26 zh

Zh2, Diff.Egy., 2022.04.27. NEPTUN: Név:

Aláírás:

$(2+5+(2+1))$ pont

1. ~~(2+1+3+3+1~~ pont)

a) Szamitsd ki a Laplace tr. definicioja alapjan!

$F(s) = \mathcal{L}(f(t)) = \mathcal{L}(\sin(-t+6))$

$F(s)=$ $\int_0^\infty e^{-st}\sin(-t+6)\,dt = \int_0^\infty e^{-st}\frac{e^{i(-t+6)}-e^{-i(-t+6)}}{2i}\,dt =$

$= \frac{e^{6i}}{2i}\int_0^\infty e^{-(s+i)t}\,dt - \frac{e^{-6i}}{2i}\int_0^\infty e^{-(s-i)t}\,dt = \frac{e^{6i}}{2i}\cdot\frac{1}{s+i} - \frac{e^{-6i}}{2i}\cdot\frac{1}{s-i}$

b) $y' + 5y = t^2$, $y(0) = 2$. Mennyi $Y(s)$? ($\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}$) Ird fel $Y(s)$ parcialis tort felbontasanak az alakjat, es add meg, hogy mennyi $y(t)$!

$(sY(s)-2) + 5Y(s) = \frac{2!}{s^{2+1}} = \frac{2}{s^3}$

$Y(s) = \frac{1}{s+5}\left(2+\frac{2}{s^3}\right) = \frac{A}{s^3} + \frac{B}{s^2} + \frac{C}{s} + \frac{D}{s+5}$

$= \frac{2!}{s^{2+1}}\cdot\frac{A}{2} + \frac{B}{s^2} + \frac{C}{s} + \frac{D}{s-(-5)}$

$y(t) = \frac{A}{2}t^2 + Bt + C + De^{-5t}$

c) Szamold ki az $f(t) = t$ es a $g(t) = 5$ fuggvenyek $h = f * g$ konvoluciojat!

$(f*g)(t) = \int_0^t f(t-\tau)g(\tau)\,d\tau = \int_0^t (t-\tau)\cdot 5\,d\tau = 5t^2 - 5\frac{t^2}{2} = \frac{5}{2}t^2$

Mennyi $\mathcal{L}(f(t))\mathcal{L}(g(t)) - \mathcal{L}(h(t))$? Eleve nulla:

valóban: $\mathcal{L}(f(t)) = \frac{1}{s^2}$, $\mathcal{L}(g(t)) = \frac{5}{s}$ $\quad \frac{1}{s^2}\cdot\frac{5}{s} = \frac{5}{2}\cdot\frac{2!}{s^{2+1}}$

$\mathcal{L}\left(\frac{5}{2}t^2\right) = \frac{5}{2}\cdot\frac{2!}{s^{2+1}}$

2. (2+2×2 + 4 pont)
Legyen

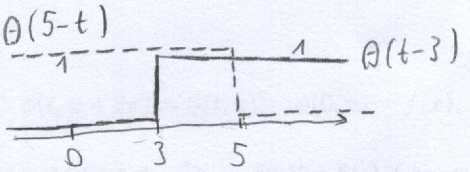
$$\begin{pmatrix}6&8\\2&4\end{pmatrix} = R+S+\lambda E, \quad \text{ahol} \quad R^T=-R,\ S^T=S,\ Tr(S)=0.$$

Mennyi R, S, λ ?

$$R=\frac{1}{2}\left(\begin{bmatrix}6&8\\2&4\end{bmatrix}-\begin{bmatrix}6&2\\8&4\end{bmatrix}\right)=\begin{bmatrix}0&3\\-3&0\end{bmatrix}$$

$$S+\lambda E=\begin{bmatrix}6&8\\2&4\end{bmatrix}-\begin{bmatrix}0&3\\-3&0\end{bmatrix}=\begin{bmatrix}6&5\\5&4\end{bmatrix}=\frac{6+4}{2}\begin{bmatrix}1&0\\0&1\end{bmatrix}+\begin{bmatrix}1&5\\5&-1\end{bmatrix}$$

Tehát $\begin{bmatrix}6&8\\2&4\end{bmatrix}=\begin{bmatrix}0&3\\-3&0\end{bmatrix}+\begin{bmatrix}1&5\\5&-1\end{bmatrix}+5\cdot\begin{bmatrix}1&0\\0&1\end{bmatrix}$



Szamitsd ki a kovetkezo Laplace transzformaltakat!

