A 6. feladatlap gyakorlatra javasolt feladatainak végeredményei, részletesen kidolgozott megoldásai

Improprius integrálok

1. Számoljuk ki a következő improprius integrálokat!

(a) $\int\limits_1^\infty \frac{1}{x}\,dx;$

(b) $\int\limits_1^\infty \frac{1}{x^2}\,dx;$

(c) $\int\limits_{-\infty}^0 e^x\,dx;$

(d) $\int\limits_0^\infty xe^{-x^2}\,dx;$

(e) $\int\limits_2^\infty \frac{1}{x\ln x}\,dx;$

(f) $\int\limits_0^\infty 3^{-10x}\,dx;$

(g) $\int\limits_0^\infty \sin x\,dx;$

(h) $\int\limits_0^\infty \frac{\sin x}{e^x}\,dx;$

(i*) $\int\limits_1^\infty \frac{1}{x\sqrt{x^2+1}}\,dx;$

(j) $\int\limits_1^\infty e^{\sqrt{x}}\,dx;$

(k) $\int\limits_{-\infty}^\infty \frac{1}{1+x^2}\,dx;$

(l) $\int\limits_{-\infty}^\infty \frac{x}{1+x^2}\,dx.$

Megoldás:

(a) $\int\limits_1^\infty \frac{1}{x}\,dx = [\ln x]_1^\infty = \lim\limits_{R\to\infty} \ln R - \ln 1 = \lim\limits_{R\to\infty} \ln R = \infty;$

(b) $\int\limits_1^\infty \frac{1}{x^2}\,dx = \int\limits_1^\infty x^{-2}\,dx = \left[\frac{-1}{x}\right]_1^\infty = \lim\limits_{R\to\infty} \frac{-1}{R} - \frac{-1}{1} = 0+1 = 1;$

(c) $\int\limits_{-\infty}^0 e^x\,dx = [e^x]_{-\infty}^0 = e^0 - \lim\limits_{R\to-\infty} e^R = 1-0 = 1;$

(d) $\int\limits_0^\infty xe^{-x^2}\,dx = \left\{\begin{array}{c} x^2=t \\ x=\sqrt{t} \\ dx = \frac{dt1}{2\sqrt{t}}dt \end{array}\right\} = \int\limits_0^\infty \sqrt{t}e^{-t}\,\frac{1}{2\sqrt{t}}\,dt = \frac{1}{2}\int\limits_0^\infty e^{-t}\,dt = \frac{1}{2}\left[-e^{-t}\right]_0^\infty =$
$\frac{1}{2}\left(\lim\limits_{R\to\infty}\left(-e^{-R}\right) - \left(-e^0\right)\right) = \frac{1}{2}(0+1) = \frac{1}{2};$

(e) $\displaystyle\int_2^\infty \frac{1}{x\ln x}\,dx = \int_2^\infty \frac{\frac{1}{x}}{\ln x}\,dx = \int_2^\infty \frac{(\ln x)'}{\ln x}\,dx = [\ln\ln x]_2^\infty = \lim_{R\to\infty}(\ln\ln R) - \ln\ln 2 = \infty;$

(f) $\displaystyle\int_0^\infty 3^{-10x}\,dx = \left[\frac{3^{-10x}}{-10\ln 3}\right]_0^\infty = \lim_{R\to\infty}\frac{3^{-10R}}{-10\ln 3} + \frac{3^0}{10\ln 3} = 0 + \frac{1}{10\ln 3} = \frac{1}{10\ln 3};$

Megjegyzés:
$\displaystyle\lim_{R\to\infty}\frac{3^{-10R}}{-10\ln 3} = -\frac{1}{10\ln 3}\lim_{R\to\infty}3^{-10R} = -\frac{1}{10\ln 3}\cdot 0 = 0.$

(g) $\displaystyle\int_0^\infty \sin x\,dx = -\left[\cos x\right]_0^\infty = -\lim_{R\to\infty}\cos R + 1$ nem létezik. Az improprius integrál divergens.

