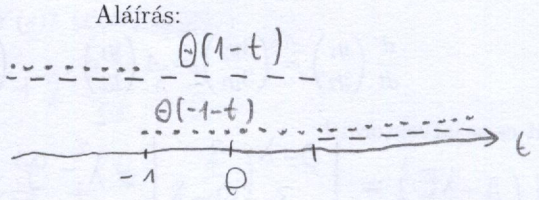
Lev.PZh, Diff.Egy., 2022.05.21. NEPTUN: Név:

Aláírás:



tehát ha $t \geq 0$, akkor

$\theta(1-t)\,\theta(-1-t) = 0$

1. (2+4+2+2 pont)

a) Szamitsd ki a Laplace tr. definicioja alapjan!

$F(s) = \mathcal{L}(f(t)) = \mathcal{L}(\theta(1-t)\theta(-1-t))$

$F(s)=$

$\mathcal{L}(f(t)) = \mathcal{L}(0) = 0$

b) $y' - 4y = (t^2-1)^2$, $y(0) = -5$. Mennyi $Y(s)$? ($\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}$) Ird fel $Y(s)$ parcialis tort felbontasanak az alakjat, es add meg, hogy mennyi $y(t)$!

$$y' - 4y = t^4 - 2t^2 + 1$$

$$\left[Y(s) - (-5)\right] - 4Y(s) = \frac{4!}{s^5} - 2\cdot\frac{2!}{s^3} + \frac{1}{s} = \frac{24}{s^5} - \frac{4}{s^3} + \frac{1}{s}$$

$$Y(s) = \frac{1}{s-4}\left[-5 + \frac{24}{s^5} - \frac{4}{s^3} + \frac{1}{s}\right]$$

$$= \frac{A}{s-4} + \frac{B}{s^5} + \frac{C}{s^4} + \frac{D}{s^3} + \frac{E}{s^2} + \frac{F}{s}$$

$$= \frac{A}{s-4} + \frac{B}{4!}\frac{4!}{s^5} + \frac{C}{3!}\frac{3!}{s^4} + \frac{D}{2!}\frac{2!}{s^3} + \frac{E}{s^2} + \frac{F}{s}$$

$$y(t) = A\,e^{4t} + \frac{B}{24}t^4 + \frac{C}{6}t^3 + \frac{D}{2}t^2 + Et + F$$

c) Legyen $x_{n+1} = 8 - x_n$, $x_0 = 13$. Mennyi x_n ?

Fixpont: $x_{fix} = 8 - x_{fix} \longrightarrow x_{fix} = 4$

$$x_n = (-1)^n(x_0 - x_{fix}) + x_{fix} = (-1)^n(13-4) + 4 = (-1)^n \cdot 9 + 4$$

d) Ird at a kovetkezo DE rendszert elsorendu idofuggetlen DE rendszerre!

$$\frac{d^2}{dt^2}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} y_1' y_2^2 \\ t^2 y_2 + y_1 t^3 \end{pmatrix}$$

$$\frac{d}{dt}\begin{bmatrix} y_1 \\ y_2 \\ v_1 \\ v_2 \\ s \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_1 y_2^2 \\ s^2 y_2 + y_1 s^3 \\ 1 \end{bmatrix}$$

4. (5+5 pont)

$y' = -y^4 + 16.$

Keresd meg a DE fixpontjait!

$$-y^4+16=0 \longrightarrow y_1=-2,\quad y_2=2$$

Ird fel a fixpontok koruli linearizalt kozelito DE-t!

$$\frac{d}{dy}(-y^4+16) = -4y^3$$

$y_1 = -2$

$$\frac{d}{dt}(y-(-2)) = \frac{d}{dt}\Delta y = -4\cdot(-2)^3\Delta y = 32\Delta y$$

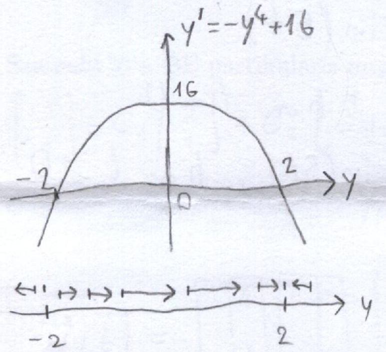
$y_2 = +2$

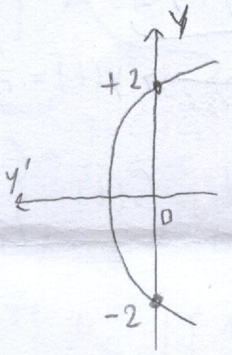
$$\frac{d}{dt}(y-2) = \frac{d}{dt}\Delta y = -4\cdot 2^3 = -32\Delta y$$