$\mathcal{L}(\theta(t-3)\theta(5-t)) =$

$$=\int_0^\infty e^{-st}\,\theta(t-3)\theta(5-t)\,dt=$$

$$=\int_3^5 e^{-st}\,dt=\left[\frac{e^{-st}}{-s}\right]_3^5=\frac{e^{-5s}}{-s}-\frac{e^{-3s}}{-s}=\frac{1}{s}\left(e^{-3s}-e^{-5s}\right)$$

$\mathcal{L}(\theta(t-3)e^{5t} =$

$$=\int_0^\infty e^{-st}\,\theta(t-3)e^{5t}\,dt=\int_3^\infty e^{-st}\cdot e^{5t}\,dt=\int_3^\infty e^{-(s-5)t}\,dt=\left[\frac{e^{-(s-5)t}}{-(s-5)}\right]_3^\infty$$

$$=\frac{e^{-(s-5)\cdot 3}}{(s-5)}$$

Legyen

$$\frac{d}{dt}\vec{y}+\begin{pmatrix}0&4\\4&0\end{pmatrix}\vec{y}=\begin{pmatrix}t\\1\end{pmatrix}, \quad \vec{y}(0)=\begin{pmatrix}5\\6\end{pmatrix}.$$

Mennyi $\vec{Y}(s)$?

$$\begin{bmatrix}sY_1(s)-5\\sY_2(s)-6\end{bmatrix}+\begin{bmatrix}0&4\\4&0\end{bmatrix}\begin{bmatrix}Y_1(s)\\Y_2(s)\end{bmatrix}=\begin{bmatrix}1/s^2\\1/s\end{bmatrix}$$

$$\left(s\cdot\begin{bmatrix}1&0\\0&1\end{bmatrix}+\begin{bmatrix}0&4\\4&0\end{bmatrix}\right)\begin{bmatrix}Y_1(s)\\Y_2(s)\end{bmatrix}=\begin{bmatrix}1/s^2+5\\1/s+6\end{bmatrix}$$

Vagyis $\begin{bmatrix}Y_1(s)\\Y_2(s)\end{bmatrix}=\begin{bmatrix}s&4\\4&s\end{bmatrix}^{-1}\begin{bmatrix}1/s^2+5\\1/s+6\end{bmatrix}$

Megjegyzés: $\begin{bmatrix}s&4\\4&s\end{bmatrix}^{-1}=\frac{1}{s^2-4^2}\begin{bmatrix}s&-4\\-4&s\end{bmatrix}$

3. (5 × 2 pont)
$y'' + 2y' + 5y = \underbrace{(t+1)^2}_{= t^2+2t+1}$, $y(0) = 2$, $y'(0) = 3$. Mennyi $Y(s)$? ($\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}$)

$$\left(s^2Y(s) - 2s - 3\right) + 2\left(sY(s) - 2\right) + 5Y(s) = \frac{2!}{s^3} + 2\cdot\frac{1}{s^2} + \frac{1}{s}$$

Vagyis

$$Y(s) = \frac{1}{s^2+2s+5}\left(2s + \underbrace{7}_{3+2\cdot(2)} + \frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}\right)$$

Oldd meg a $G'' + 2G' + 5G = \delta(t)$ egyenletet, ahol $G(t) = 0$, ha $t < 0$!

$t < 0$, $G(t) = 0$

$t \approx 0$, $G(t) \approx R(t)$, ahol $\left(R(t) = 0, \text{ha } t<0 \text{ és } R''(t) = \delta(t)\right)$

Vagyis $R(t) = \begin{cases} 0 & \text{ha } t<0 \\ t & \text{ha } t>0 \end{cases}$

$G(0^+) = R(0^+) = 0$

$G'(0^+) = R'(0^+) = 1$

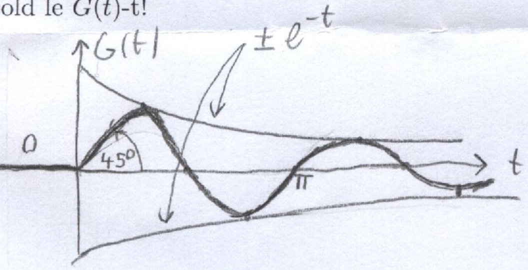
$t > 0$ Oldd meg: $y'' + 2y + 5y = 0$, $y(0) = 0$, $y'(0) = 1$.

$\lambda^2 + 2\lambda + 5 = 0 \rightarrow \lambda_{1,2} = -1 \pm 2i$, így $y(t) = e^{-t}\left(C_1\cos(2t) + C_2\sin(2t)\right)$

$\left.\begin{matrix} y(0)=0 \\ y'(0)=1 \end{matrix}\right\} \rightarrow C_1 = 0$, $C_2 = \frac{1}{2}$, vagyis $y(t) = \frac{1}{2}e^{-t}\sin(2t)$

Tehát $G(t) = \begin{cases} 0, & \text{ha } t<0 \\ \frac{1}{2}e^{-t}\sin(2t), & \text{ha } t>0 \end{cases}$

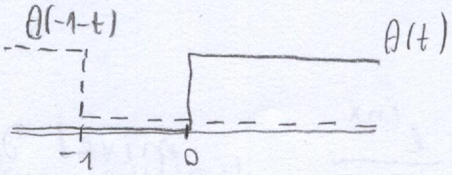
Rajzold le $G(t)$-t!



