ZH 1. Diff.Eq. 2022.03.17. NEPTUN:

Nev: Alairas:

1. (4+(3+3) pont)
a)

$$\frac{d}{dx}y = f(x,y) = x^2 + y - 2;$$

Ird fel y masodrendu Taylor polinomjat az $x = 0$ pont korul, ha $y(0) = 3$!

$y(0) = 3$

$y'(0) = f(0,3) = 0^2 + 3 - 2 = 1$

$y''(x) = (\partial_x + f\partial_y) f(x,y) =$

$= (\partial_x + (x^2+y-2)\partial_y)(x^2+y-2)$

$= (2x) + (x^2+y-2)\cdot 1$

$y''(0) = 2\cdot 0 + (0^2+3-2)\cdot 1 = 1$

$y(x) \approx y(0) + y'(0)\cdot x + \frac{1}{2!}y''(0)\cdot x^2 =$

$= 3 + 1\cdot x + \frac{1}{2}\cdot 1\cdot x^2 = 3 + x + \frac{x^2}{2}$

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b) Alkalmazd az Euler, illetve a Heun modszert a kovetkezo DE-re $\Delta t = 0.01$ lepeskozzel!

$$\frac{d}{dt}y = y^2 + t^2, \qquad y(2) = 3.$$

Mit josol Euler es Heun $y(2.01)$ ertekere?

Euler:

$k_1 = 3^2 + 2^2 = 13$

$y(2.01) = 3 + 13\cdot 0.01 = 3.13$

Heun:

$k_2 = 3.13^2 + 2.01^2$

$y(2.01) = 3 + \frac{1}{2}\left([3^2+2^2] + [3.13^2+2.01^2]\right)\cdot 0.01$

2. (4+3+3 pont)

$$\frac{d}{dt}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} -y_1 + 2y_2 \\ 2y_1 - y_2 \end{pmatrix} = A\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \qquad \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

Keresd meg A sajatertekeit es sajatvektorait!

$$A = \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix}$$

Sajátérték: $0 = \det(A - \lambda E) = \begin{vmatrix} -1-\lambda & 2 \\ 2 & -1-\lambda \end{vmatrix} = (-1-\lambda)^2 - 2\cdot 2 = \lambda^2 + 2\lambda - 3$

$\longrightarrow \lambda_1 = 1, \ \lambda_2 = -3$

Sajátvektorok:

$\lambda_1 = 1$ $\quad A - \lambda E = \begin{bmatrix} -1-1 & 2 \\ 2 & -1-1 \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}$

$\begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\left.\begin{matrix} -2x + 2y = 0 \\ 2x - 2y = 0 \end{matrix}\right\} \to x = y \qquad \vec{V}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Vagy ennek valahány szorosa, kivéve a $\vec{0}$ vektort

$\lambda_2 = -3$ $\quad A - \lambda E = \begin{bmatrix} -1-(-3) & 2 \\ 2 & -1-(-3) \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$

$\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\left.\begin{matrix} 2x + 2y = 0 \\ 2x + 2y = 0 \end{matrix}\right\} \to x = -y, \qquad \vec{V}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Ird fel a DE altalanos megoldasat!

$$\vec{y}_{ált}(t) = C_1 e^{1\cdot t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 e^{-3\cdot t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Szamold ki a DE partikularis megoldasat!

$$\begin{bmatrix} 1 \\ 3 \end{bmatrix} = C_1 e^{1\cdot 0}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 e^{-3\cdot 0}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = C_1\begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\longrightarrow \left.\begin{matrix} 1 = C_1 + C_2 \\ 3 = C_1 - C_2 \end{matrix}\right\} \to C_1 = \frac{1+3}{2} = 2, \ C_2 = \frac{1-3}{2} = -1$$

$$\vec{y}_{part}(t) = 2\cdot e^{t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} - 1\cdot e^{-3t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2e^{t} - 3e^{-3t} \\ 2e^{t} + 3e^{-3t} \end{bmatrix}$$

Vagy!

$$\vec{y}_{part}(t) = \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}}_{} \underbrace{\begin{bmatrix} e^{1\cdot t} & 0 \\ 0 & e^{-3\cdot t} \end{bmatrix}}_{} \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1}}_{S^{-1}} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$S = \begin{bmatrix} \vec{V}_1 & \vec{V}_2 \end{bmatrix} \qquad e^{tD} = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix}$$

3. ((3+2)+2+3 pont)
3a)

$$A = \begin{pmatrix} -1 & 0 \\ 6 & -1 \end{pmatrix}$$

Mennyi e^{tA}?

