LevZh, Diff.Egy., 2024.05.21. NEPTUN: :

Név: Aláírás:

(2+2+4+2 pont)

1.a. Legyen $y'(t) = t^7$, $y(5) = -5$. Fejezd ki $y(8)$-at hatarozott integralas segitsegevel!

$$y(8) = -5 + \int_5^8 s^7 \, ds$$

1.b. Legyen

$$\frac{d^2}{dt^2}\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} 3y_2(t) + 2\frac{d}{dt}y_2(t) \\ y_1^2(t)y_2(t)\frac{d}{dt}y_1(t) \end{pmatrix}.$$

Irj fel egy elsorendu DE-t amelyik ekvivalens ezzel az egyenlettel!

$$\frac{d}{dt}\begin{bmatrix} y_1 \\ y_2 \\ v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ 3y_2 + 2v_2 \\ y_1^2 y_2 v_1 \end{bmatrix}$$

1.c. Legyen

$$\frac{d}{dt}y = (y-1)(1-t), \quad y(1) = 3.$$

Ird fel $y($[redacted] 1 $+ \Delta t)$ masodrendu Taylor polinomjat!

$$y(1+\Delta t) \approx y(1) + \dot{y}(1)\Delta t + \frac{1}{2}\ddot{y}(1)\Delta t^2$$

$$y(1) = 3$$

$$\dot{y}(1) = (3-1)(1-1) = 0$$

$$\ddot{y}(t) = \left(\partial_t + (y-1)(1-t)\partial_y\right)\left[(y-1)(1-t)\right] = -(y-1) + (y-1)(1-t)(1-t)$$

$$\ddot{y}(1) = -(3-1) + (3-1)(1-1)^2 = -2$$

Vagyis

$$y(1+\Delta t) \approx 3 + 0\cdot\Delta t + \frac{-2}{2!}\Delta t^2 = 3 - \Delta t^2$$

1.d. Legyen $x_{n+1} - 3x_n + 4 = 0$, $x_1 = 33$. Mennyi x_n ?

$$x_{n+1} = 3x_n - 4$$

1 Fixpont: $x_{fix} = 3x_{fix} - 4 \longrightarrow x_{fix} = 2$

2 $x_n = 3^{n-1}(33-2) + 2 = 3^{n-1}\cdot 31 + 2$

2. (4+1+3+2 pont) Legyen

$$\frac{d}{dt}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} y_1 + y_2 \\ y_1 + y_2 \end{pmatrix} = A\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \qquad \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{illetve} \quad \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Keresd meg A sajatertekeit es sajatvektorait!

Sajátérték: $0 = \det(A-\lambda E) = \begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 1^2 = \lambda^2 - 2\lambda$

tehát $\lambda_1 = 0, \quad \lambda_2 = 2$

$\lambda_1 = 0$: $\begin{bmatrix} 1-0 & 1 \\ 1 & 1-0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$x+y=0$ sajátaltér: $\left\{\begin{bmatrix} x \\ -x \end{bmatrix}\right\}$

$\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\lambda_2 = 2$: $\begin{bmatrix} 1-2 & 1 \\ 1 & 1-2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$-x+y=0$, sajátaltér: $\left\{\begin{bmatrix} x \\ x \end{bmatrix}\right\}$

$\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Ird fel a DE altalanos megoldasat!

$$\vec{y}_{ált}(t) = C_1 \cdot e^{0t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + C_2 \cdot e^{2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} C_2 e^{2t} + C_1 \\ C_2 e^{2t} - C_1 \end{bmatrix}$$

Szamold ki a DE partikularis megoldasait midket kezdeti feltetel mellett!

