Zh1, Diff.Egy., 2025.03.27. NEPTUN: :

Név: Aláírás:

(2+2+4+2 pont)

1.a. Legyen $y'(t) = \sin(3t^4)$, $y(5) = -5$. Fejezd ki $y(8)$-at hatarozott integralas segitsegevel!

$$y(8) = -5 + \int_5^8 \sin(3s^4)\,ds$$

1.b. Legyen

$$\frac{d^2}{dt^2}\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} y_1(t) + \frac{d}{dt}(y_2(t)y_1(t)) \\ y_1(t) - y_1(t)\frac{d}{dt}y_1(t) \end{pmatrix}.$$

Irj fel egy elsorendu DE-t amelyik ekvivalens ezzel az egyenlettel!

$$\frac{d}{dt}\begin{bmatrix} y_1 \\ y_2 \\ v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ y_1 + v_2 y_1 + y_2 v_1 \\ y_1 - y_1 v_1 \end{bmatrix}$$

$$\frac{d}{dt}(y_2 y_1) = \dot{y}_2 y_1 + y_2 \dot{y}_1 = v_2 y_1 + y_2 v_1$$

1.c. Legyen

$$\frac{d}{dt}y = (1 - y^2)(t^2 + 1), \quad y(3) = 2.$$

Ird fel $y(3 + \Delta t)$ masodrendu Taylor polinomjat!

$$y(3) = 2$$

$$\dot{y}(3) = (1-2^2)(3^2+1) = -30$$

$$\ddot{y}(t) = \left(\partial_t + [(1-y^2)(t^2+1)]\partial_y\right)[(1-y^2)(t^2+1)] =$$

$$= [(1-y^2)(2t)] + [(1-y^2)(t^2+1)][-2y]$$

$$\ddot{y}(3) = 1182$$

Tehát $y(3+\Delta t) \approx 2 - 30\Delta t + \overset{591}{\frac{1182}{2!}}\Delta t^2$

1.d. Legyen $x_{n+1} = 4 - 3x_n$, $x_1 = 344$. Mennyi x_n ?

Fixpont: $x_{fix} = 4 - 3x_{fix} \longrightarrow x_{fix} = 1$

$$x_n = (-3)^{n-1}(344 - 1) + 1$$

Mivel x_1 volt megadva x_0 helyett: $\overbrace{1, 2, 3, \ldots n}^{n-1 \text{ lépes}}$

2. (4+1+3+2 pont) Legyen

$$\frac{d}{dt}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} y_1 - 3y_2 \\ -3y_1 + y_2 \end{pmatrix} = A\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \qquad \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ illetve } \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Keresd meg A sajatertekeit es sajatvektorait!

$$\lambda: \begin{vmatrix} 1-\lambda & -3 \\ -3 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 9 = 0 \longrightarrow \lambda_1 = 4, \quad \lambda_2 = -2$$

$$\lambda_1 = 4: \begin{bmatrix} 1-4 & -3 \\ -3 & 1-4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-3x - 3y = 0 \rightarrow y = -x$$

$$\vec{V}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\lambda_2 = -2: \begin{bmatrix} 1-(-2) & -3 \\ -3 & 1-(-2) \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$3x + 3y = 0 \longrightarrow y = x$$

$$\vec{V}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Ird fel a DE altalanos megoldasat!

$$\vec{y}_{ált}(t) = C_1 e^{4t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + C_2 e^{-2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Szamold ki a DE partikularis megoldasait midket kezdeti feltetel mellett!

$$\text{I}: \quad C_1\begin{bmatrix} 1 \\ -1 \end{bmatrix} + C_2\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{aligned} C_1 + C_2 &= 1 \\ -C_1 + C_2 &= 0 \end{aligned} \longrightarrow C_1 = \frac{1}{2}, \quad C_2 = -\frac{1}{2}$$

$$\vec{y}^{\,\text{I}}_{part}(t) = \frac{1}{2}e^{4t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \frac{1}{2}e^{-2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}e^{4t} + \frac{1}{2}e^{-2t} \\ -\frac{1}{2}e^{4t} + \frac{1}{2}e^{-2t} \end{bmatrix}$$

$$\text{II}: \quad C_1\begin{bmatrix} 1 \\ -1 \end{bmatrix} + C_2\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{aligned} C_1 + C_2 &= 0 \\ -C_1 + C_2 &= 1 \end{aligned} \longrightarrow C_1 = -\frac{1}{2}, \quad C_2 = \frac{1}{2}$$

$$\vec{y}^{\,\text{II}}_{part}(t) = -\frac{1}{2}e^{4t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \frac{1}{2}e^{-2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Mennyi e^{tA} ?