$$\exp\left[t\begin{pmatrix} -1 & 0 \\ 6 & -1 \end{pmatrix}\right] = \exp\left[t\left\{\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 6 & 0 \end{pmatrix}\right\}\right] =$$

$$= \exp\left[t\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\right] \exp\left[t\begin{pmatrix} 0 & 0 \\ 6 & 0 \end{pmatrix}\right] =$$

$$= \begin{pmatrix} e^{-1.t} & 0 \\ 0 & e^{-1.t} \end{pmatrix} \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + t \cdot \begin{pmatrix} 0 & 0 \\ 6 & 0 \end{pmatrix} + \frac{t^2}{2!}\underbrace{\begin{pmatrix} 0 & 0 \\ 6 & 0 \end{pmatrix}^2}_{\text{Zéró mátrix}} + \ldots \right\} = \begin{pmatrix} e^{-1.t} & 0 \\ 0 & e^{-1.t} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 6t & 1 \end{pmatrix}$$

$$= e^{-t}\begin{pmatrix} 1 & 0 \\ 6t & 1 \end{pmatrix}$$

Ird fel a kovetkezo DE megoldasat!

$$\frac{d}{dt}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} -y_1 \\ 6y_1 - y_2 \end{pmatrix}, \qquad \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$\vec{y}_{part}(0) = e^{tA}\vec{y}(0) = e^{-t}\begin{pmatrix} 1 & 0 \\ 6t & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} e^{-t} \\ e^{-t}(6t+3) \end{pmatrix}$$

3b) Ird at ezt a DE-t egy elsorendu rendszerre

$$\frac{d^2}{dt^2}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \dot{y}_1{}^2 - y_2 \\ 2\dot{y}_2 - 3\dot{y}_1 \end{pmatrix}$$

Legyen $v_1 = \dot{y}_1$, $v_2 = \dot{y}_2$, ekkor

$$\frac{d}{dt}\begin{bmatrix} y_1 \\ y_2 \\ v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_1^2 - y_2 \\ 2v_2 - 3v_1 \end{bmatrix}$$

3c)
Legyen $x_{n+1} = 1.2x_n - 20$, $\quad x_0 = 1234$. Mennyi ekkor x_n ?

Fixpont: $x_{n+1} = x_n$, $\quad x_{fix} = 1.2x_{fix} - 20 \longrightarrow 20 = 0.2x_{fix} \longrightarrow x_{fix} = 100$

$$x_n = 1.2^n(1234 - 100) + 100 = 1.2^n \cdot 1134 + 100$$

4a. (1+1+1+2 pont)

$y' = (-y^2+4).$ $= f(y)$

Keresd meg a DE fixpontjait!

$-y^2+4=0 \longrightarrow y_1=-2, \quad y_2=2$

Ird fel a fixpontok koruli linearizalt kozelito DE-t!

$$\frac{df(y)}{dy} = \frac{d(-y^2+4)}{dy} = -2y$$

$y_1=-2$:

$\Delta y = y-(-2)$

$\frac{d}{dt}\Delta y = -2\cdot(-2)\cdot\Delta y = 4\Delta y$

y_1

$y_2=2$

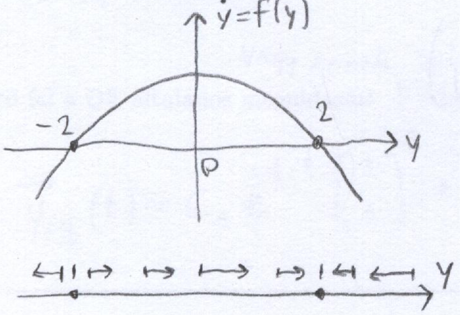
$\Delta y = y-2$

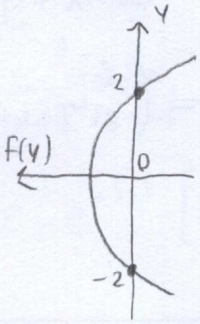
$\frac{d}{dt}\Delta y = -2\cdot 2\,\Delta y = -4\Delta y$