(h) Parciális integrálással:

$$\int \frac{\sin x}{e^x}\,dx = \int e^{-x}\sin x\,dx = -e^{-x}\sin x - \int -e^{-x}\cos x\,dx =$$

$$= -e^{-x}\sin x + \int e^{-x}\cos x\,dx = -e^{-x}\sin x + \left(-e^{-x}\cos x - \int -e^{-x}\left(-\sin x\right)\,dx\right) =$$

$$= -e^{-x}\sin x - e^{-x}\cos x - \int e^{-x}\sin x\,dx,$$

így

$$\int \frac{\sin x}{e^x}\,dx = \int e^{-x}\sin x\,dx = -\frac{1}{2}e^{-x}\left(\sin x + \cos x\right).$$

Tehát:

$$\int_0^\infty \frac{\sin x}{e^x}\,dx = \left[-\frac{1}{2}e^{-x}\left(\sin x + \cos x\right)\right]_0^\infty = \lim_{R\to\infty}\left(-\frac{1}{2}e^{-R}\left(\sin R + \cos R\right)\right) + \frac{1}{2}1 = \frac{1}{2},$$

mert $\lim_{R\to\infty} e^{-R} = 0$, $\sin R + \cos R$ pedig korlátos, így $\lim_{R\to\infty} -\frac{1}{2}e^{-R}\left(\sin R + \cos R\right) = 0$.

(i) $\displaystyle\int_1^\infty \frac{1}{x\sqrt{x^2+1}}\,dx = \left\{\begin{array}{c} x = \operatorname{sh} t \\ dx = \operatorname{ch} t\,dt \end{array}\right\} = \int_{\operatorname{arsh} 1}^\infty \frac{1}{\operatorname{sh} t\sqrt{\operatorname{sh}^2 t + 1}}\operatorname{ch} t\,dt = \int_{\operatorname{arsh} 1}^\infty \frac{1}{\operatorname{sh} t \operatorname{ch} t}\operatorname{ch} t\,dt =$

$$= \int_{\operatorname{arsh} 1}^\infty \frac{1}{\operatorname{sh} t}\,dt = \int_{\ln\left(1+\sqrt{2}\right)}^\infty \frac{2}{e^t - \frac{1}{e^t}}\,dt = \left\{\begin{array}{c} e^t = u \\ t = \ln u \\ dt = \frac{1}{u}\,du \end{array}\right\} = \int_{1+\sqrt{2}}^\infty \frac{2}{u - \frac{1}{u}}\frac{1}{u}\,du =$$

$$= \int_{1+\sqrt{2}}^\infty \frac{2}{u^2-1}\,du = \int_{1+\sqrt{2}}^\infty \frac{1}{u-1} - \frac{1}{u+1}\,du = \left[\ln|u-1| - \ln|u+1|\right]_{1+\sqrt{2}}^\infty =$$

$$= \left[\ln\left|\frac{u-1}{u+1}\right|\right]_{1+\sqrt{2}}^\infty = \lim_{R\to\infty}\ln\left|\frac{R-1}{R+1}\right| - \ln\left|\frac{\sqrt{2}}{\sqrt{2}+2}\right| =$$

$$= \lim_{R\to\infty} \ln \left| \frac{1-\frac{1}{R}}{1+\frac{1}{R}} \right| - \ln\frac{\sqrt{2}}{\sqrt{2}+2} = \ln 1 - \ln\frac{\sqrt{2}}{\sqrt{2}+2} = -\ln\frac{\sqrt{2}}{\sqrt{2}+2};$$

(j) $\displaystyle\int_1^\infty e^{\sqrt{x}}\,dx = \left[2e^{\sqrt{x}}\left(\sqrt{x}-1\right)\right]_1^\infty = \lim_{R\to\infty} 2e^{\sqrt{R}}\left(\sqrt{R}-1\right) - 0 = \infty;$

(k) $\displaystyle\int_{-\infty}^\infty \frac{1}{1+x^2}\,dx = [\operatorname{arctg} x]_{-\infty}^\infty = \lim_{R\to\infty} \operatorname{arctg} R - \lim_{r\to-\infty} \operatorname{arctg} r = \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi;$