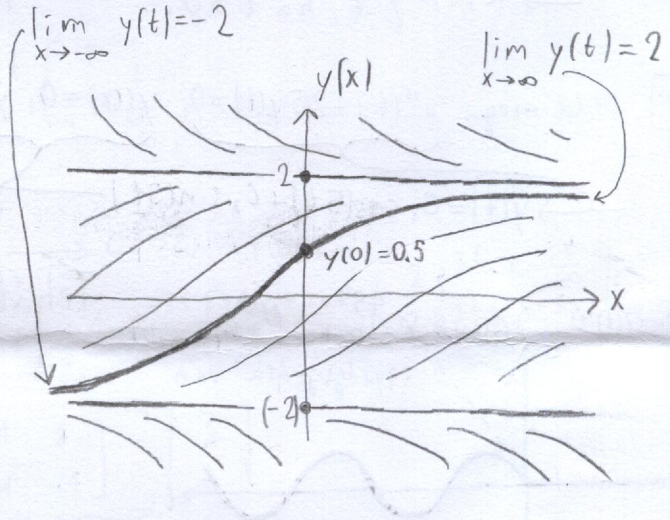
Ha $y(0) = 0.5$, mennyi

$\lim_{x\to\infty} y(x) = 2$ $\qquad \lim_{x\to-\infty} y(x) = -2$

Vazold a DE megoldasgorbeit!







3b. (5 pont)

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} (y_1+3)(y_2-4) \\ y_2-6. \end{pmatrix}$$

Keresd meg a DE fixpontait!

$$(y_1+3)(y_2-4)=0$$

$$y_2-6=0 \longrightarrow y_2=6 \longrightarrow (y_1+3)(6-4)=0 \longrightarrow y_1=-3$$

Tehát a fixpont: $\vec{Z} = \begin{bmatrix} -3 \\ 6 \end{bmatrix}$

Ird fel a fixpont koruli linearizalt kozelito DE-t!

$$\mathrm{Jac} = \begin{bmatrix} \partial_{y_1}[(y_1+3)(y_2-4)] & \partial_{y_2}[(y_1+3)(y_2-4)] \\ \partial_{y_1}(y_2-6) & \partial_{y_2}(y_2-6) \end{bmatrix}$$

$$= \begin{bmatrix} y_2-4 & y_1+3 \\ 0 & 1 \end{bmatrix}$$

$$\mathrm{Jac}(\vec{Z}) = \begin{bmatrix} 6-4 & -3+3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\frac{d}{dt}\begin{bmatrix} y_1-(-3) \\ y_2-6 \end{bmatrix} = \frac{d}{dt}\begin{bmatrix} \Delta y_1 \\ \Delta y_2 \end{bmatrix} = $$

$$= \mathrm{Jac}(\vec{Z})\vec{\Delta y} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \Delta y_1 \\ \Delta y_2 \end{bmatrix}$$

$$= \begin{bmatrix} 2\Delta y_1 \\ \Delta y_2 \end{bmatrix}.$$

2. ((3+1+2)+4 pont)

A)

$$\frac{d}{dt}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 3y_2 \\ 3y_1 \end{pmatrix} = A\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \qquad \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

Keresd meg A sajatertekeit es sajatvektorait!

$$A = \begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix} \qquad \det(A-\lambda E) = \begin{vmatrix} 0-\lambda & 3 \\ 3 & 0-\lambda \end{vmatrix} = \lambda^2 - 9 \longrightarrow \lambda_1 = 3,\ \lambda_2 = -3$$

$$\lambda_1 = 3: \begin{bmatrix} 0-3 & 3 \\ 3 & 0-3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{matrix} -3x+3y=0 \\ 3x-3y=0 \end{matrix} \rightarrow x = y$$

$$\vec{V}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\lambda_2 = -3: \begin{bmatrix} 0-(-3) & 3 \\ 3 & 0-(-3) \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$3x+3y=0 \longrightarrow y = -x$$

$$\rightarrow \vec{V}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Ird fel a DE altalanos megoldasat!

$$\vec{y}_{ált}(t) = C_1 e^{3t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 e^{-3t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Szamold ki a DE partikularis megoldasat!

$$\begin{bmatrix} 3 \\ 1 \end{bmatrix} = C_1\begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2\begin{bmatrix} 1 \\ -1 \end{bmatrix} \longrightarrow \begin{matrix} 3 = C_1 + C_2 \\ 1 = C_1 - C_2 \end{matrix} \longrightarrow C_1 = 2,\ C_2 = 1$$

$$\text{tehát } \vec{y}_{part}(t) = 2e^{3t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 1\cdot e^{-3t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2e^{3t} - e^{-3t} \\ 2e^{3t} + e^{-3t} \end{bmatrix}$$