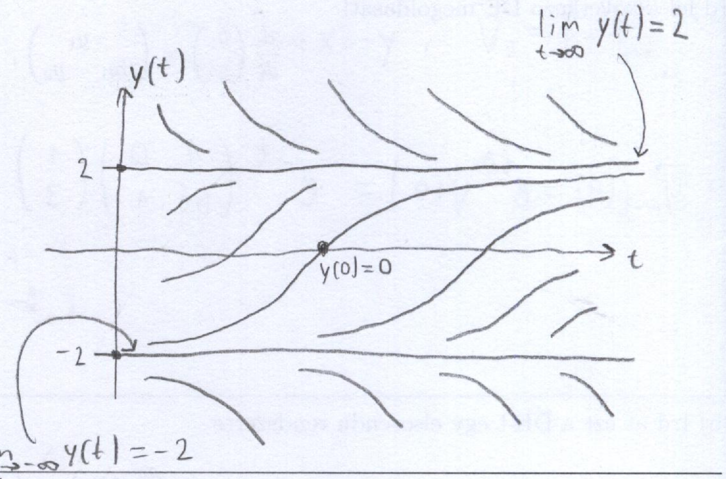
Ha $y(0)=0$, mennyi

$\lim_{x\to\infty} y(x) =?$ 2 and $\lim_{x\to-\infty} y(x) = ?$ -2

Vazold az $y(x)$ megoldasait a DE-nek!







4b. (2+3 pont)

$$\frac{d}{dt}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} -y_1+2 \\ 2y_2(-y_1-3y_2). \end{pmatrix}$$

Keresd meg a DE fixpontjait!

$$\left.\begin{aligned} -y_1+2&=0 \\ 2y_2(-y_1-3y_2)&=0 \end{aligned}\right\} \to y_1=2 \xrightarrow{\text{vagy}} \begin{cases} y_2=0 \\ (-2-3y_2)=0 \to y_2=-\frac{2}{3} \end{cases}$$

Tehát:

$\vec{z}_A = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

$\vec{z}_B = \begin{bmatrix} 2 \\ -2/3 \end{bmatrix}$

Ird fel a fixpont koruli linearizalt kozelito DE-t!

Jacobi mátrix:

$$Jac = \begin{bmatrix} \partial_{y_1}(-y_1+2) & \partial_{y_2}(-y_1+2) \\ \partial_{y_1}[2y_2(-y_1-3y_2)] & \partial_{y_2}[2y_2(-y_1-3y_2)] \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ -2y_2 & -2y_1-12y_2 \end{bmatrix}$$

$$Jac(\vec{z}_A) = \begin{bmatrix} -1 & 0 \\ 0 & -4 \end{bmatrix} \qquad Jac(\vec{z}_B) = \begin{bmatrix} -1 & 0 \\ +\frac{4}{3} & -12 \end{bmatrix}$$

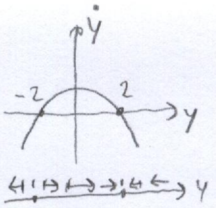
$$\frac{d}{dt}\begin{bmatrix} y_1-2 \\ y_2-0 \end{bmatrix} = \frac{d}{dt}\overrightarrow{\Delta y} = \begin{bmatrix} -1 & 0 \\ 0 & -4 \end{bmatrix}\overrightarrow{\Delta y}$$

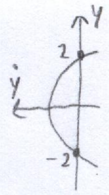
$$\frac{d}{dt}\begin{bmatrix} y_1-2 \\ y_2+\frac{2}{3} \end{bmatrix} = \frac{d}{dt}\begin{bmatrix} \Delta y_1 \\ \Delta y_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ +\frac{4}{3} & 4 \end{bmatrix}\begin{bmatrix} \Delta y_1 \\ \Delta y_2 \end{bmatrix}$$

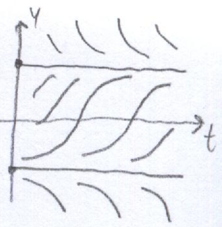
$y_2-(-\frac{2}{3})$

Fixpoints: $-y^2+4=0$

$\longrightarrow y_1=-2,\ y_2=2$







4a. (1+1+1+2 pont)
$y' = (-y^2+4). = f(y)$
Find the fixed points of the DE!

Lin DE: $\frac{df}{dy} = \frac{d(-y^2+4)}{dy} = -2y$

$\Delta y_1 = y-(-2)$ $\qquad$ $\Delta y_2 = y-2$

$\frac{d}{dt}\Delta y_1 = -2(-2)\Delta y_1 = 4\Delta y_1$ $\qquad$ $\frac{d}{dt}\Delta y_2 = -2\cdot 2\Delta y_2 = -4\Delta y_2$

$\lim\limits_{t\to\infty} y(t) = 2$

$\lim\limits_{t\to-\infty} y(t) = -2$

If $y(0)=0$, compute
$\lim_{x\to\infty} y(x) =?$ and $\lim_{x\to-\infty} y(x) = ?$
Sketch the $y(x)$ solution curves of the DE!