$\vec{y}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$: $C_1\begin{bmatrix} 1 \\ -1 \end{bmatrix} + C_2\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\left.\begin{matrix} C_1 + C_2 = 1 \\ -C_1 + C_2 = 0 \end{matrix}\right\} C_1 = C_2 = \frac{1}{2}$

vagy $\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$\vec{y}^{I}(t) = \frac{1}{2}e^{0t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \frac{1}{2}e^{2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 + e^{2t} \\ -1 + e^{2t} \end{bmatrix}$$

$\vec{y}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$: $C_1\begin{bmatrix} 1 \\ -1 \end{bmatrix} + C_2\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\left.\begin{matrix} C_1 + C_2 = 0 \\ -C_1 + C_2 = 1 \end{matrix}\right\} C_2 = \frac{1}{2}, \ C_1 = -\frac{1}{2}$

vagy $\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$\vec{y}^{II}(t) = -\frac{1}{2}e^{0t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \frac{1}{2}e^{2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -1 + e^{2t} \\ 1 + e^{2t} \end{bmatrix}$$

Mennyi e^{tA} ?

$$e^{tA} = S e^{tD} S^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} e^{0 \cdot t} & 0 \\ 0 & e^{2 \cdot t} \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}^{-1}$$

vagy

$$e^{tA} = \begin{bmatrix} | & | \\ \vec{y}^{I}(t) & \vec{y}^{II}(t) \\ | & | \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 + e^{2t} & -1 + e^{2t} \\ -1 + e^{2t} & 1 + e^{2t} \end{bmatrix}$$

3

(([redacted]+2)+([redacted]+2+3) pont)

3a. Legyen

$$\frac{d}{dt}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 2y_1 + 3y_2 \\ 2y_2 \end{pmatrix} = A\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

Mennyi e^{tA} ?

$$\exp\left(t\begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}\right) = \exp\left(\begin{bmatrix} 2t & 0 \\ 0 & 2t \end{bmatrix} + \begin{bmatrix} 0 & 3t \\ 0 & 0 \end{bmatrix}\right) =$$

$$= \exp\begin{bmatrix} 2t & 0 \\ 0 & 2t \end{bmatrix} \cdot \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 3t \\ 0 & 0 \end{bmatrix} + \underbrace{\frac{1}{2!}\begin{bmatrix} 0 & 3t \\ 0 & 0 \end{bmatrix}^2 + \ldots}_{\text{zero mátrix}}\right)$$

$$= \exp\begin{bmatrix} 2t & 0 \\ 0 & 2t \end{bmatrix} \cdot \begin{bmatrix} 1 & 3t \\ 0 & 1 \end{bmatrix} = e^{2t} \cdot \begin{bmatrix} 1 & 3t \\ 0 & 1 \end{bmatrix}$$

Mi az elozo DE partikularis megoldasa az $(y_1(0), y_2(0))^T = (4, 5)$ kezdeti feltetel mellett?

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = e^{2t} \cdot \begin{bmatrix} 1 & 3t \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ 5 \end{bmatrix} = e^{2t}\begin{bmatrix} 4 + 15t \\ 5 \end{bmatrix}$$

3.b. Legyen $x_{n+1} - x_n = 77$, $x_2 = 33$. Mennyi x_n ?

Számtani sor: $x_n = 33 + (n-2) \cdot 77$

3.d Legyen

$$\frac{d}{dt}y = (y-1)(1-t), \quad y(3) = 1.$$

Mit josol Heun modszere $y(3.001)$ ertekere?

Euler: $k_1 = (1-1)(1-3) = 0$

becslés: $y(3.001) = 1 + 0 \cdot 0.001 = 1$

Heun: $k_2 = (1-1) \cdot (1-3.001) = 0$

becslés: $y(3.001) = 1 + \frac{1}{2}(0+0) \cdot 0.001 = 1$

4a. (5 pont)
$\frac{dy}{dt} = y^3 - y$.
Keresd meg a DE fixpontjait!

$$y^3-y=0 = y(y^2-1) = y(y-1)(y+1)$$

$$y_1=-1, \quad y_2=0, \quad y_3=1$$

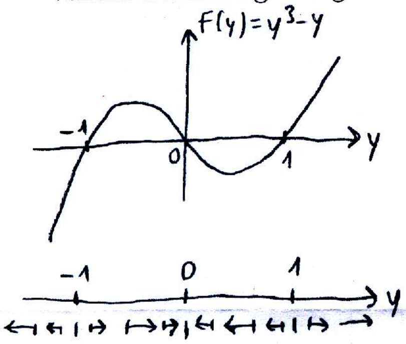
Ird fel a fixpontok koruli linearizalt kozelito DE-t!