(l) $\displaystyle\int_{-\infty}^\infty \frac{x}{1+x^2}\,dx = \frac{1}{2}\int_{-\infty}^\infty \frac{2x}{1+x^2}\,dx = \frac{1}{2}\left[\ln\left(1+x^2\right)\right]_{-\infty}^\infty =$

$\displaystyle= \lim_{R\to\infty}\frac{1}{2}\ln\left(1+R^2\right) - \lim_{r\to-\infty}\frac{1}{2}\ln\left(1+r^2\right) = \infty-\infty$, tehát az improprius integrál nem létezik.

2. Számoljuk ki a következő improprius integrálokat!

(a) $\displaystyle\int_0^1 \frac{1}{x}\,dx;$

(b) $\displaystyle\int_0^1 \frac{1}{\sqrt{x}}\,dx;$

(c) $\displaystyle\int_0^1 \ln x\,dx;$

(d) $\displaystyle\int_0^{\frac{\pi}{2}} \operatorname{tg} x\,dx;$

(e) $\displaystyle\int_0^2 \frac{x}{\sqrt{4-x^2}}\,dx;$

(f) $\displaystyle\int_0^2 \frac{1}{\operatorname{sh}^2 x}\,dx;$

(g) $\displaystyle\int_0^3 \frac{1}{\sqrt{9-x}}\,dx;$

(h) $\displaystyle\int_0^{\frac{\pi}{4}} \frac{1}{\cos^2 2x}\,dx.$

Megoldás:

(a) $\displaystyle\int_0^1 \frac{1}{x}\,dx = \lim_{\varepsilon\to0}\int_{0+\varepsilon}^1 \frac{1}{x}\,dx = \lim_{\varepsilon\to0}\left[\ln x\right]_\varepsilon^1 = \ln 1 - \lim_{\varepsilon\to0}\ln\varepsilon = 0-(-\infty) = \infty.$

Az improprius integrál divergens.

(b) $\displaystyle\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{\varepsilon\to0}\int_\varepsilon^1 \frac{1}{\sqrt{x}}\,dx = \int_\varepsilon^1 x^{\frac{-1}{2}}\,dx = \lim_{\varepsilon\to0}\left[\frac{x^{\frac{1}{2}}}{\frac{1}{2}}\right]_0^1 = \lim_{\varepsilon\to0}\left[2\sqrt{x}\right]_0^1 = 2 - 2\lim_{\varepsilon\to0}\sqrt{\varepsilon} =$

$= 2-0 = 2.$ Az improprius integrál konvergens.

(c) $\int\limits_0^1 \ln x\,dx = \lim\limits_{\varepsilon\to 0}\int\limits_\varepsilon^1 \ln x\,dx = \lim\limits_{\varepsilon\to 0}\left[x\ln x - x\right]_\varepsilon^1 = (\ln 1 - 1) - \lim\limits_{\varepsilon\to 0}(\varepsilon\ln\varepsilon - \varepsilon) =$

$= -1 - \lim\limits_{\varepsilon\to 0}\varepsilon\ln\varepsilon + \lim\limits_{\varepsilon\to 0+0}\varepsilon = -1 - \lim\limits_{\varepsilon\to 0}\varepsilon\ln\varepsilon = -1$, mert

$$\lim_{\varepsilon\to 0}\varepsilon\ln\varepsilon = (0\cdot -\infty) = \lim_{\varepsilon\to 0}\frac{\ln\varepsilon}{\frac{1}{\varepsilon}} = \left(\frac{-\infty}{\infty}\right) = \lim_{\varepsilon\to 0+0}\frac{\frac{1}{\varepsilon}}{\frac{-1}{\varepsilon^2}} = \lim_{\varepsilon\to 0}(-\varepsilon) = 0.$$

