alpha\leftarrow\beta} & W_{\alpha\leftarrow\gamma} \\ W_{\beta\leftarrow\alpha} & W_{\beta\leftarrow\beta} & W_{\beta\leftarrow\gamma} \\ W_{\gamma\leftarrow\alpha} & \boxed{W_{\gamma\leftarrow\beta}} & W_{\gamma\leftarrow\gamma} \end{bmatrix}$$

Here for example the $w_{\gamma\leftarrow\beta} = 1-e^{6t}$ matrix element gives the chance that if the state was β at time $t=0$, then the state is γ at time t.

The sum of the matrix elements in the columns of e^{tA} is always one.

Steady state: $\vec{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Zero eigenvalue of A. One eigenvalue of e^{tA}.

Numerical approximation

$$\frac{d}{dt}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 2 & -6 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = A\vec{y}, \quad \exp(tA) = \begin{bmatrix} e^{-2t} & 0 \\ \frac{1}{2}e^{-2t}-\frac{1}{2}e^{-6t} & e^{-6t} \end{bmatrix}$$

Euler method: $\vec{y}(0) = \begin{bmatrix} 4 \\ 7 \end{bmatrix}$, $\vec{y}_n \approx \vec{y}(n\Delta t)$

$$\vec{y}_{n+1} = \vec{y}_n + \Delta t\cdot A\vec{y}_n = (E+\Delta t\cdot A)\vec{y}_n = \begin{bmatrix} 1-2\Delta t & 0 \\ 2\Delta t & 1-6\Delta t \end{bmatrix}\vec{y}_n$$

$$A\vec{v}_i = \lambda_i\vec{v}_i \longleftrightarrow (E+\Delta t\cdot A)\vec{v}_i = (1+\Delta t\lambda_i)v_i$$

A, $E+\Delta t A$: same eigenvectors

$$\vec{y}(T) \approx \vec{y}_{T/\Delta t} = (E+\Delta t\cdot A)^{T/\Delta t} = \begin{bmatrix} 1-2\Delta t & 0 \\ 2\Delta t & 1-6\Delta t \end{bmatrix}\begin{bmatrix} 4 \\ 7 \end{bmatrix} =$$

$$= \begin{bmatrix} 1-2\Delta t & 0 \\ 2\Delta t & 1-6\Delta t \end{bmatrix}^{T/\Delta t}(2\vec{v}_1+5\vec{v}_2) = 2\cdot(1-2\Delta t)^{T/\Delta t}\cdot\vec{v}_1 + 5\cdot(1-6\Delta t)^{T/\Delta t}\cdot\vec{v}_2$$

$$\approx 2\cdot e^{-2T}\vec{v}_1 + 5\cdot e^{-6T}\vec{v}_2 = \vec{y}(T).$$

$\approx$ o.k. ha $\Delta t \to 0$.

Indeed $\lim\limits_{\Delta t\to 0}(1-2\Delta t)^{T/\Delta t} = \lim\limits_{n\to\infty}\left(1+\frac{-2}{n}\right)^{n\cdot T} = e^{-2T}$.

① $\det(A-\lambda E) = 0 = (-\lambda)^n + a_{n-1}\lambda^{n+1} + \ldots + a_0$

How to compute a_k?

Strategy: Compute a_k when A diagonal

Express a_k with $Tr(A) \ldots Tr(A^n)$

Since $Tr(A) = Tr(SAS^{-1})$, the expression works in general, too.

Examples:

ⓐ $$\det(A-\lambda E) = \begin{vmatrix} d_1-\lambda & 0 \\ 0 & d_2-\lambda \end{vmatrix} = (d_1-\lambda)(d_2-\lambda) = \lambda^2 - \underbrace{(d_1+d_2)}_{Tr A}\lambda + \underbrace{d_1 d_2}_{\det A = \frac{1}{2}(Tr(A)^2 - Tr(A^2))}$$

So for 2×2 A, we have $\det(A-\lambda E) = \lambda^2 - Tr(A)\lambda + \frac{1}{2}(Tr(A)^2 - Tr(A^2))$

ⓑ 3dim: $$\begin{vmatrix} d_1-\lambda & & \\ & d_2-\lambda & \\ & & d_3-\lambda \end{vmatrix} = -\lambda^3 - (d_1d_2 + d_1d_3 + d_2d_3)\lambda + (d_1+d_2+d_3)\lambda^2 + d_1d_2d_3$$

Set $s_1 = d_1+d_2+d_3$, $s_2 = d_1d_2+d_1d_3+d_2d_3$, $s_3 = d_1d_2d_3$, $t_1 = d_1+d_2+d_3$, $t_2 = d_1^2+d_2^2+d_3^2$, $t_3 = d_1^3+d_2^3+d_3^3$

Then $s_1 = t_1$, $s_2 = \frac{1}{2}t_1^2 - \frac{1}{2}t_2$, $s_3 = \frac{1}{6}t_1^3 - \frac{1}{2}t_1t_2 + \frac{1}{3}t_3$

So $$\det(A-\lambda E) = -\lambda^3 + Tr(A)\lambda^2 - \left(\frac{1}{2}Tr(A)^2 - \frac{1}{2}Tr(A^2)\right)\cdot\lambda + \underbrace{\left(\frac{1}{6}Tr(A)^3 - \frac{1}{2}Tr(A)\cdot Tr(A^2) + \frac{1}{3}Tr(A^3)\right)}_{\det A}$$

② e^A numerical computation

Jordan decomposition is numerically unstable

$$\begin{bmatrix} 0 & 10^{-20} \\ 0 & 10^{-10} \end{bmatrix} \sim \begin{bmatrix} 0 & 0 \\ 0 & 10^{-10} \end{bmatrix}, \quad \begin{bmatrix} 0 & 10^{-20} \\ 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

only three multiplication

ⓐ $B \leftarrow A/2^s$ ⓑ Approximate e^B, ⓒ $e^A = (e^B)^{2^s}$ (for example $X^8 = X^{2^3} = ((X^2)^2)^2$)

③ Critically damped harmonic oscillator: $\ddot{y} + 4\dot{y} + 4y = 0$

$$\frac{d}{dt}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -4 & -4 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \quad \begin{vmatrix} 0-\lambda & 1 \\ -4 & -4-\lambda \end{vmatrix} = 0 \longrightarrow \lambda = -2$$

Observe that $\begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = -2\cdot\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = -2\cdot\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Search for similar vectors for A:

$$\begin{bmatrix} 0 & 1 \\ -4 & -4 \end{bmatrix}\vec{v}_1 = -2\vec{v}_1, \quad \begin{bmatrix} 0 & 1 \\ -4 & -4 \end{bmatrix}\vec{v}_2 = -2\vec{v}_2 + \vec{v}_1 \longrightarrow \vec{v}_1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} -1/2 \\ 0 \end{bmatrix}$$

Then $$A = \begin{bmatrix} 0 & 1 \\ -4 & -4 \end{bmatrix} = \begin{bmatrix} -1 & -1/2 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} -1 & -1/2 \\ 2 & 0 \end{bmatrix}^{-1}.$$

