Zh1, Diff.Egy., 2024.04.18. NEPTUN: :

Név: Aláírás:

(2+2+4+2 pont)

1.a. Legyen $y'(t) = t^7/(1-t^2)$, $y(5) = -5$. Fejezd ki $y(8)$-at hatarozott integralas segitsegevel!

$$y(8) = -5 + \int_5^8 \frac{t^7}{1-t^2}\,dt$$

1.b. Legyen

$$\frac{d^2}{dt^2}\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} y_2(t) + \frac{d}{dt}y_2(t) \\ y_1(t)y_2(t) - \frac{d}{dt}y_1(t) \end{pmatrix}.$$

Irj fel egy elsorendu DE-t amelyik ekvivalens ezzel az egyenlettel!

$$\frac{d}{dt}\begin{bmatrix} y_1 \\ y_2 \\ V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} V_1 \\ V_2 \\ y_2 + V_2 \\ y_1 y_2 - V_1 \end{bmatrix}$$

1.c. Legyen

$$\frac{d}{dt}y = (y^2-1)(1-t^2), \quad y(3) = 2.$$

Ird fel $y(3+\Delta t)$ masodrendu Taylor polinomjat!

$$y(3) = 2$$

$$\dot{y}(3) = (2^2-1)(1-3^2) = -24$$

$$\ddot{y} = \left[\partial_t + (y^2-1)(1-t^2)\partial_y\right]\left[(y^2-1)(1-t^2)\right] =$$

$$= \left[(y^2-1)(-2t)\right] + \left[(y^2-1)(1-t^2)^2 \cdot (2y)\right]$$

$$\ddot{y}(3) = \left[(2^2-1)(-2\cdot 3)\right] + \left[(2^2-1)(1-3^2)\cdot(2\cdot 2)\right] = 750$$

$$y(3+\Delta t) \approx 2 + (-24)\Delta t + \frac{1}{2}\cdot 750 \cdot \Delta t^2 = 2 - 24\Delta t - 375\Delta t^2$$

1.d. Legyen $x_{n+1} = 3x_n + 4$, $x_1 = 344$. Mennyi x_n ?

Fixpont: $x_{fix} = 3x_{fix} + 4 \longrightarrow x_{fix} = -2$

$$x_n = 3^{n-1}\left(344 - (-2)\right) + (-2)$$

$$= 3^{n-1} \cdot 346 - 2$$

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2. (4+1+3+2 pont) Legyen

$$\frac{d}{dt}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 5y_1 - 3y_2 \\ -3y_1 + 5y_2 \end{pmatrix} = A\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \qquad \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{illetve} \quad \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Keresd meg A sajatertekeit es sajatvektorait!

$$A = \begin{bmatrix} 5 & -3 \\ -3 & 5 \end{bmatrix} \qquad |A - \lambda E| = \begin{vmatrix} 5-\lambda & -3 \\ -3 & 5-\lambda \end{vmatrix} = (5-\lambda)^2 - 9 = \lambda^2 - 10\lambda + 16 = 0$$

tehát $\lambda_1 = 2, \quad \lambda_2 = 8$

$\lambda_1 = 2 \quad A - 2\cdot E = \begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix}$

$\begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

tehát $u = v$,

sajátaltér: $\left\{\begin{bmatrix} u \\ u \end{bmatrix}\right\}$, $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda_2 = 8 \quad A - 8E = \begin{bmatrix} -3 & -3 \\ -3 & -3 \end{bmatrix}$

$\begin{bmatrix} -3 & -3 \\ -3 & -3 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

tehát $-u = v$

sajátaltér: $\left\{\begin{bmatrix} u \\ -u \end{bmatrix}\right\}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Ird fel a DE altalanos megoldasat!

A sajátrendszere:

$\lambda_1 = 2, \quad \lambda_2 = 8$

$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

így az általános megoldás

$$\vec{y}_{ált}(t) = c_1 \cdot e^{2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{8t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Szamold ki a DE partikularis megoldasait midket kezdeti feltetel mellett!

