DE.test.I.18.march.28

1. (a) (4 points)
   Let $\frac{d}{dt}y = (y^2 - 4)y$.
   i. Find the fixed points!
   ii. Write down the linearized equations around the fixed points!
   iii. If $y(0) = 1.34$, then how much are $\lim_{t\to\infty} y(t)$ and $\lim_{t\to-\infty} y(t)$?
   iv. Plot the solution curves of the DE!

(i) $f(y) = (y^2-4)y$

$f(y)=0 \Rightarrow y_1 = -2, \quad y_2 = 0, \quad y_3 = 2$

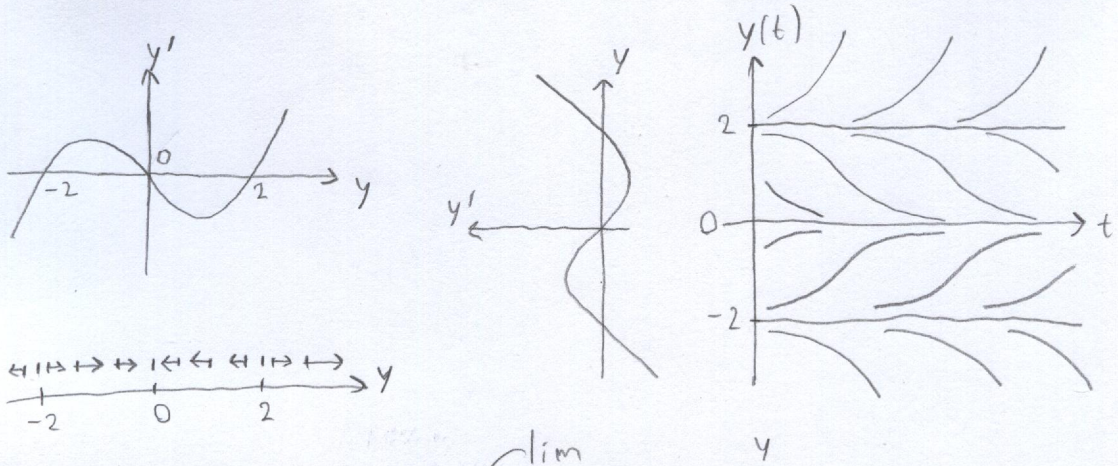
(ii) $\frac{df(y)}{dy} = 3y^2 - 4 = f'(y)$

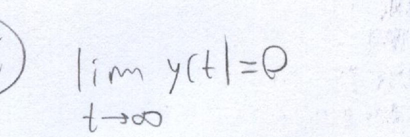
y_1: $\frac{d}{dt}(y-(-2)) = \frac{d}{dt}\Delta y = 8\Delta y \qquad 8 = f'(-2) = 3\cdot(-2)^2 - 4$

y_2: $\frac{d}{dt}(y-0) = \frac{d}{dt}\Delta y = -4\Delta y \qquad -4 = f'(0) = 3\cdot 0^2 - 4$

y_3: $\frac{d}{dt}(y-2) = \frac{d}{dt}\Delta y = 8\Delta y \qquad 8 = f'(2) = 3\cdot 2^2 - 4$

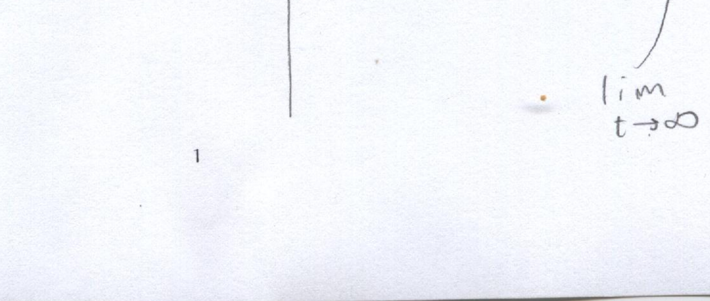
(iv)





(iii) $\lim_{t\to\infty} y(t) = 0$

$\lim_{t\to-\infty} y(t) = 2$



(b) (2+4 points)

$$\frac{d}{dt}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} (y_2 - 2) \\ (y_1 + 4)(y_2 - 5). \end{pmatrix}$$

i. Find the fixed point(s)!

ii. Write down the linearized equation(s) around the fixed point(s)!