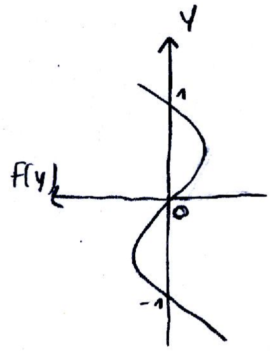
$\frac{d}{dy}(y^3-y)=3y^2-1$ | $\frac{d}{dt}(y-(-1)) = \frac{d}{dt}\Delta y = \underbrace{2}_{3\cdot(-1)^2-1}\Delta y$ | $\frac{d}{dt}(y-0) = \frac{d}{dt}\Delta y = \underbrace{-1}_{3\cdot 0^2-1}\cdot\Delta y$ | $\frac{d}{dt}(y-1) = \frac{d}{dt}\Delta y = \underbrace{2}_{3\cdot 1^2-1}\Delta y$

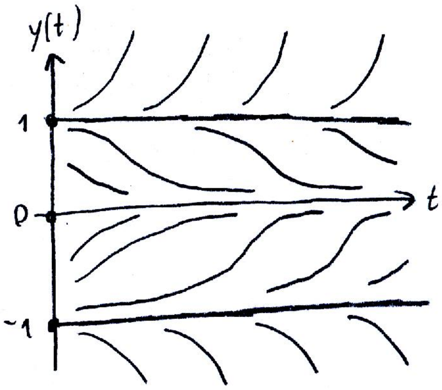
Ha $y(0) = 0.34$, akkor mennyi

$\lim_{x\to\infty} y(x) = 0$ $\qquad\qquad$ $\lim_{x\to-\infty} y(x) = 1$

Vazold a DE megoldasgorbeit!







4b. (5 pont)

$$\frac{d}{dt}\begin{pmatrix}y_1\\ y_2\end{pmatrix} = \begin{pmatrix} y_2-1 \\ (y_2-4)(5-y_2).\end{pmatrix}$$

Keresd meg a DE fixpontjat!

$y_2-1=0 \longrightarrow y_2=1$, de ekkor
$(y_2-4)(5-y_2)=0$ $\qquad (1-4)(5-1)\neq 0$,
így nincs fixpont

Ird fel a fixpont koruli linearizalt kozelito DE-t!

Mivel nincs fixpont, nincs mit linearizálni

$$\frac{d}{dt}\begin{bmatrix}y_1\\ y_2\end{bmatrix} = \begin{bmatrix}y_2-1\\ (y_1-4)(5-y_2)\end{bmatrix}$$

Fixpont: $y_2-1=0 \longrightarrow y_2=1 \longrightarrow$
$(y_1-4)(5-y_2)=0 \longrightarrow (y_1-4)(5-1)=0 \longrightarrow$
$\longrightarrow y_1=4$

tehát a fixpont $\vec{z}=\begin{bmatrix}4\\ 1\end{bmatrix}$

Jacobi mátrix:

$$Jac = \begin{bmatrix}\partial_{y_1}(y_2-1) & \partial_{y_2}(y_2-1)\\ \partial_{y_1}[(y_1-4)(5-y_2)] & \partial_{y_2}[(y_1-4)(5-y_2)]\end{bmatrix} =$$

$$= \begin{bmatrix}0 & 1\\ 5-y_2 & -(y_1-4)\end{bmatrix}, \quad Jac(\vec{z}) = \begin{bmatrix}0 & 1\\ 4 & 0\end{bmatrix}$$

Lin. DE:

$$\frac{d}{dt}\begin{bmatrix}y_1-4\\ y_2-1\end{bmatrix} = \frac{d}{dt}\begin{bmatrix}\Delta y_1\\ \Delta y_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ 4 & 0\end{bmatrix}\begin{bmatrix}\Delta y_1\\ \Delta y_2\end{bmatrix}$$

bármelyik jó megoldás