Ird fel a $y'' + 2y' + 5y = f(t)$ egyenlet megoldasat, ha $y(t) = f(t) = 0$ amikor $t < 0$!

$$y(t) = (G*f)(t) = \int_{-\infty}^{\infty} G(t-\tau)f(\tau)\,d\tau = \int_0^t \frac{1}{2}e^{-(t-\tau)}\sin(2\cdot(t-\tau))\,d\tau$$

Szamitsd ki az $\mathbb{R}$-en adott $\underbrace{\theta(t)\theta(-1-t)}_{=0}$ fuggveny Fourier transzformaltjat!



Mivel a függvény azonosan nulla, így a Fourier transzformáltja is nulla

4. (4+(2+4) pont)

$\to 4x_2^5$

A) Legyen $L = \dot{x}_1 x_2 + 3x_2\dot{x}_2 + 4x_2^2 x_2^3$. Ird fel az L Lagrange fuggvenyhez tartozo Euler-Lagrange egyenleteket!

$$\text{EL:}\quad \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} - \frac{\partial L}{\partial x_i} = 0$$

$$x_1:\quad \frac{d}{dt}\dot{x}_2 - 0 = 0 \qquad \text{vagyis } \ddot{x}_2 = 0$$

$$x_2:\quad \frac{d}{dt}(\dot{x}_1 + 3x_2) - (3\dot{x}_2 + 20x_2^4) = 0, \qquad \text{vagyis } \ddot{x}_1 + 3\dot{x}_2 - 3\dot{x}_2 - 20x_2^4 = \ddot{x}_1 - 20x_2^4 = 0$$

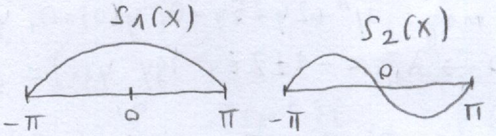
B) Legyen

$$\partial_t\phi(t,x) = \partial_x^2\phi(t,x), \quad \phi(t,x+2\pi) = \phi(t,x), \quad \phi(0,x) = f(x),$$

ahol $f(x) = -sgn(x) + 1$ a $[-\pi,\pi]$ intervalumon

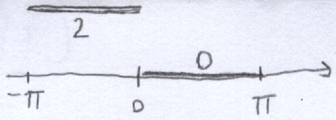
1. Ird fel egy ortonormalt bazisat $L^2([-\pi,\pi],dx)$-nek!

pl. $e_n(x) = \frac{e^{inx}}{\sqrt{2\pi}}$, $n \in \mathbb{Z}$, vagy pl. $s_n(x) = \frac{\sin\left(\frac{n}{2}(x+\pi)\right)}{\sqrt{\pi}}$, $n = 1, 2, 3, \dots$



2. Mennyi $\phi(t,x)$?

$f(x)$:



$$\hat{f}(x) = \sum_{n\in\mathbb{Z}} \hat{f}_n \frac{e^{inx}}{\sqrt{2\pi}},$$

$$\text{ahol } \hat{f}_n = \left(\frac{e^{inx}}{\sqrt{2\pi}}, f(x)\right) = \int_{-\pi}^{\pi} \frac{e^{-inx}}{\sqrt{2\pi}} f(x)\,dx = \int_{-\pi}^{0} \frac{e^{-inx}}{\sqrt{2\pi}} \cdot 2\,dx$$

ha $n \neq 0$

$$= \frac{2}{\sqrt{2\pi}\cdot(-in)}\left[e^{-inx}\right]_{-\pi}^{0} = \frac{\sqrt{2}\,i}{\sqrt{\pi}\,n}\underbrace{\left(1 - e^{i\pi n}\right)}_{\begin{cases} 0, & \text{ha } n \text{ páros} \\ 2, & \text{ha } n \text{ páratlan} \end{cases}}$$

$$\hat{f}_0 = 2\sqrt{2\pi}$$

Ekkor

$$\varphi(t,x) = \sum_{n\in\mathbb{Z}} \hat{f}_n e^{-n^2 t} \frac{e^{inx}}{\sqrt{2\pi}}, \qquad \text{mivel } \partial_x^2\left(\frac{e^{inx}}{\sqrt{2\pi}}\right) = -n^2\left(\frac{e^{inx}}{\sqrt{2\pi}}\right)$$