Consequently

$$e^{tA} = \begin{bmatrix} -1 & -1/2 \\ 2 & 0 \end{bmatrix}\cdot\left(e^{-2t}\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} -1 & -1/2 \\ 2 & 0 \end{bmatrix}^{-1} = e^{2t}\begin{bmatrix} 1+2t & t \\ -4t & 1-2t \end{bmatrix}.$$

3 dim rotation

angular velocity vector $\vec{a}$: $\frac{d}{dt}\vec{r}(t) = \vec{a} \times \vec{r}(t)$

$$\frac{d}{dt}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}a_1\\a_2\\a_3\end{bmatrix} \times \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{vmatrix}\bar{i} & \bar{j} & \bar{k}\\ a_1 & a_2 & a_3\\ x_1 & x_2 & x_3\end{vmatrix} = (a_2x_3 - a_3x_2)\,\bar{i} - (a_1x_3 - a_3x_1)\,\bar{j} + (a_1x_2 - a_2x_1)\,\bar{k}$$

$$= \begin{bmatrix}0 & -a_3 & a_2\\ a_3 & 0 & -a_1\\ -a_2 & a_1 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = R_{\vec{a}}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$$

3d vector $\vec{a} \longleftrightarrow R_{\vec{a}}$ 3x3 antisymm. matrix

bijection

Example: $\vec{e}_3 = \begin{bmatrix}0\\0\\1\end{bmatrix}$, $R_{\vec{e}_3} = \begin{bmatrix}\boxed{\begin{matrix}0 & -1\\ 1 & 0\end{matrix}} & \begin{matrix}0\\0\end{matrix}\\ \begin{matrix}0 & 0\end{matrix} & \boxed{0}\end{bmatrix}$

$R_{\vec{e}_3}$ block diagonal, so compute the exp-mat. for the blocks

$$\exp\left(t\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\right) = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} + t\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix} + \frac{t^2}{2!}\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}^2 + \frac{t^3}{3!}\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}^3 + \frac{t^4}{4!}\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}^4 + \dots$$

$$= \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} + t\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix} - \frac{t^2}{2!}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} - \frac{t^3}{3!}\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix} + \frac{t^4}{4!}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} + \dots$$

$$= \left(1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \dots\right)\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} + \left(t - \frac{t^3}{3!} + \dots\right)\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix} = \begin{bmatrix}\cos t & -\sin t\\ \sin t & \cos t\end{bmatrix}$$

since $\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix} \sim 90^\circ$ rotation, we have $\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}^{23} = \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}^{5\cdot 4+3} = \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}^{2+1} = -\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}$

analogy: $i^2 = -1$

$\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}^2 = \begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix} = -1\cdot E$

As $\exp(t\cdot[0]) = [1]$, we have

$$R_{\vec{e}_3} = \begin{bmatrix}\boxed{\begin{matrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{matrix}} & \begin{matrix}0\\0\end{matrix}\\ \begin{matrix}0 & 0\end{matrix} & \boxed{1}\end{bmatrix}$$

Antisymmetric matrices generate rotations (preserve lengths)

$$A = -A^T \iff |\exp(tA)\vec{y}| = |\vec{y}| \quad \text{for all } \vec{y}.$$

Indeed $\frac{d}{dt}|\vec{y}(t)|^2 = \frac{d}{dt}(\vec{y}(t), \vec{y}(t)) = \left(\frac{d}{dt}\vec{y}(t), \vec{y}(t)\right) + \left(\vec{y}(t), \frac{d}{dt}\vec{y}(t)\right)$

$$= (A\vec{y}, \vec{y}) + (\vec{y}, A\vec{y}) = \left([A + A^T]\vec{y}, \vec{y}\right) = (0\cdot\vec{y}, \vec{y}) = 0$$

Moreover $A = -A^T \iff R = e^{tA}$, $R^{-1} = e^{-tA} = e^{tA^T} = \left(e^{tA}\right)^T = R^T$

Indeed for orthogonal matrices the inverse and the transposed matrixes are the same.

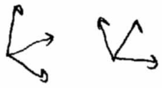
Control theory, reachability

Problem: Given are: a ball, two robotic arms can rotate the ball around the x and y axes.

Can we rotate the ball around the z axis?

(1) State space: $SO(3)$

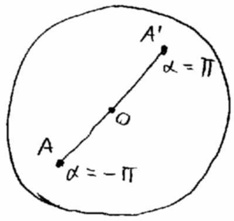
rigid body's orientation:



rotation of an orthonormed básis, the set (group) of these is called $SO(3)$ — 3dim orthogonal matrices, det = 1, no reflection

$SO(3)$ is a 3dim manifold:

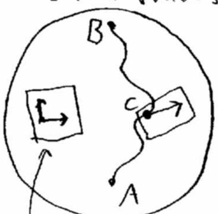
rotation ~ axis + angle



$SO(3) \simeq$ 3dim solid ball of radius π where the antipodal A and A' points on the surface represent the same rotation, so the should be identifed with each other

(2) Reachability

3dim state space



can move only in 2 directions

Does a curve between A and B, such that its tangent vectors are in the 2dim subspaces, exist?

Initial state of the ball: $E = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}$

Possible rotations:

$$S = e^{tX} = \exp(t R_{\vec{e}_1}), \quad T = e^{tY} = \exp(t R_{\vec{e}_2})$$

$$S^{-1} = e^{-tX} \qquad T^{-1} = e^{-tY}$$

$$STS^{-1}T^{-1}E \longleftarrow TS^{-1}T^{-1}E \longleftarrow S^{-1}T^{-1}E \longleftarrow T^{-1}E \longleftarrow E$$

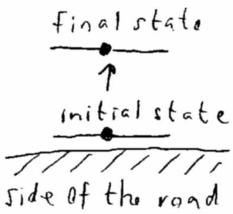
rotation around the x axis by angle t

$-t$ rotation around y

initial state

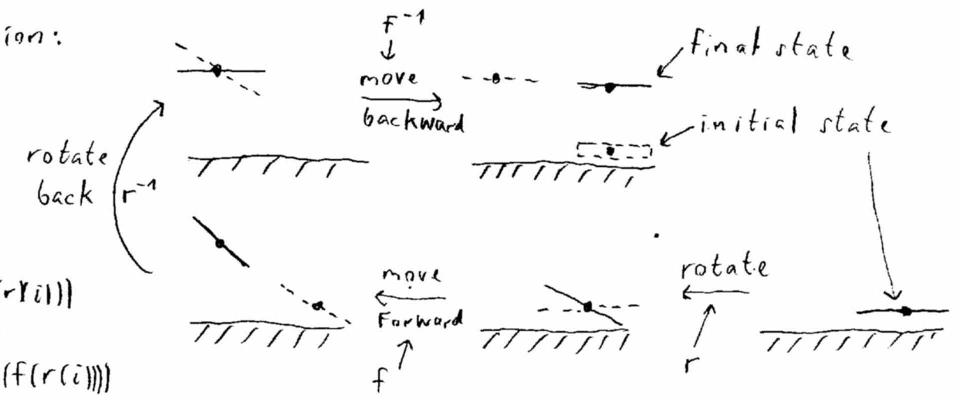
Remark: The $STS^{-1}T^{-1}$ combination might look a bit strange, but it is similar to what drivers do when the move a car to the middle of the road from the side of it.