I. $\vec{y}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \begin{matrix} c_1 + c_2 = 1 \\ c_1 - c_2 = 0 \end{matrix}$

tehát $c_1 = \frac{1}{2}, \; c_2 = \frac{1}{2}$

$$\vec{y}_I(t) = \frac{1}{2}e^{2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{1}{2}e^{8t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} e^{2t} + e^{8t} \\ e^{2t} - e^{8t} \end{bmatrix}$$

II. $\vec{y}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \begin{matrix} c_1 + c_2 = 0 \\ c_1 - c_2 = 1 \end{matrix}$

tehát $c_1 = \frac{1}{2}, \; c_2 = -\frac{1}{2}$

$$\vec{y}_{II}(t) = \frac{1}{2}e^{2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{1}{2}e^{8t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} e^{2t} - e^{8t} \\ e^{2t} + e^{8t} \end{bmatrix}$$

Mennyi e^{tA} ?

$$e^{tA} = \begin{bmatrix} | & | \\ \vec{y}_I(t) & \vec{y}_{II}(t) \\ | & | \end{bmatrix} = \frac{1}{2}\begin{bmatrix} e^{2t} + e^{8t} & e^{2t} - e^{8t} \\ e^{2t} - e^{8t} & e^{2t} + e^{8t} \end{bmatrix}$$

vagy

$$e^{tA} = S e^{tD} S^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} e^{2t} & 0 \\ 0 & e^{8t} \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1}$$

((2+2)+($3+3$ ~~1+2+3~~) pont)

3a. Legyen

$$\frac{d}{dt}\begin{pmatrix}y_1\\ y_2\end{pmatrix} = \begin{pmatrix}2y_1\\ 3y_1+2y_2\end{pmatrix} = A\begin{pmatrix}y_1\\ y_2\end{pmatrix}$$

$2 \cdot E$

mivel $\begin{bmatrix}2&0\\0&2\end{bmatrix}\begin{bmatrix}0&0\\3&0\end{bmatrix} = \begin{bmatrix}0&0\\3&0\end{bmatrix}\begin{bmatrix}2&0\\0&2\end{bmatrix}$

Mennyi e^{tA} ?

$$A = \begin{bmatrix}2&0\\3&2\end{bmatrix}$$

$$\exp(tA) = \exp\left(t\cdot\begin{bmatrix}2&0\\0&2\end{bmatrix} + t\begin{bmatrix}0&0\\3&0\end{bmatrix}\right) = \exp\left(t\cdot\begin{bmatrix}2&0\\0&2\end{bmatrix}\right)\cdot\exp\left(t\cdot\begin{bmatrix}0&0\\3&0\end{bmatrix}\right) =$$

$$= \begin{bmatrix}e^{2t}&0\\0&e^{2t}\end{bmatrix}\cdot\left(\begin{bmatrix}1&0\\0&1\end{bmatrix} + t\cdot\begin{bmatrix}0&0\\3&0\end{bmatrix} + \frac{t^2}{2}\underbrace{\begin{bmatrix}0&0\\3&0\end{bmatrix}^2}_{\begin{bmatrix}0&0\\3&0\end{bmatrix}^2 = \begin{bmatrix}0&0\\0&0\end{bmatrix}} + \dots\right)$$

$$= e^{2t}\begin{bmatrix}1&0\\3t&1\end{bmatrix},$$

Mi az elozo DE partikularis megoldasa az $(y_1(0), y_2(0))^T = (4,5)$ kezdeti feltetel mellett?

$$y(t) = e^{tA}\begin{bmatrix}4\\5\end{bmatrix} = e^{2t}\begin{bmatrix}1&0\\3t&1\end{bmatrix}\begin{bmatrix}4\\5\end{bmatrix} = e^{2t}\begin{bmatrix}4\\12t+5\end{bmatrix}$$

3.b. Legyen $f(x) = 1/\sqrt{x}$, $x_0 = 16$. Ird fel $f(16+\Delta x)$ linearis approximaciojat, illletve adj egy felso becslest ennek a hibajara, ha $\Delta x = 1/10$!

$$f'(x) = -\frac{1}{2}x^{-3/2}, \quad f''(x) = \left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)x^{-5/2} = \frac{3}{4}\frac{1}{\sqrt[2]{x^5}}$$

lin. approx: $f(16+\Delta x) \approx \frac{1}{\sqrt{16}} + \left(-\frac{1}{2}\right)\frac{1}{\sqrt{16^3}}\Delta x = \frac{1}{4} - \frac{1}{128}\Delta x$

$$\text{lin.approx.hiba} = \left|\frac{1}{\sqrt{16.1}} - \left[\frac{1}{4} - \frac{1}{128}\cdot 0.1\right]\right| \leq$$

$$\leq \frac{1}{2}\cdot 0.1^2 \cdot \max_{x\in[16,16.1]}\left|\frac{3}{4}\cdot\frac{1}{\sqrt[2]{x^5}}\right| = \frac{1}{2}\cdot 0.1^2\cdot\frac{3}{4}\cdot\frac{1}{\sqrt[2]{16^5}} = \frac{3}{2^{13}}\cdot 0.1^2$$

3.d Legyen

$$\frac{d}{dt}y = (y^2-1)(1-t^2), \quad y(3) = 2.$$

Mit josol Heun modszere $y(3.001)$ ertekere?