(i) $\begin{pmatrix} y_2 - 2 \\ (y_1+4)(y_2-5) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Longrightarrow$ $y_2 - 2 = 0 \longrightarrow y_2 = 2$

$(y_1+4)(2-5) = 0 \longrightarrow y_1 = -4$

Fixed point: $P = \begin{pmatrix} -4 \\ 2 \end{pmatrix}$

(ii) Jacobian matrix:

$$Jac = \begin{pmatrix} \partial_{y_1}(y_2-2) & \partial_{y_2}(y_2-2) \\ \partial_{y_1}[(y_1+4)(y_2-5)] & \partial_{y_2}[(y_1+4)(y_2-5)] \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ y_2-5 & y_1+4 \end{pmatrix}$$

$$Jac(P) = \begin{pmatrix} 0 & 1 \\ -3 & 0 \end{pmatrix}$$

Lin. Eq.:

$$\frac{d}{dt}\begin{pmatrix} y_1 - (-4) \\ y_2 - 2 \end{pmatrix} = \frac{d}{dt}\begin{pmatrix} \Delta y_1 \\ \Delta y_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -3 & 0 \end{pmatrix}\begin{pmatrix} \Delta y_1 \\ \Delta y_2 \end{pmatrix}$$

2. (2+3+3+2 points)

(a) What is the solution of $y'(t) = 2 + \delta(t)$, $y(-3) = 4$?

(b) What is the solution of $y''(t) = -3\delta(t)$, $y(-3) = 0$, $y(3) = 0$?

(c) What is the retarded solution of $G''(t) = 4G(t) + \delta(t)$?

(d) Express the solution of the DE

$$y''(t) = 4y(t) + f(t), \quad y(t) = f(t) = 0, \text{ if } t << 0,$$

with G !

(a) $y'(t) = 2 + \delta(t) \longrightarrow y(t) = 2t + H(t) + C$

$y(-3) = 4 \longrightarrow 2\cdot(-3) + \underbrace{H(-3)}_{0} + C = 4 \longrightarrow C = 10$

$$y(t) = 2t + H(t) + 10 = \begin{cases} 2t+10 & \text{if } t<0 \\ 2t+11 & \text{if } t>0 \end{cases}$$

(b) $y'' = -3\delta(t) \longrightarrow y(t) = -3K(t) + C_1 t + C_2$, where $K(t) = \begin{cases} 0, & \text{if } t<0 \\ t, & \text{if } t>0 \end{cases}$

$y(-3) = 0 \longrightarrow -3\cdot 0 + C_1\cdot(-3) + C_2 = 0$

$y(3) = 0 \longrightarrow -3\cdot 3 + C_1 \cdot 3 + C_2 = 0$

$-3C_1 + C_2 = 0$

$3C_1 + C_2 = 9$

$y(t) = -3K(t) + \frac{3}{2}t + \frac{9}{2}$

$C_1 = \frac{9}{6} = \frac{3}{2}$, $C_2 = \frac{9}{2}$

(c) $G''(t) = 4G(t) + \delta(t)$

$t<0$ $G(t) = 0$ (retarded sol.) so $G(0^-) = G'(0^-) = 0$

$t \approx 0$ $G(t) \approx K(t) \longrightarrow G(0^+) = 0$, $G'(0^+) = 1$

$t>0$ $G(t) = y^G(t)$, where y^G solves: $(y^G)'' = 4y^G$, $y^G(0) = 0$, $(y^G)'(0) = 1$

so $y^G(t) = C_1 e^{2t} + C_2 e^{-2t}$, $C_1 + C_2 = 0$, $2C_1 - 2C_2 = 1$

so $y^G(t) = \frac{1}{4}(e^{2t} - e^{-2t})$

Consequently

$$G(t) = \begin{cases} 0 & \text{if } t<0 \\ \frac{1}{4}(e^{2t} - e^{-2t}) & \text{if } t>0 \end{cases}$$

(d) $y(t) = \int_{-\infty}^{\infty} G(t-s) f(s)\, ds = (G * f)(t) = \int_{-\infty}^{t} \frac{1}{4}\left(e^{2(t-s)} - e^{-2(t-s)}\right) f(s)\, ds$

3. (4+2+4 pont) Let

$$\frac{d}{dt}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 2y_1 - 3y_2 \\ 3y_1 + 2y_2 \end{pmatrix} = A\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \quad \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \text{respectively} \quad \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

(a) Find the eigenvectors and eigenvalues of A !

(b) Write down the generel solution of the DE!

(c) Compute the two particular solutions!

(a) $A = \begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix}$. char. eq.: $\det(A - \lambda E) = 0 = \begin{vmatrix} 2-\lambda & -3 \\ 3 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 + 9$

$= \lambda^2 - 4\lambda + 13 \longrightarrow \lambda_{1,2} = \frac{4 \pm \sqrt{16-42}}{2} = 2 \pm 3i$

$\lambda_1 = 2+3i$

$$\begin{pmatrix} 2-(2+3i) & -3 \\ 3 & 2-(2+3i) \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$-3iu - 3v = 0$

$\overline{v}_1 = \begin{pmatrix} 1 \\ -i \end{pmatrix}$

$\lambda_2 = 2-3i$

$$\begin{pmatrix} 2-(2-3i) & -3 \\ 3 & 2-(2-3i) \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$3iu - 3v = 0$

