Problems in Modern Analysis

1. **Metric Spaces**

   a) Show that $d_1(x,y) = \sqrt{|x-y|}$ defines a metric on the set of all real numbers.

   b) Does $d_2(x,y) = |x-2y|$ define a metric on the set of all real numbers? And $d_3(x,y) = \dfrac{|x-y|}{1+|x-y|}$?

   c) Prove that l^p is a metric space!

2. **Convergence and Cauchy Sequence in Metric Spaces**

   a) If (x_n) and (y_n) are Cauchy sequences in a metric space (X,d), show that (a_n), where $a_n = d(x_n, y_n)$, converges. Give illustrative examples!

   b) Let (X,d) be a metric space and $(x_n) : \mathbb{N} \to X$. If (x_n) is Cauchy sequence and has a convergent subsequence, show that (x_n) is convergent with the limit x.

   c) If d_1 and d_2 are metrics on the same set X and there are positive numbers α and β such that for all $x, y \in X$,

   $$\alpha d_1(x,y) \leq d_2(x,y) \leq \beta d_1(x,y),$$

   show that the Cauchy sequences in (X,d_1) and (X,d_2) are the same.

3. **Vector Spaces, Banach Spaces, Hilbert spaces**

   a) If $S \subset V$ be a linear subspace of a vector space show that the relation on V

   $$v_1 \sim v_2 \quad \Leftrightarrow \quad v_1 - v_2 \in S$$

   is an equivalence relation and that the set of equivalence classes, denoted usually V/S, is a vector space in a natural way.

   b) Can every metric on a vector space be obtained from a norm? Reason for your answer!

   c) Show that an normed space L is a Banach space if and only if from the convergence of $\sum \|x_i\|$ we get the convergence of the serie $\sum x_i$ in L.

   d) Prove that the space l^p with $p \neq 2$ is not a Hilbert space.