Problem:



Solution:

i
$\downarrow$
$r(i)$
$\downarrow$
$f(r(i))$
$\downarrow$
$r^{-1}(f(r(i)))$
$\downarrow$
$f^{-1}(r^{-1}(f(r(i))))$



$S = e^{tX}$, $S^{-1} = e^{-tX}$, $T = e^{tY}$, $T^{-1} = e^{-tY}$

$$STS^{-1}T^{-1} = \left(E + tX + \frac{t^2}{2!}X^2 + \dots\right)\left(E + tY + \frac{t^2}{2}Y^2 + \dots\right)\left(E - tX + \frac{t^2}{2}X^2 + \dots\right)\left(E - tY + \frac{t^2}{2}Y^2 + \dots\right)$$

$$= E + t^2(XY - YX) + t^3\cdot(\dots) + \dots = E + t^2[X,Y] + \dots \approx \exp(t^2\cdot[X,Y]) \quad \text{if } t \approx 0$$

$S^{\pm 1}, T^{\pm 1}$ rotation $\Longrightarrow$ $STS^{-1}T^{-1}$ rotation, too $\Longrightarrow$ $[X,Y] = XY - YX$ antisymmetric

commutator of X and Y

In our case:

$$X = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad XY - YX = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = R_{\vec{e}_3} = Z$$

So by rotating only around the x and y axises, we can produce a rotation around the z axis, so we can move in the third direction of the 3dim state space.

Consequently we can reach all the points of the state space.

In general

$$A = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & -b_3 & b_2 \\ b_3 & 0 & -b_1 \\ -b_2 & b_1 & 0 \end{bmatrix}, \quad AB - BA = [A,B] = \begin{bmatrix} 0 & -(a_1b_2 - a_2b_1) & a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 & 0 & -(a_2b_3 - a_3b_2) \\ -(a_3b_1 - a_1b_3) & a_2b_3 - a_3b_2 & 0 \end{bmatrix}$$

$$\vec{a} \longleftrightarrow A, \quad \vec{b} \longleftrightarrow B, \qquad \vec{a} \times \vec{b} \longleftrightarrow AB - BA = [A,B]$$

Remark: Lie group : $SO(3)$: 3dim orthogonal matrices with $\det = 1$

Lie algebra : $so(3)$: 3dim antisymmetric matrices,

operation: commutator (Lie-bracket)

$A, B \longrightarrow [A,B] = AB - BA$

multiplication table: $[X,Y] = Z$, $[Z,X] = Y$, $[Y,Z] = X$

$[Y,X] = -Z$, $[X,Z] = -Y$, $[Z,Y] = -X$

$[X,X] = 0$, $[Y,Y] = 0$, $[Z,Z] = 0$

Properties of the bilinear commutator: $[A,B] = -[B,A]$, $[A,[B,C]] = [[A,B],C] + [B,[A,C]]$

Jacobi identity, similar to the Leibnitz rule: $\frac{d}{dx}(f\cdot g) = \left(\frac{d}{dx}f\right)\cdot g + f\cdot\left(\frac{d}{dx}g\right)$

More Lie groups, algebras:

Lie algebra: $gl(n)$: $n \times n$ matrices, Lie group: $GL^+(n) = \{ e^{tX} \mid X \in gl(n) \}$

Roughly a Lie group is a closed subset of som $GL(n)$, for example the matrices with determinant $= 1$, that is called $SL(n)$

One can "almost" recover the Lie group from its Lie algebra.

"almost": $(\mathbb{R}^1, +) = \Big\{ \underbrace{\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}}_{T(t)} \Big| t \in \mathbb{R} \Big\}$, $r_1 = \left\{ t\cdot\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\}$, $SO(2) = \Big\{ \underbrace{\begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}}_{R(t)} \Big| t \in [0, 2\pi) \Big\}$, $so(2) = \left\{ t\cdot\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right\}$

Same trivial Lie-alg, different Lie groups. But for small positive t, multipliation is the same:

$$T(t_1)T(t_2) = \begin{bmatrix} 1 & t_1 + t_2 \\ 0 & 1 \end{bmatrix} = T(t_1 + t_2), \qquad R(t_1)R(t_2) = R(t_1 + t_2)$$

SL(n), volume preserving transzformations

Problem: How much is $\det(\exp(tA))$?

Solution: If A is diagonal, then

$$\det\left(t\cdot\begin{bmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{bmatrix}\right) = \det\begin{bmatrix} e^{td_1} & & 0 \\ & \ddots & \\ 0 & & e^{td_n} \end{bmatrix} = e^{t(d_1+\dots+d_n)} = e^{t\cdot Tr(A)}$$

Trace

Here $Tr(A) = \sum_i A_{ii}$, the sum of diagonal elements

$$Tr(AB) = Tr(BA) = \sum_i\left(\sum_j A_{ij}B_{ji}\right) = \sum_i\left(\sum_j B_{ij}A_{ji}\right)$$

$Tr(SAS^{-1}) = Tr(S^{-1}SA) = Tr(A)$, so $Tr(A)$ is independent of the coord. basis

If A is diagonalizable: $\det(\exp(tA)) = \det(\exp(t\cdot SDS^{-1})) = \det(S e^{tD} S^{-1})$

$$= \det(S)\det e^{tD}\det(S^{-1}) = e^{t\cdot Tr(A)}$$

Alternative solution:

$$\det(\exp((t+\Delta t)A)) = \det e^{\Delta t\cdot A}\cdot\det e^{tA}$$

$$\det e^{\Delta t A} = \det\left(E + \Delta t\cdot A + \frac{\Delta t^2}{2}\dots\right) \approx \det\begin{bmatrix} 1+\Delta t\cdot a_{11} & & \Delta t\dots \\ & \ddots & \\ \Delta t\dots & & 1+\Delta t\cdot a_{nn} \end{bmatrix}$$

$$= 1 + \Delta t(a_{11}+\dots+a_{nn}) + \Delta t^2\dots \approx 1 + \Delta t\cdot Tr(A)$$

So $\frac{d}{dt}\det(e^{tA}) = Tr(A)\cdot\det(e^{tA})$, $\det(e^{0\cdot A}) = \det E = 1$,

consequently $\det(e^{tA}) = e^{t\cdot Tr(A)}$

Exercise: Compute $\det\left(\exp\left(t\cdot\begin{bmatrix}1&2\\3&4\end{bmatrix}\right)\right)$!

Solution: $\det e^{t\cdot\begin{bmatrix}1&2\\3&4\end{bmatrix}} = \exp\left(t\cdot Tr\begin{bmatrix}1&2\\3&4\end{bmatrix}\right) = \exp(t\cdot(1+4)) = e^{5t}$

$SL(n)$ is the Lie group of the $n\times n$ matrices with determinant equal to 1.

Lie group: $SL(n)$

$$\{g\in GL(n) \mid \det g = 1\}$$

Lie algebra: $sl(n)$

$$\{X\in Mat_n(\mathbb{R}) \mid Tr X = 0\}$$

Linear elasticity

$\boxed{8_{\text{V}}}$

deformation: $\vec{r} \to \vec{f}(\vec{r})$

translation: $\vec{r} \to \vec{r} + \vec{a}$ makes no stress, so assume that $\vec{f}(\vec{0}) = \vec{0}$.