Vagy

$$\vec{y}_{part}(t) = \Bigg[\underset{\vec{V}_1}{\boxed{\begin{matrix} 1 \\ 1 \end{matrix}}}\ \underset{\vec{V}_2}{\boxed{\begin{matrix} 1 \\ -1 \end{matrix}}}\Bigg]\begin{bmatrix} e^{3t} & 0 \\ 0 & e^{-3t} \end{bmatrix}\underbrace{\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1}}_{= \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}}\begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

B) Ird fel a kovetkezo DE megoldasat!

$$\frac{d}{dt}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 3y_1 \\ 3y_1 + 3y_2 \end{pmatrix} = A\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \qquad \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$$A = \begin{bmatrix} 3 & 0 \\ 3 & 3 \end{bmatrix} = 3\cdot\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix}$$

$$\exp(tA) = \exp\left(t\cdot\left\{3\cdot\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix}\right\}\right) = \exp\left(3t\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\cdot\exp\left(t\begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix}\right)$$

$$= e^{3t}\cdot\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + t\begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix} + \frac{t^2}{2}\underbrace{\begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix}^2}_{= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}}\right) = e^{3t}\begin{bmatrix} 1 & 0 \\ 3t & 1 \end{bmatrix}$$

Tehát

$$y(t) = e^{3t}\begin{bmatrix} 1 & 0 \\ 3t & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 3e^{3t} \\ 9te^{3t} + e^{3t} \end{bmatrix}$$

3. (5 × 2 pont)
$y'' + 25y = (t^2+1)^2$, $y(0) = 2$, $y'(0) = 3$. Mennyi $Y(s)$? ($\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}$)

$$\left(s^2 Y(s) - 2s - 3\right) + 25\,Y(s) = \frac{4!}{s^5} + 2\cdot\frac{2!}{s^3} + \frac{1}{s}$$

$$Y(s) = \frac{1}{s^2+25}\left(2s + 3 + \frac{24}{s^5} + \frac{4}{s^3} + \frac{1}{s}\right)$$

Oldd meg a $G'' + 25G = \delta(t)$ egyenletet, ahol $G(t) = 0$, ha $t < 0$!

$\boxed{t<0}$ Mivel retardált megoldást keresünk, így ekkor $G(t) = 0$

$\boxed{t \approx 0}$ $G''(t) \approx R(t)$, ahol $R(t) = 0$, ha $t < 0$ és $R''(t) = \delta(t)$

$\longrightarrow R(t) = \begin{cases} 0, & \text{ha } t<0 \\ t, & \text{ha } t>0. \end{cases}$ Továbbá $G(0^+) = R(0^+) = 0$

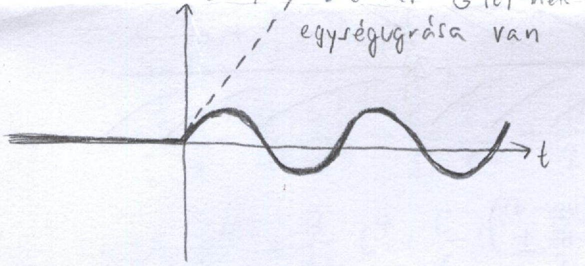
$G'(0^+) = R'(0^+) = 1$

$\boxed{t>0}$ Oldd meg: $y''(t) + 25y(t) = 0$, $y(0) = 0$, $y'(0) = 1$ (a kezdeti feltételek innen származnak)

$\longrightarrow y(t) = C_1 \cos(5t) + C_2 \sin(5t) \longrightarrow y(t) = \frac{1}{5}\sin(5t)$

Tehát $G(t) = \begin{cases} 0, & \text{ha } t<0 \\ \frac{1}{5}\sin(5t), & \text{ha } t>0 \end{cases}$

Rajzold le $G(t)$-t!



Mennyi $G(t)$ Laplace transzformaltja?

$$\mathcal{L}(G(t)) = \frac{1}{s^2+25}$$ ← ($Y(s)$ szorzófaktora az első részben)

Vagy mivel $\mathcal{L}(\sin(at)) = \frac{a}{s^2+a^2}$, így $\mathcal{L}\left(\frac{1}{5}\sin(5t)\right) = \frac{1}{5}\,\frac{5}{s^2+25} = \frac{1}{s^2+25}$

Ird fel a $y'' + 25y = f(t)$ egyenlet megoldasat, ha $y(t) = f(t) = 0$ amikor $t < 0$!

$$y(t) = (G * f)(t) = \int_{-\infty}^{\infty} G(t-\tau) f(\tau)\, d\tau = \int_0^t \frac{1}{5}\sin[5(t-\tau)]\, f(\tau)\, d\tau$$