$$e^{tA} = \begin{array}{c} \begin{array}{cc} \vec{V}_1 & \vec{V}_2 \end{array} \\ \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \end{array} \begin{bmatrix} e^{4t} & 0 \\ 0 & e^{-2t} \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} =$$

vagy

$$= e^{tA} = \begin{bmatrix} | & | \\ \vec{y}^{\,\text{I}}_{part}(t) & \vec{y}^{\,\text{II}}_{part}(t) \\ | & | \end{bmatrix} = \frac{1}{2}\begin{bmatrix} e^{4t} + e^{-2t} & -e^{-4t} + e^{2t} \\ -e^{+4t} + e^{2t} & e^{4t} + e^{2t} \end{bmatrix}$$

((2+2)+(1+2+3) pont)

3a. Legyen

$$\frac{d}{dt}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 2y_1 + 3y_2 \\ 2y_2 \end{pmatrix} = A\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

Mennyi e^{tA} ?

$$\exp\left[t\begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix}\right] = \exp\left[\begin{bmatrix} 2t & 0 \\ 0 & 2t \end{bmatrix} + \begin{bmatrix} 0 & 3t \\ 0 & 0 \end{bmatrix}\right] =$$

$$= \exp\left(\begin{bmatrix} 2t & 0 \\ 0 & 2t \end{bmatrix}\right) \circ \exp\left(\begin{bmatrix} 0 & 3t \\ 0 & 0 \end{bmatrix}\right) =$$

$$= \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{2t} \end{bmatrix}\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 3t \\ 0 & 0 \end{bmatrix} + \frac{1}{2!}\begin{bmatrix} 0 & 3t \\ 0 & 0 \end{bmatrix}^2 + \dots\right) = e^{2t}\begin{bmatrix} 1 & 3t \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} e^{2t} & 3te^{2t} \\ 0 & e^{2t} \end{bmatrix}$$

Mi az elozo DE partikularis megoldasa az $(y_1(0), y_2(0))^T = (4, 5)$ kezdeti feltetel mellett?

$$\vec{y}_{part}(t) = \begin{bmatrix} e^{2t} & 3te^{2t} \\ 0 & e^{2t} \end{bmatrix}\begin{bmatrix} 4 \\ 5 \end{bmatrix} = e^{2t}\begin{bmatrix} 4+15t \\ 5 \end{bmatrix}$$

3.b. Legyen $f(x) = 1/\sqrt[3]{x}$, $x_0 = 27$. Ird fel $f(27 + \Delta x)$ linearis approximaciojat, illletve adj egy felso becslest ennek a hibajara, ha $\Delta x = 1/10$!

$$f(x) = x^{-1/3}, \quad f'(x) = -\frac{1}{3}x^{-4/3}, \quad f''(x) = \frac{4}{9}x^{-7/3}$$

$$f(27) = \frac{1}{3}, \quad f'(27) = -\frac{1}{243}, \quad f''(27) = \frac{4}{19683}$$

$$f(27+\Delta x) \approx \frac{1}{3} - \frac{1}{243}\Delta x \qquad (\text{lin. approx.})$$

$$\left| f(27+\Delta x) - \left[\frac{1}{3} - \frac{1}{243}\cdot 0.1\right] \right| \leq \frac{1}{2}\cdot 0.1^2 \cdot \frac{4}{19683} \qquad \max_{z\in[27,27.1]} \left|\frac{4}{9}z^{-7/3}\right| =$$

$$= \frac{4}{9}27^{-7/3} = \frac{4}{3^5}$$

3.d Legyen

$$\frac{d}{dt}y = (1 - y^2)(t^2 + 1), \quad y(3) = 2.$$

Mit josol Heun modszere $y(3.001)$ ertekere?

$$k_1 = (1-2^2)(3^2+1) = -30$$

$$y_1 = y(3) + k_1 \cdot 0.001 = 1.97$$

$$k_2 = (1-1.97^2)(3.001^2+1)$$

$$y(3.001) \approx y_2 = 2 + \frac{1}{2}\left[[-30] + [(1-1.97^2)(3.001^2+1)]\right]\cdot 0.001$$

4a. (5 pont)
$\frac{dy}{dt} = y(4-y^2)$. $= 4y - y^3 = y(4-y^2) = y(2+y)(2-y)$, $\frac{d(4y-y^3)}{dy} = 4-3y^2$
Keresd meg a DE fixpontjait!