Then for small $\vec{r} \approx \vec{0}$, we have

$$\mathrm{Jac} = \frac{\partial \vec{f}}{\partial \vec{r}}, \quad \vec{f}(\vec{r}) \approx \mathrm{Jac}(\vec{0}) \cdot \vec{r} = J\vec{r}.$$ This is elasticity theory.

Linear elasticity: $J = E + tA, \quad t \approx 0$

For example: dimension $\sim 1m$, deformation $\sim 1mm$, def. gradient $\sim \frac{1mm}{1m}$, $t = 0.001$, A: element of order 1

Problem: Given A, compute the stress tensor

Strategy: $J = E + tA \approx (E + tR)(E + t\cdot\lambda E)(E + tS)$ ← order is irrelevant in the first order of t

Transformations: $E + tR \approx$ rotation, $R = -R^T$, zero stress

$E + t\lambda E \approx$ (de)compression by the factor $(1 + t\cdot\lambda)$

$E + tS \approx$ (de)compression in orthogonal directions with zero net volume change: $Tr(S) = 0$

$$J = E + tA \approx (E + tR)\cdot(E + t\cdot\lambda E)\cdot(E + tS) \approx E + t(R + \lambda E + S)$$

zero trace

antisymmetric

Exercise: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = R + \lambda E + S$, where $R = -R^T$, $Tr\,S = 0$. Compute λ, R, S!

Solution: $R = \frac{A - A^T}{2}, \quad \lambda = \frac{1}{dim} Tr(A), \quad S = \frac{A + A^T}{2} - \lambda \cdot E$

$$R = \frac{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}}{2} = \begin{bmatrix} 0 & -1/2 \\ 1/2 & 0 \end{bmatrix}, \quad \lambda = \frac{1}{2}(1+4) = \frac{5}{2}, \quad S = \frac{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}}{2} - \frac{5}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{3}{2} & \frac{5}{2} \\ \frac{5}{2} & \frac{3}{2} \end{bmatrix}$$

Indeed

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 0 & -1/2 \\ 1/2 & 0 \end{bmatrix} + \frac{5}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -3/2 & 5/2 \\ 5/2 & 3/2 \end{bmatrix}.$$

deformation: $\vec{r} \longrightarrow \vec{F}(\vec{r}) = \vec{r} + \vec{U}(\vec{r})$, $\|\nabla \vec{U}\| \ll 1$

$$Jac = \frac{\partial \vec{F}}{\partial \vec{r}} = E + \frac{\partial \vec{U}}{\partial \vec{r}} = E + \mathcal{A}\,, \quad (Jac)_{ij} = \delta_{ij} + \frac{\partial U_i}{\partial x_j}$$

$$\mathcal{A} = \mathcal{R} + \Lambda E + S$$

$$\mathcal{R} = \frac{\mathcal{A} - \mathcal{A}^T}{2}$$

$$\Lambda = \frac{Tr(\mathcal{A})}{dim}\,. \qquad Tr(\mathcal{A}) = \sum_i \frac{\partial U_i}{\partial x_i} = div\,\vec{U}$$

$$S = \frac{\mathcal{A} + \mathcal{A}^T}{2} - \frac{Tr(\mathcal{A})}{dim}$$

Homogeneous, isotropic material, Hook's Law, 3dim

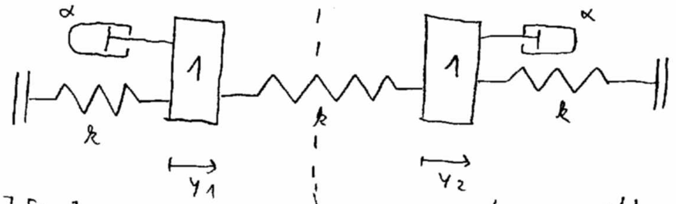
σ: stress tensor

$$\sigma = 2\mu\left(\frac{\mathcal{A} + \mathcal{A}^T}{2}\right) + \lambda Tr(\mathcal{A}) \cdot E$$

$$= 2\mu S + (2\mu + 3\lambda) \cdot \frac{1}{3} E$$

λ, μ: Lamé parameters

Symmetry



$\longleftrightarrow$ symmetric with respect to reflection

Problem:

$$\frac{d}{dt}\begin{bmatrix} y_1\\ y_2\\ v_1\\ v_2\end{bmatrix} = \left[\begin{array}{cc|cc} & & 1 & \\ & & & 1\\ \hline -2k & k & -a & \\ k & -2k & & -a\end{array}\right]\begin{bmatrix} y_1\\ y_2\\ v_1\\ v_2\end{bmatrix} = A\vec{z}$$

Solution: ① Find the eigensystem of A, compute e^{tA}, and we are done.

② Alternative solution: use symmetry to simplify the computation

Reflection: $$\mathcal{P}(\vec{z}) = \mathcal{P}\left(\begin{bmatrix} y_1\\ y_2\\ v_1\\ v_2\end{bmatrix}\right) = \begin{bmatrix} -y_2\\ -y_1\\ -v_2\\ -v_1\end{bmatrix} = \left[\begin{array}{cc|cc} & -1 & & \\ -1 & & & \\ \hline & & & -1\\ & & -1 & \end{array}\right]\begin{bmatrix} y_1\\ y_2\\ v_1\\ v_2\end{bmatrix} = P\vec{z}$$

Evolution: $$\phi_t(\vec{z}) = e^{tA}\cdot\vec{z}$$

Symmetry: $$\phi_t(\mathcal{P}(\vec{z})) = \mathcal{P}(\phi_t(\vec{z}))$$
$$e^{tA}P\vec{z} = Pe^{tA}\vec{z}$$
$$AP = PA\ ,\ A = PAP^{-1}$$

$$\begin{tikzcd} \vec{z} \arrow[rr, "\phi_t"] \arrow[d, "\mathcal{P}"'] & & e^{tA}\vec{z} \arrow[d, "\mathcal{P}"] \\ P\vec{z} \arrow[r, "\phi_t"'] & e^{tA}P\vec{z} \arrow[r, phantom, "="] & Pe^{tA}\vec{z} \end{tikzcd}$$

Strategy: Find common eigenvectors for A and P

eigenvalues of P: $(P-\lambda E) = (\lambda^2-1)^2 = 0 \longrightarrow \lambda = \pm 1$ (automatic, since $P^2 = E$)

$\lambda = 1$ eigen subspace: $\left\{\begin{bmatrix} x\\ -x\\ y\\ -y\end{bmatrix}\right\}$ $\qquad$ $\lambda = -1$ eigensubspace: $\left\{\begin{bmatrix} x\\ x\\ y\\ y\end{bmatrix}\right\}$

$$A\begin{bmatrix} x\\ -x\\ y\\ -y\end{bmatrix} = \begin{bmatrix} y\\ -y\\ -3kx-ay\\ 3kx+ay\end{bmatrix}, \qquad \begin{pmatrix} x\\ y\end{pmatrix} \longrightarrow \begin{pmatrix} y\\ -3kx-ay\end{pmatrix} = \begin{pmatrix} 0 & 1\\ -3k & -a\end{pmatrix}\begin{pmatrix} x\\ y\end{pmatrix}$$

$$A\begin{bmatrix} x\\ x\\ y\\ y\end{bmatrix} = \begin{bmatrix} y\\ y\\ -kx-ay\\ -kx-ay\end{bmatrix}, \qquad \begin{pmatrix} x\\ y\end{pmatrix} \longrightarrow \begin{pmatrix} y\\ -kx-ay\end{pmatrix} = \begin{pmatrix} 0 & 1\\ -k & a\end{pmatrix}\begin{pmatrix} k\\ y\end{pmatrix}.$$