4b. (2+3 pont)

$$\frac{d}{dt}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} -y_1+2 \\ 2y_2(-y_1-3y_2). \end{pmatrix}$$

$\text{Jac} = \begin{bmatrix} -1 & 0 \\ -2y_2 & -2y_1-12y_2 \end{bmatrix}$

Find the fixed points of the DE!

Fixpoints: $\vec{z}_A = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$, $\vec{z}_B = \begin{bmatrix} 2 \\ -2/3 \end{bmatrix}$ $\quad$ $\text{Jac}(\vec{z}_A) = \begin{bmatrix} -1 & 0 \\ 0 & -4 \end{bmatrix}$ $\quad$ $\text{Jac}(\vec{z}_B) = \begin{bmatrix} -1 & 0 \\ -\frac{4}{3} & 4 \end{bmatrix}$

Write down the linearized DE around the fixed points!

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1. (4+(3+3) pont)
a)

$y(0)=3$

$y'(0)=0^2+3-2=1$

$y''(x) = (\partial_x + (x^2+y-2)\partial_y)(x^2+y-2) = 2x+(x^2+y-2)\cdot 1$

$y''(0)=1$

$y(x) = 3+1\cdot x+\frac{1}{2!}x^2$

$$\frac{d}{dx}y = f(x,y) = x^2+y-2;$$

How much is y'' ? Write down the second order Taylor polynomial of $y(x)$ around $x=0$, if $y(0)=3$!

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b) Apply the Euler and the Heun methods for the following DEs with stepsize $\Delta x = 0.01$!

$$\frac{d}{dt}y = y^2+t^2, \qquad ,y(2)=3.$$

Euler: $k_1 = 3^2+2^2 = 13$

$y(2.01) = 3+13\cdot 0.01 = 3.13$

What are the predictions for $y(2.01)$?
Euler:
Heun: $y(2.01) = 3+\frac{1}{2}(13+(3.13^2+2.01^2))\cdot 0.01$

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2. (4+3+3 pont)

$$\frac{d}{dt}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} -y_1+2y_2 \\ 2y_1-y_2 \end{pmatrix} = A\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \qquad \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

Eigenvals: $0 = \begin{vmatrix} -1-\lambda & 2 \\ 2 & -1-\lambda \end{vmatrix} \longrightarrow$

$\lambda_1 = 1,\ \lambda_2 = -3$

$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\quad$ $\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Find the eigenvalues and eigenvectors of A ! $\quad y_{gen}(t) = c_1 e^{1\cdot t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-3t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Write down the general solution of the DE! $\quad \begin{pmatrix} 1 \\ 3 \end{pmatrix} = c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ -1 \end{bmatrix} \longrightarrow c_1=2,\ c_2=-1$

Compute the particular solution!

$y_{part}(t) = 2e^{1\cdot t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} - 1\cdot e^{-3t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

((3+2)+2+3 pont)
3a)

$e^{tA} = \exp\left(t\cdot\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\right)\cdot\exp\left(t\cdot\begin{bmatrix} 0 & 0 \\ 6 & 0 \end{bmatrix}\right) = e^{-t}\cdot\begin{bmatrix} 1 & 0 \\ 6t & 1 \end{bmatrix}$

$$A = \begin{pmatrix} -1 & 0 \\ 6 & -1 \end{pmatrix}$$

How much is e^{tA}?

Express the solution of the following DE

$$\frac{d}{dt}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} -y_1 \\ 6y_1-y_2 \end{pmatrix}, \qquad \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$\vec{y} = e^{-t}\begin{bmatrix} 1 & 0 \\ 6t & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix}$

with e^{tA} !

3b) Express the following DE as a first order system!

$$\frac{d^2}{dt^2}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \dot{y_1}^2 - y_2 \\ 2\dot{y}_2 - 3\dot{y}_1 \end{pmatrix}$$

$\frac{d}{dt}\begin{bmatrix} y_1 \\ y_2 \\ v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_1^2 - y_2 \\ 2v_2 - 3v_1 \end{bmatrix}$

3c)
Let $x_{n+1} = 1.2x_n - 20$, $\quad x_0 = 1234$. How much is x_n ?

Fixpoint: $x_f = 1.2x_f - 20 \longrightarrow x_f = 100$

$x_n = 1.2^n(1234-100)+100 = 1.2^n\cdot 1134+100$