$y_1 = 0, \quad y_2 = 2, \quad y_3 = -2$

Ird fel a fixpontok koruli linearizalt kozelito DE-t!

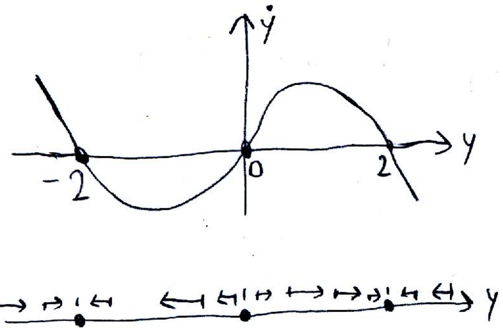
$\Delta y = y - 0, \quad \frac{d}{dt}\Delta y = \overset{4-3\cdot 0^2}{4}\cdot\Delta y$ | $\Delta y = y-2, \quad \frac{d}{dt}\Delta y = \overset{4-3\cdot 2^2}{-8}\,\Delta y$ | $\Delta y = y-(-2), \quad \frac{d}{dt}\Delta y = \overset{4-3\cdot(-2)^2}{-8}\,\Delta y$

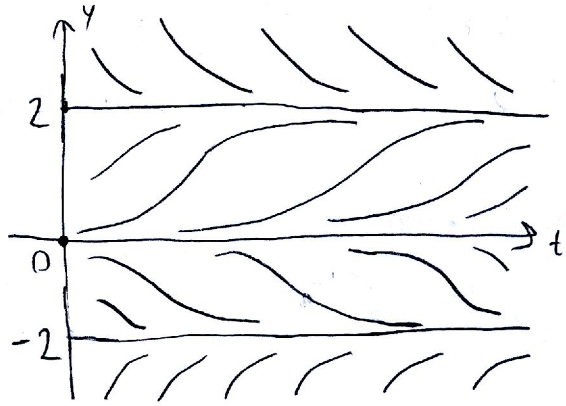
Ha $y(0) = -1.34$, mennyi

$\lim_{x\to\infty} y(x) = -2$ $\qquad\qquad$ $\lim_{x\to-\infty} y(x) = 0$

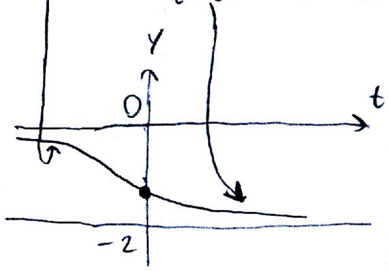
$\lim_{t\to-\infty} y(t) = 0$

Vazold a DE megoldasgorbeit!





$\lim_{t\to\infty} y(t) = -2$



4b. (5 pont)

$$\frac{d}{dt}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} (y_1-4)(5-y_2) \\ y_2 y_1. \end{pmatrix}$$

Keresd meg a DE fixpontjat!

$$\vec{z}_A = \begin{bmatrix} 4 \\ 0 \end{bmatrix}, \vec{z}_B = \begin{bmatrix} 0 \\ 5 \end{bmatrix}, \quad Jac = \begin{bmatrix} \partial_{y_1}[(y_1-4)(5-y_2)] & \partial_{y_2}[(y_1-4)(5-y_2)] \\ \partial_{y_1}[y_2 y_1] & \partial_{y_2}[y_2 y_1] \end{bmatrix} = \begin{bmatrix} 5-y_2 & 4-y_1 \\ y_2 & y_1 \end{bmatrix}$$

Ird fel a fixpont koruli linearizalt kozelito DE-t!

$$Jac(\vec{z}_A) = \begin{bmatrix} 5 & 0 \\ 0 & 4 \end{bmatrix} \qquad\qquad Jac(\vec{z}_B) = \begin{bmatrix} 0 & 4 \\ 5 & 0 \end{bmatrix}$$

$$\vec{\Delta y} = \vec{y} - \begin{bmatrix} 4 \\ 0 \end{bmatrix} \qquad\qquad \vec{\Delta y} = \vec{y} - \begin{bmatrix} 0 \\ 5 \end{bmatrix}$$

$$\frac{d}{dt}\vec{\Delta y} = \begin{bmatrix} 5 & 0 \\ 0 & 4 \end{bmatrix}\vec{\Delta y} \qquad\qquad \frac{d}{dt}\vec{\Delta y} = \begin{bmatrix} 0 & 4 \\ 5 & 0 \end{bmatrix}\vec{\Delta y}$$