Problems in Modern Analysis

Metric Spaces

1. Let $\mathbb{R}^2$ be the set of all ordered pairs of real numbers and

   a) $d_1 : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be a function defined as

   $$d_1(x, y) = |x_1 - y_1| + |x_2 - y_2|, \qquad \forall x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2.$$

   Show that d_1 is a metric on $\mathbb{R}^2$.

   b) Does $d_2(x, y) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$ define a metric on the set of all ordered pairs of real numbers? Prove your answer.

2. Let $\mathcal{I}$ denot the collection of closed intervals of $\mathbb{R}$:

   $$\mathcal{I} = \{[a, b] \mid a, b \in \mathbb{R}, a \leq b\}.$$

   The function $d : \mathcal{I} \times \mathcal{I} \to \mathbb{R}$ given by follows:

   $$d(I, J) = \max\{|c - a|, |d - b|\}, \qquad \forall I = [a, b], J = [c, d] \in \mathcal{I}.$$

   Show that $(\mathcal{I}, d)$ is a metric space.

3. Prove that sum of two metrics is again a metric.

Convergence and Cauchy Sequence in Metric Spaces

4. If (x_n) and (y_n) are Cauchy sequences in a metric space (X, d), show that (a_n), where $a_n = d(x_n, y_n)$, converges. Give an illustrative example.

5. Is boundedness of a sequence in a metric space sufficient for the sequence to be Cauchy? Give an illustrative example.

6. Prove that if $x_n \to x$ and $y_n \to y$ in a metric space (X, d), then $d(x_n, y_n) \to d(x, y)$.

Normed Spaces, Banach Spaces, Hilbert spaces

7. Show that the discrete metric on a vector space $X \neq \{0\}$ cannot be obtained from a norm.

8. If in a normed space X, absolute convergence of any series always implies convergence of that series, show that X is complete.

9. Show that in a Banach space an absolutely convergent series is convergent.

10. Prove that l^2 is Hilbert space with inner-product

    $$\langle x, y \rangle = \sum_{n=1}^{\infty} \overline{x}_n y_n.$$

    Show further that l^1 and l^∞ are not inner-product spaces.