$\overline{v}_2 = \begin{pmatrix} 1 \\ i \end{pmatrix}$

(b) $\overline{y}_{gen}(t) = C_1 e^{(2+3i)t}\begin{pmatrix} 1 \\ -i \end{pmatrix} + C_2 e^{(2-3i)t}\begin{pmatrix} 1 \\ i \end{pmatrix}$

(c) $\begin{pmatrix} 1 \\ 0 \end{pmatrix} = C_1\begin{pmatrix} 1 \\ -i \end{pmatrix} + C_2\begin{pmatrix} 1 \\ i \end{pmatrix}$

$$\left.\begin{aligned} C_1 + C_2 &= 1 \\ -iC_1 + iC_2 &= 0 \end{aligned}\right\} \longrightarrow \begin{aligned} C_1 &= 1/2 \\ C_2 &= 1/2 \end{aligned}$$

$y(t) = \frac{1}{2}e^{(2+3i)t}\begin{pmatrix} 1 \\ -i \end{pmatrix} + \frac{1}{2}e^{(2-3i)t}\begin{pmatrix} 1 \\ i \end{pmatrix}$

$= e^{2t}\begin{pmatrix} \cos(3t) \\ \sin(3t) \end{pmatrix}$

$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = C_1\begin{pmatrix} 1 \\ -i \end{pmatrix} + C_2\begin{pmatrix} 1 \\ i \end{pmatrix}$

$$\left.\begin{aligned} C_1 + C_2 &= 0 \\ -iC_1 + iC_2 = 1 \longleftrightarrow C_1 - C_2 &= i \end{aligned}\right\} \longrightarrow \begin{aligned} C_1 &= i/2 \\ C_2 &= -i/2 \end{aligned}$$

$y(t) = \frac{i}{2}e^{(2+3i)t}\begin{pmatrix} 1 \\ -i \end{pmatrix} - \frac{i}{2}e^{(2-3i)t}\begin{pmatrix} 1 \\ i \end{pmatrix}$

$= e^{2t}\begin{pmatrix} -\sin(3t) \\ \cos(3t) \end{pmatrix}$

4. ((3+2)+5 points)

(a) Let A be the same as it was in the previous excercise.

- How much is e^{tA} ?
- What is the solution of $\frac{d}{dt}\bar{y}(t) = A\bar{y}(t) + \bar{f}(t)$, $\bar{y}(t) = \bar{f}(t) = 0$, if $t << 0$, ?

(b) Let

$$\frac{d}{dt}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} y_1 + 3y_2 \\ y_2 \\ 7y_3 \end{pmatrix} = B\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

- How much is e^{tB} ?
- What is the solution of the previous DE with initial condition $(y_1(0), y_2(0))^T = \cancel{(5,4)}$ $(5,4,3)$

(a) • $$e^{tA} = \begin{pmatrix} e^{2t}\cos(3t) & -e^{2t}\sin(3t) \\ e^{2t}\sin(3t) & e^{2t}\cos(3t) \end{pmatrix}$$

or $$e^{tA} = \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}\begin{pmatrix} e^{(2+3i)t} & 0 \\ 0 & e^{(2-3i)t} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}^{-1}$$

• $$\bar{y}(t) = \int_{-\infty}^{t} e^{(t-s)A}\bar{f}(s)\,ds = \int_{-\infty}^{t} e^{2(t-s)}\begin{pmatrix} \cos(3(t-s)) & -\sin(3(t-s)) \\ \sin(3(t-s)) & \cos(3(t-s)) \end{pmatrix}\begin{pmatrix} f_1(s) \\ f_2(s) \end{pmatrix} ds$$

(b) • $$B = \left[\begin{array}{cc|c} 1 & 3 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 7 \end{array}\right]$$ block diagonal matrix

$$e^{t\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}} = e^{\begin{bmatrix} t & 0 \\ 0 & t \end{bmatrix}} \cdot e^{\begin{bmatrix} 0 & 3t \\ 0 & 0 \end{bmatrix}} = \begin{pmatrix} e^t & 0 \\ 0 & e^t \end{pmatrix}\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 3t \\ 0 & 0 \end{pmatrix} + \frac{1}{2}\underbrace{\begin{pmatrix} 0 & 3t \\ 0 & 0 \end{pmatrix}^2}_{=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}} + \dots\right]$$

$$= \begin{pmatrix} e^t & 3te^t \\ 0 & e^t \end{pmatrix}$$

$$e^{tB} = \left[\begin{array}{cc|c} e^t & 3te^t & 0 \\ 0 & e^t & 0 \\ \hline 0 & 0 & e^{7t} \end{array}\right]$$

• $$\bar{y}(t) = \begin{bmatrix} e^t & 3te^t & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^{7t} \end{bmatrix}\begin{bmatrix} 5 \\ 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 5e^t + 12te^t \\ 4e^t \\ 3e^{7t} \end{bmatrix}$$