Let $a=3$, $k=5/12$, then $\begin{bmatrix} 0 & 1\\ -3\cdot\frac{5}{12} & -3\end{bmatrix} \longrightarrow \lambda_1 = -\frac{5}{2}, \vec{v}_1 = \begin{bmatrix} -2/5\\ 1\end{bmatrix}, \lambda_2 = -\frac{1}{2}, \vec{v}_2 = \begin{bmatrix} -2\\ 1\end{bmatrix}$

So for A:

$$\lambda_1 = -\frac{5}{2}, \vec{v}_1 = \begin{bmatrix} -2/5\\ 2/5\\ 1\\ -1\end{bmatrix}, \qquad \lambda_2 = -\frac{1}{2}, \vec{v}_2 = \begin{bmatrix} -2\\ 2\\ 1\\ -1\end{bmatrix}$$

$\boxed{11^* \ \text{V}}$

For the $P\vec{v} = -\vec{v}$ subspace:

$$\begin{bmatrix} 0 & 1 \\ -\frac{5}{12} & -3 \end{bmatrix} \longrightarrow \lambda_1 = -2.85, \quad \vec{v}_1 = \begin{bmatrix} -0.35 \\ 1 \end{bmatrix}$$

$$\lambda_2 = -0.14, \quad \vec{v}_2 = \begin{bmatrix} -6.84 \\ 1 \end{bmatrix}$$

So for A:

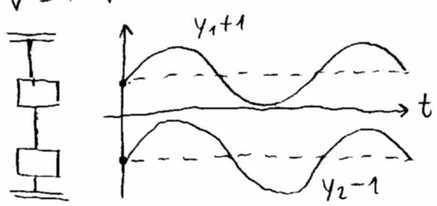
$$\lambda_3 = -2.85, \quad \vec{v}_3 = \begin{bmatrix} -0.35 \\ -0.35 \\ 1 \\ 1 \end{bmatrix}, \qquad \lambda_4 = -0.14, \quad \vec{v}_4 = \begin{bmatrix} -6.84 \\ -6.84 \\ 1 \\ 1 \end{bmatrix}$$

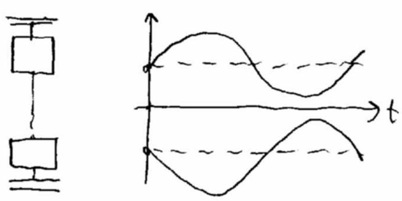
So e^{tA}:

$$e^{tA} = \left[\begin{array}{cc|cc} -\frac{2}{5} & -2 & -0.35 & -6.84 \\ \frac{2}{5} & -2 & -0.35 & -6.84 \\ \hline 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{array}\right] \left[\begin{array}{cc|cc} e^{-\frac{5}{2}t} & & & \\ & e^{-\frac{1}{2}t} & & \\ \hline & & e^{-2.85t} & \\ & & & e^{-0.14t} \end{array}\right] \left[\begin{array}{cc|cc} -\frac{2}{5} & * & * & * \\ * & * & * & * \\ \hline * & * & * & * \\ * & * & * & 1 \end{array}\right]^{-1}$$

Oscillation modes (actually this is an overdamped system)

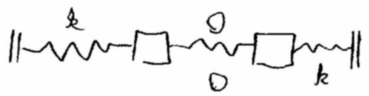
$P\vec{v} = 1 \cdot \vec{v}$ $\qquad\qquad$ $P\vec{v} = -1 \cdot \vec{v}$



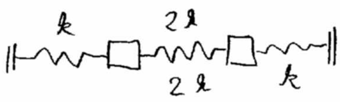


Effective spring constant:

k:



$3k$:



Why was it reasonable to search for the eigenvalues of A in the subspaces

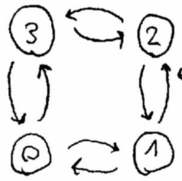
$$S_{\pm 1} = \{\vec{v} \mid P\vec{v} = \pm 1\} \ ?$$

For example if $\vec{v} \in S_{+1}$, then

$$P\vec{v} = 1 \cdot \vec{v} \Longrightarrow P(A\vec{v}) = AP\vec{v} = A \cdot 1 \cdot \vec{v} = 1 \cdot (A\vec{v}),$$ so if $v \in S_{+1}$, then $Av \in S_{+1}$, too.

Theorem: Assume that A, B are diagonalizable matrices, and $AB = BA$.
Then there exist a common eigenvector basis for A and B.

Discrete Fourier Transformation   DFT, FFT



Transition probability in $\Delta t \approx 0$ time is $w = 1 \cdot \Delta t$ on the arrows, zero otherwise.

The system is symmetric with respect to the

$\pi = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 3 & 0 & 1 & 2 \end{pmatrix}$ permutation

$$\frac{d}{dt}\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -2 & 1 & & 1 \\ 1 & -2 & 1 & \\ & 1 & -2 & 1 \\ 1 & & 1 & -2 \end{bmatrix}\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix} = A\vec{y}, \qquad \mathcal{P}(\vec{y}) = \mathcal{P}\left(\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix}\right) = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_0 \end{bmatrix} = \begin{bmatrix} & 1 & & \\ & & 1 & \\ & & & 1 \\ 1 & & & \end{bmatrix}\vec{y} = P\vec{y}$$

Evolution: $\phi_t(\vec{y}) = e^{tA}\vec{y}$

Symmetry: $\mathcal{P}(\vec{y}) = P\vec{y}$

$$\mathcal{P}(\phi_t(\vec{y})) = \phi_t(\mathcal{P}(\vec{y}))$$
$$Pe^{tA}\vec{y} = e^{tA}P\vec{y}$$
$$PA = AP$$

Strategy: ① Find the eigensubspaces of P

② In this case the are onedimensional, so they provide the eigenvectors of A

① $P^4 = E$, so if $P\vec{v} = \lambda\vec{v}$, $\longrightarrow P^4\vec{v} = \lambda^4\vec{v} = E\vec{v} = 1 \cdot \vec{v} \longrightarrow \lambda^4 = 1$

potential eigenvalues: $\varepsilon^0 = 1, \varepsilon^1, \varepsilon^2, \varepsilon^3$, where $\varepsilon = e^{2\pi i/4} = \cos\frac{2\pi}{4} + i \cdot \sin\frac{2\pi}{4} = i$

Normalized eigenvectors:

$\lambda_0 = \varepsilon^0 = 1$, $\lambda_1 = \varepsilon^1 = i$, $\lambda_2 = \varepsilon^2 = -1$, $\lambda_3 = \varepsilon^3 = -i$

$$\vec{V}_0 = \frac{1}{\sqrt{4}}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{4}}\begin{bmatrix} \varepsilon^{0\cdot 0} \\ \varepsilon^{1\cdot 0} \\ \varepsilon^{2\cdot 0} \\ \varepsilon^{3\cdot 0} \end{bmatrix}, \quad \vec{V}_1 = \frac{1}{\sqrt{4}}\begin{bmatrix} \varepsilon^{0\cdot 1} \\ \varepsilon^{1\cdot 1} \\ \varepsilon^{2\cdot 1} \\ \varepsilon^{3\cdot 1} \end{bmatrix}, \quad \vec{V}_2 = \begin{bmatrix} \varepsilon^{0\cdot 2} \\ \varepsilon^{1\cdot 2} \\ \varepsilon^{2\cdot 2} \\ \varepsilon^{3\cdot 2} \end{bmatrix}, \quad \vec{V}_3 = \frac{1}{\sqrt{4}}\begin{bmatrix} \varepsilon^{0\cdot 3} \\ \varepsilon^{1\cdot 3} \\ \varepsilon^{2\cdot 3} \\ \varepsilon^{3\cdot 3} \end{bmatrix}$$

These are orthogonal vectors, for example

$$(\vec{V}_1, \vec{V}_3) = \frac{1}{\sqrt{4}} \cdot \frac{1}{\sqrt{4}}\left(\overline{\varepsilon^{0\cdot 1}} \cdot \varepsilon^{0\cdot 3} + \overline{\varepsilon^{1\cdot 1}} \cdot \varepsilon^{1\cdot 3} + \overline{\varepsilon^{2\cdot 1}} \cdot \varepsilon^{2\cdot 3} + \overline{\varepsilon^{3\cdot 1}} \cdot \varepsilon^{3\cdot 3}\right)$$
$$= \frac{1}{4}\left(1 \cdot 1 + (-i)\cdot(-i) + (-1)\cdot(-1) + i \cdot i\right) = 0$$

This is automatic, since P is a normal matrix: $PP^* = P^*P$, where $P^* = \overline{P}^T$

Actually $P^* = P^T = P^3 = P^{-1}$, so $P^*P = PP^*$

① P orthogonal matrix $\longrightarrow P^{-1} = P^T$

② P real, so $\overline{P} = P$, consequently $P^* = P^T$

③ $P^4 = E$, so $P(P^3) = E$, consequently $P^{-1} = P^3$

② $\vec{V}_0 \dots \vec{V}_3$ are eigenvectors of A, the eigenvalues are $\boxed{13^*_{\underline{V}}}$

$$A\vec{V}_k = \left[\begin{array}{cc|cc} -2 & 1 & & 1 \\ 1 & -2 & 1 & \\ \hline & 1 & -2 & 1 \\ 1 & & 1 & -2 \end{array}\right] \begin{bmatrix} \varepsilon^{0\cdot k} \\ \varepsilon^{1\cdot k} \\ \varepsilon^{2\cdot k} \\ \varepsilon^{3\cdot k} \end{bmatrix} = \begin{bmatrix} \varepsilon^{0\cdot k} - 2\varepsilon^{1\cdot k} + \varepsilon^{2\cdot k} \\ * \\ * \\ * \end{bmatrix} = (\varepsilon^{-k} - 2 + \varepsilon^{k}) \begin{bmatrix} * \\ \varepsilon^{1\cdot k} \\ * \\ * \end{bmatrix}$$

So A's eigensystem:

$$\lambda_k = (\varepsilon^{-k} - 2 + \varepsilon^{k}), \quad \vec{V}_k = \frac{1}{\sqrt{4}} \begin{bmatrix} \varepsilon^{0\cdot k} \\ \varepsilon^{1\cdot k} \\ \varepsilon^{2\cdot k} \\ \varepsilon^{3\cdot k} \end{bmatrix}, \quad k = 0, 1, 2, 3$$

A is diagonalized by S: with inverse $S^{-1} = S^* = \bar{S}^T = \bar{S}$

$$(S)_{k\ell} = \frac{1}{\sqrt{4}} e^{k\cdot \ell} \qquad (S^{-1})_{k\ell} = \bar{S}_{\ell k} = \frac{1}{\sqrt{4}} \overline{\varepsilon^{\ell\cdot k}} = \frac{1}{\sqrt{4}} \varepsilon^{-\ell\cdot k}$$

$$\lambda_k = 2\left(\cos\left(\tfrac{2\pi}{4}\cdot k\right) - 1\right)$$

k	0	1	2	3
λ_k	0	-2	-4	-2

The exponential matrix is

$$e^{tA} = S e^{tD} S^{-1} = \underbrace{\frac{1}{\sqrt{4}} \left[\begin{array}{cc|cc} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ \hline 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{array}\right]}_{S} \left[\begin{array}{cc|cc} 1 & & & \\ & e^{-2t} & & \\ \hline & & e^{-4t} & \\ & & & e^{-2t} \end{array}\right] \cdot \overbrace{\frac{1}{\sqrt{4}} \cdot \left[\begin{array}{cc|cc} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ \hline 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{array}\right]}^{S^{-1} = \bar{S}}$$

DFT: $4 \leftrightarrow N$, $\varepsilon = e^{2\pi i/N} = \cos\frac{2\pi}{N} + i\cdot\sin\frac{2\pi}{N}$

orthonormed basis of $\mathbb{C}^N$: $\vec{V}_k = \begin{bmatrix} \varepsilon^{0\cdot k} \\ \varepsilon^{1\cdot k} \\ \vdots \\ \varepsilon^{(N-1)\cdot k} \end{bmatrix}$, $k = 0, 1 \dots N-1$

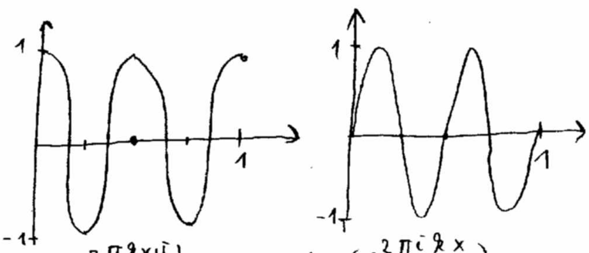
$x \in [0,1]$

$\vec{V}_k$ can be obtained as the sampling of the function $V_k(x)$ at the points: $\Delta x = \frac{1}{N}$, $x_m = m\cdot\Delta x$, $m = 0, 1 \dots N-1$, plus a normalization by $\frac{1}{\sqrt{N}}$.