Euler jóslata: $k_1 = (2^2-1)(1-3^2) = -24$

$\to y_1 = 2 + (-24)\cdot 0.001 = 1.976$

Heun jóslat: $k_2 = (1.976^2-1)(1-3.001^2)$

$\to y_2 = 2 + \frac{1}{2}\left[(-24) + (1.976^2-1)(1-3.001^2)\right]\cdot 0.001$

?

4a. (5 pont)

$\frac{dy}{dt} = 8 - y^3. \; = f(y)$

Keresd meg a DE fixpontjait!

$$f(y_{Fix}) = 0 = 8 - y_{fix}^3 \longrightarrow y_{fix} = 2$$

Ird fel a fixpontok koruli linearizalt kozelito DE-t!

$$\Delta y = y - 2$$

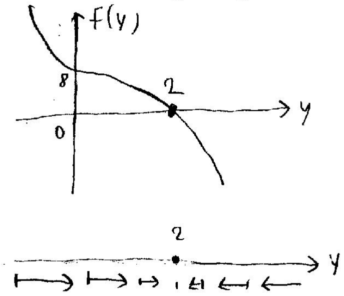
$$\frac{df}{dy} = f' = \frac{d(8-y^3)}{dy} = -3y^2, \quad f'(y_{Fix}) = -3 \cdot 2^2 = -12 \qquad \frac{d}{dt}\Delta y = -12 \cdot \Delta y$$

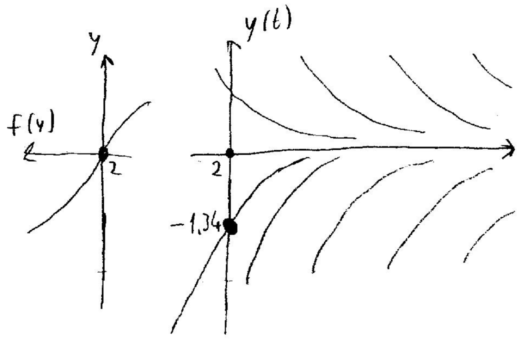
Ha $y(0) = -1.34$, mennyi

$\lim_{x\to\infty} y(x) = 2$ $\qquad\qquad$ $\lim_{x\to-\infty} y(x) = -\infty$

pontosabban $y(x)$ nem létezik túl nagy negatív x-ekre.

Vazold a DE megoldasgorbeit!





4b. (5 pont)

$$\frac{d}{dt}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} y_2 y_1 \\ (y_1 - 4)(5 - y_2). \end{pmatrix}$$

Keresd meg a DE fixpontjat!

$$y_2 y_1 = 0 \longrightarrow (a)\; y_1 = 0 \quad \text{vagy} \quad (b)\; y_2 = 0$$

$$(a)\; y_1 = 0 \longrightarrow (0-4)(5-y_2) = 0 \longrightarrow y_2 = 5$$

$$(b)\; y_2 = 0 \longrightarrow (y_1 - 4)(5-0) = 0 \longrightarrow y_1 = 4$$

tehát a két fixpont: $Z_A = \begin{bmatrix} 0 \\ 5 \end{bmatrix}$ $\quad Z_B = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$

Ird fel a fixpont koruli linearizalt kozelito DE-t!

$$Jac = \begin{bmatrix} \partial_{y_1}[y_1 y_2] & \partial_{y_2}[y_1 y_2] \\ \partial_{y_1}[(y_1-4)(5-y_2)] & \partial_{y_2}[(y_1-4)(5-y_2)] \end{bmatrix} = \begin{bmatrix} y_2 & y_1 \\ 5-y_2 & 4-y_1 \end{bmatrix}$$

$$Jac(Z_A) = \begin{bmatrix} 5 & 0 \\ 0 & 4 \end{bmatrix} \qquad Jac(Z_B) = \begin{bmatrix} 0 & 4 \\ 5 & 0 \end{bmatrix}.$$

$$\vec{\Delta y} = \begin{bmatrix} y_1 - 0 \\ y_2 - 5 \end{bmatrix} \qquad \vec{\Delta y} = \begin{bmatrix} y_1 - 4 \\ y_2 - 0 \end{bmatrix}$$

$$\frac{d}{dt}\vec{\Delta y} = \begin{bmatrix} 5 & 0 \\ 0 & 4 \end{bmatrix} \vec{\Delta y} \qquad \frac{d}{dt}\vec{\Delta y} = \begin{bmatrix} 0 & 4 \\ 5 & 0 \end{bmatrix} \cdot \vec{\Delta y}$$