Problems in Modern Analysis

1. **Metric Spaces**

   a) Show that $d_1(x, y) = |x - y|$ defines a metric on the set of all real numbers.

   b) Does $d_2(x, y) = (x - y)^2$ define a metric on the set of all real numbers? And $d_3(x, y) = |x^2 - y^2|$? Give reason for your answer!

   c) Let (X, d) be a metric space and

   $$\tilde{d} : X \times X \to \mathbb{R}, \qquad \tilde{d}(x, y) := \frac{d(x, y)}{1 + d(x, y)}.$$

   Show that $(X, \tilde{d})$ is a metric space.

2. **Convergence and Cauchy Sequence in Metric Spaces**

   a) Let (X, d) be a metric space and $(x_n), (y_n) : \mathbb{N} \to X$. If (x_n) and (y_n) are Cauchy sequences in a metric space (X, d), show that the real sequence (a_n), where $a_n = d(x_n, y_n)$, converges.

   b) Let (X, d) be a metric space and $(x_n) : \mathbb{N} \to X$. If (x_n) is a Cauchy sequence and has a convergent subsequence, show that (x_n) is convergent.

3. **Vector Spaces, Banach Spaces, Hilbert spaces**

   a) Show that if V is a vector space and $S \subset V$ is a subset which is closed under addition and scalar multiplication:

   $$(\text{that is:} \quad v_1, v_2 \in S, \lambda \in \mathbb{K} \quad \Longrightarrow \quad v_1 + v_2 \in S \quad \text{and} \quad \lambda v_1 \in S),$$

   then S is a vector space as well.

   b) Can every metric on a vector space be obtained from a norm? Give reason for your answer!

   c) Show that a normed space L is a Banach space if and only if from the convergence of $\sum \|x_i\|$ we get the convergence of the serie $\sum x_i$ in L.

   d) Prove that the space l^p with $p \neq 2$ is not a Hilbert space.