① function $V_k(x) = e^{2\pi i k x} = \cos(2\pi k x) + i\cdot\sin(2\pi k x)$

② sampling at $x = 0\cdot\Delta x, 1\cdot\Delta x, \dots (N-1)\cdot\Delta x$

③ $(\vec{V}_k)_m = \frac{1}{\sqrt{N}} \cdot V_k(x_m) = \frac{1}{\sqrt{N}} e^{2\pi i k x_m} = \frac{1}{\sqrt{N}} \varepsilon^{k\cdot m}$



$\mathrm{Re}(e^{2\pi k x\cdot i}) = \cos(2\pi k x)$. $\mathrm{Im}(e^{2\pi i k x}) = \sin(2\pi k x)$

$k = 2$

$\mathbb{C}^N$ vector space (finite dimensional Hilbert space)

Inner product: $(\vec{a}, \vec{b}) = \sum_{k=0}^{N-1} \bar{a}_k b_k$

standard basis: $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ldots \vec{e}_{N-1} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$

Fourier basis:
$\varepsilon = e^{2\pi i/N}$ $\quad \vec{V}_k = \frac{1}{\sqrt{N}} \begin{bmatrix} \varepsilon^{0\cdot k} \\ \varepsilon^{1\cdot k} \\ \vdots \\ \varepsilon^{(N-1)\cdot k} \end{bmatrix}, \quad k = 0, 1, \ldots N-1$

It is easy to expand a vector in an orthonormed basis:

$$\vec{U} = U_0 \vec{e}_1 + \ldots + U_{N-1} \vec{e}_{N-1} = \begin{bmatrix} U_0 \\ \vdots \\ U_{N-1} \end{bmatrix}_e = y_0 \vec{V}_1 + \ldots + y_{N-1} \vec{V}_{N-1} = \begin{bmatrix} y_0 \\ \vdots \\ y_{N-1} \end{bmatrix}_V$$

$$(\vec{V}_k, \vec{U}) = y_k = \frac{1}{\sqrt{N}} \left[\overline{\varepsilon^{0\cdot k}}, \overline{\varepsilon^{1\cdot k}}, \ldots, \overline{\varepsilon^{(N-1)\cdot k}} \right] \begin{bmatrix} U_0 \\ \vdots \\ U_{N-1} \end{bmatrix}$$, since $(\vec{V}_k, V_l) = 0$ if $k \neq l$

So
$$\begin{bmatrix} y_0 \\ \vdots \\ y_{N-1} \end{bmatrix}_V = \frac{1}{\sqrt{N}} \begin{bmatrix} \varepsilon^{-0\cdot 0} & \varepsilon^{-1\cdot 0} & \cdots & \varepsilon^{-(N-1)\cdot 0} \\ \varepsilon^{-0\cdot k} & \varepsilon^{-1\cdot k} & \cdots & \varepsilon^{-(N-1)\cdot 1} \\ \vdots & \vdots & & \vdots \\ \varepsilon^{-0\cdot(N-1)} & \varepsilon^{-1\cdot(N-1)} & \cdots & \varepsilon^{-(N-1)\cdot(N-1)} \end{bmatrix} \begin{bmatrix} U_0 \\ U_1 \\ \vdots \\ U_{N-1} \end{bmatrix}_e = \mathcal{F}(U) = FU$$

Inverse Fourier tr:

$$\vec{U}_e = \begin{bmatrix} | & & | \\ \vec{V}_1 & \ldots & \vec{V}_n \\ | & & | \end{bmatrix} \vec{y}_V = F^{-1} \vec{y}_V = F^* \vec{y}_V = \bar{F} \vec{y}_V = \mathcal{F}^{-1}(\vec{y}_V) = \frac{1}{\sqrt{N}} \begin{bmatrix} \varepsilon^{0\cdot 0} & \cdots & \varepsilon^{(N-1)\cdot 0} \\ \vdots & & \vdots \\ \varepsilon^{0\cdot(N-1)} & \cdots & \varepsilon^{(N-1)(N-1)} \end{bmatrix} \vec{y}_V$$

We have $\|\vec{U}\|^2 = \|\mathcal{F}(\vec{U})\|^2 = \sum_{k=0}^{N-1} |U_k|^2 = \sum_{k=0}^{N-1} |y_k|^2$. So $\mathcal{F}$ is unitary, preserves the inner product. Remark: usually $\frac{1}{\sqrt{N}}$ is omitted in $\mathcal{F}$, compensated by $\frac{1}{\sqrt{N}} \to \frac{1}{N}$ in $\mathcal{F}^{-1}$

$\mathbb{C}^N$ vector space + inner product | $\mathcal{H} = L^2([0,1], dx)$ Hilbert space

inner product: $(\vec{a}, \vec{b}) = \sum_m \bar{a}_m b_m$

$$\vec{V}_k = \frac{1}{\sqrt{N}} \begin{bmatrix} \varepsilon^{0 \cdot k} \\ \varepsilon^{1 \cdot k} \\ \vdots \\ \varepsilon^{(N-1) \cdot k} \end{bmatrix}$$

vector $\longleftrightarrow$ function $\quad V_k(x) = e^{2\pi i k x} \quad x \in [0,1]$

values of V_k at $x_m = m \cdot \Delta x$, $\Delta x = 1/N$

$$(\vec{V}_k)_m = \frac{1}{\sqrt{N}} \varepsilon^{k \cdot m} \longleftrightarrow V_k(x_m) = \exp\left(2\pi i k \cdot \frac{m}{N}\right) = e^{2\pi i k x_m}$$

$$\vec{f} = \frac{1}{\sqrt{N}} \begin{bmatrix} f(0 \cdot \Delta x) \\ f(1 \cdot \Delta x) \\ \vdots \\ f((N-1) \cdot \Delta x) \end{bmatrix} \longleftrightarrow f(x), \quad x \in [0,1]$$

sample f at x_m,
$x_m = 0, \Delta x, \dots (N-1)\Delta x$

$$(\vec{f}, \vec{g}) = \sum_m \bar{f}_m g_m = \sum_m \overline{f(x_m)}\, g(x_m) \cdot \frac{1}{N} \longleftrightarrow \int_0^1 \overline{f(x)}\, g(x)\, dx = (f, g)$$

Orthonormed basis:

$(\vec{V}_k, \vec{V}_l) = \delta_{k,l}$, $l, k = 0, 1 \dots N-1$

$$(e^{2\pi i k x}, e^{2\pi i l x}) = \int_0^1 e^{-2\pi i k x} \cdot e^{2\pi i l x}\, dx = \int_0^1 e^{2\pi i (l-k)x}\, dx = \delta_{k,l} \qquad k, l \in \mathbb{Z}$$

DFT:

$$\mathcal{F}: \vec{X} \to \begin{bmatrix} x_0 \\ \vdots \\ x_{N-1} \end{bmatrix} \to \mathcal{F}(\vec{X}) = \begin{bmatrix} (\vec{V}_0, \vec{X}) \\ \vdots \\ (\vec{V}_{N-1}, \vec{X}) \end{bmatrix}$$

Fourier transform $\mathcal{F}: L^2([0,1], dx) \to l_2$

$$\mathcal{F}: f(x) \to \hat{f}_k = (e^{2\pi i k x}, f(x)) = \int_0^1 e^{-2\pi i k x} f(x)\, dx$$

Inverse DFT:

$$\mathcal{F}^{-1}: \vec{y} = \begin{bmatrix} y_0 \\ \vdots \\ y_{N-1} \end{bmatrix} \to \mathcal{F}^{-1}(\vec{y}) = \begin{bmatrix} (\vec{V}_0, \vec{y}) \\ \vdots \\ (\vec{V}_{N-1}, \vec{y}) \end{bmatrix}$$

Inverse Fourier tr.

$$\mathcal{F}^{-1}: \hat{f} \to \sum_{k \in \mathbb{Z}} \hat{f}_k \cdot e^{2\pi i k x}$$

derivation: $f'(x_m) \approx \frac{1}{2\Delta x}(f(x_{m+1}) - f(x_{m-1}))$

$$\frac{d}{dx} \longleftrightarrow \frac{1}{2\Delta x}(P - P^{-1}) = D$$

$$D\vec{V}_k = \left[\frac{1}{2\Delta x}(\varepsilon - \varepsilon^{-1})\right]\vec{V}_k = \left[\frac{1}{\Delta x}\sin(2\pi k \Delta x) \cdot i\right]\vec{V}_k = \lambda_k \vec{V}_k$$

if $k \ll N$, $\Delta x \approx 0$, then

$\lambda_k \approx 2\pi k i$

Derivation $D = \frac{d}{dx}$

$$\frac{d}{dx}\left[e^{2\pi i k x}\right] = 2\pi k i \left[e^{2\pi i k x}\right]$$

eigen vector
eigenvalue of $\frac{d}{dx}$

$A\vec{V} = \lambda \vec{V}$

$A \longleftrightarrow \frac{d}{dx}$
$\vec{V} \longleftrightarrow e^{2\pi i k x}$
$\lambda \longleftrightarrow 2\pi k i$

same

Summary

(1) $e^{tA} = \begin{bmatrix} | & & | \\ \vec{y}^1(t) & \ldots & \vec{y}^n(t) \\ | & & | \end{bmatrix}$, where $\vec{y}^1$ solves $\dot{\vec{y}} = A\vec{y}$, $\vec{y}(0) = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \vec{e}_1$

$\vdots$

$\vec{y}^n$ solves $\dot{\vec{y}} = A\vec{y}$, $\vec{y}(0) = \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix} = \vec{e}_n$

(2) Sum of the columns of A is zero $\longleftrightarrow$ A generates a stochastic process. Steady state $\longleftrightarrow$ zero eigenvector of A

(3) The $\vec{a}$ angular velocity rotations are generated by $A = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}$,

$A = -A^T \longrightarrow (e^{tA}\vec{y}, e^{tA}\vec{y}) = (\vec{y}, \vec{y})$

(4) $\frac{d}{dt} \det(e^{tA}) = \mathrm{Tr}(A) \cdot \det(e^{tA}) \longrightarrow \det(e^{tA}) = e^{t \cdot \mathrm{Tr}(A)}$

Volume preserving transformation group: $\mathrm{Tr}(A) = 0$

(5) Symmetry: $\frac{d}{dt}\vec{y} = A\vec{y}$ invariant with respect to the $\vec{y} \to P\vec{y}$ transformation if $PA = AP$. If A, P diagonalizable $\to$ have common eigenvector basis

(6) $P \begin{bmatrix} x_0 \\ \vdots \\ x_{N-1} \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{N-1} \\ x_0 \end{bmatrix}$. Orthonormed eigensystem: $\lambda_k = \varepsilon^k$, $(\vec{V}_k)_l = \varepsilon^{kl}$ $\quad \varepsilon = e^{2\pi i/N}$, $k = 0, 1, \ldots N-1$

Inverse DFT: $\mathcal{F}^{-1} = \begin{bmatrix} | & & | \\ \vec{V}_0 & \ldots & \vec{V}_{N-1} \\ | & & | \end{bmatrix}$, DFT: $\mathcal{F} = (\mathcal{F}^{-1})^{-1} = \overline{(\mathcal{F}^{-1})}$

(7) Analogies of $\mathbb{C}^N$ and $\mathcal{H} = L^2([0,1], dx)$ $\qquad \Delta x = \frac{1}{N}$

$f(x)$	$\vec{f}$, $(\vec{f})_m = \frac{1}{\sqrt{N}} f(x_m)$, $x_m = \frac{m}{N} = m\Delta x$
$(f, g) = \int_0^1 \overline{f(x)} g(x) dx$	$(\vec{f}, \vec{g}) = \sum_m \overline{f_m} g_m$
$\frac{d}{dx} f$	$\frac{1}{2N}(P - P^{-1}) = D$
$e^{2\pi i k x}$	$\vec{V}_k$, $(\vec{V}_k)_m = e^{2\pi i k x_m}$
$\frac{d}{dx} e^{2\pi i k x} = 2\pi i k\, e^{2\pi i k x}$	$D\vec{V}_k = \underbrace{\left[\frac{1}{\Delta x} \sin(2\pi k \Delta x) \cdot i\right]}_{\approx 2\pi i k \text{ if } k \ll N} \vec{V}_k$

Sample problems

(1) How much is $\det\left(\exp\left(t\cdot\begin{bmatrix}3&4\\5&6\end{bmatrix}\right)\right)=d(t)$?

Solution: $d(t)=e^{t\cdot \mathrm{Tr}\begin{bmatrix}3&4\\5&6\end{bmatrix}}=e^{t\cdot(3+6)}=e^{9t}$

(2) $A=\begin{bmatrix}1&4&4\\3&0&2\\2&1&2\end{bmatrix}=R+\lambda E+S$, where R is antysymmetryc and S has zero trace. Compute R, S, λ !

Solution:

$$R=\frac{A-A^T}{2}=\frac{1}{2}\begin{bmatrix}0&1&2\\-1&0&1\\-2&-1&0\end{bmatrix},\quad \lambda=\frac{1}{3}\mathrm{Tr}(A)=\frac{1}{3}(1+0+2)=1,\quad S=\frac{A+A^T}{2}-\frac{1}{3}\cdot\overset{\lambda}{1}\cdot E=\begin{bmatrix}0&7/2&3\\7/2&-1&3/2\\3&3/2&1\end{bmatrix}$$

(3) $P\begin{bmatrix}x_0\\x_1\\x_2\end{bmatrix}=\begin{bmatrix}x_1\\x_2\\x_0\end{bmatrix}$. Find the eigensystem of P !

Solution:

$$\varepsilon=e^{2\pi i/3}=\cos 120^\circ+i\cdot\sin 120^\circ=-\frac{1}{2}+\frac{\sqrt{3}}{2}i,\quad P^3=E,$$

$$\lambda_0=1,\quad \lambda_1=\varepsilon,\quad \lambda_2=\varepsilon^2$$

$$\vec V_0=\frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix},\quad \vec V_2=\frac{1}{\sqrt{3}}\begin{bmatrix}1\\\varepsilon^1\\\varepsilon^2\end{bmatrix},\quad \vec V_2=\begin{bmatrix}1\\\varepsilon^{1\cdot2}\\\varepsilon^{2\cdot2}\end{bmatrix}$$

(4) Find the eigensystem of $A=\begin{bmatrix}-2&1&1\\1&-2&1\\1&1&-2\end{bmatrix}$!

Solution: $P=\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}$, $PA=AP$, so the eigenvalues of P are

$1,\ \varepsilon=e^{2\pi i/3}=\cos\frac{2\pi}{3}+i\cdot\sin\frac{2\pi}{3}=-\frac{1}{2}+\frac{\sqrt{3}}{2}i$

with eigenvectors

$$\vec V_0=\frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix},\quad \vec V_1=\frac{1}{\sqrt{3}}\begin{bmatrix}1\\\varepsilon\\\varepsilon^2\end{bmatrix},\quad \vec V_2=\frac{1}{\sqrt{3}}\begin{bmatrix}1\\\varepsilon^2\\\varepsilon^4\end{bmatrix}$$

On this eigenvector basis the eigenvalues of A are

$$\lambda_0=0,\quad \lambda_1=2\cos 120^\circ-2=-3,\quad \lambda_3=2\cos(2\cdot 120^\circ)-2=-3$$