(d) $\int\limits_0^{\frac{\pi}{2}} \operatorname{tg} x\,dx = \lim\limits_{\varepsilon\to 0}\int\limits_0^{\frac{\pi}{2}-\varepsilon}\operatorname{tg} x\,dx = -\lim\limits_{\varepsilon\to 0}\int\limits_0^{\frac{\pi}{2}-\varepsilon}\frac{-\sin x}{\cos x}\,dx = -\lim\limits_{\varepsilon\to 0}\left[\ln(\cos x)\right]_0^{\frac{\pi}{2}-\varepsilon} =$

$= -\left(\lim\limits_{\varepsilon\to 0}\ln\left(\cos\left(\frac{\pi}{2}-\varepsilon\right)\right) - \ln 1\right) = +\infty$. Az improprius integrál divergens.

(e) $\int\limits_0^2 \frac{x}{\sqrt{4-x^2}}\,dx = \lim\limits_{\varepsilon\to 0}\int\limits_0^{2-\varepsilon}\frac{x}{\sqrt{4-x^2}}\,dx = \left(-\frac{1}{2}\right)\lim\limits_{\varepsilon\to 0}\int\limits_0^{2-\varepsilon}(-2x)\left(4-x^2\right)^{\frac{-1}{2}}\,dx =$

$= \left(-\frac{1}{2}\right)\lim\limits_{\varepsilon\to 0}\left[\frac{(4-x^2)^{\frac{1}{2}}}{\frac{1}{2}}\right]_0^{2-\varepsilon} = -\lim\limits_{\varepsilon\to 0}\left[\sqrt{4-x^2}\right]_0^{2-\varepsilon} = -\left(\lim\limits_{\varepsilon\to 0}\sqrt{4-(2-\varepsilon)^2} - 2\right) = 2.$

Az improprius integrál konvergens.

(f) $\int\limits_0^2 \frac{1}{\operatorname{sh}^2 x}\,dx = \lim\limits_{\varepsilon\to 0}\int\limits_\varepsilon^2\frac{1}{\operatorname{sh}^2 x}\,dx = \lim\limits_{\varepsilon\to 0}\left[-\operatorname{cth} x\right]_\varepsilon^2 = -\operatorname{cth} 2 + \lim\limits_{\varepsilon\to 0}\operatorname{cth}\varepsilon = -\operatorname{cth} 2 + \infty = +\infty.$

Az improprius integrál divergens.

(g) Nem improprius integrál.

$$\int\limits_0^3 \frac{1}{\sqrt{9-x}}\,dx = \int\limits_0^3 (9-x)^{\frac{-1}{2}}\,dx = \left[\frac{2\,(9-x)^{\frac{1}{2}}}{-1}\right]_0^3 = \left[-2\sqrt{9-x}\right]_0^3 = -2\sqrt{6}+6.$$

(h) $\int\limits_0^{\frac{\pi}{4}} \frac{1}{\cos^2 2x}\,dx = \lim\limits_{\varepsilon\to 0}\int\limits_0^{\frac{\pi}{4}-\varepsilon}\frac{1}{\cos^2 2x}\,dx = \lim\limits_{\varepsilon\to 0}\left[\frac{\operatorname{tg} 2x}{2}\right]_0^{\frac{\pi}{4}-\varepsilon} = \lim\limits_{\varepsilon\to 0}\left(\operatorname{tg}\left(\frac{\pi}{2}-2\varepsilon\right)\right) - 0 = \infty.$

Az improprius integrál divergens.

3. Határozzuk meg az $f : \mathbb{R}^+ \to \mathbb{R}$, $f(x) = x\ln x$ függvény grafikonja és az x-tengely által közrezárt véges síkrész területét!

Megoldás: $\mu(T) = -\int\limits_0^1 x\ln x\,dx = -\lim\limits_{\varepsilon\to 0}\int\limits_\varepsilon^1 x\ln x\,dx.$

Parciális integrálással:

$$\int x\ln x\,dx = \left\{ \begin{array}{ll} u(x)=\ln x & u'=\frac{1}{x} \\ v(x)=\frac{x^2}{2} & v'(x)=x \end{array} \right\} = \frac{x^2}{2}\ln x - \frac{1}{2}\int x\,dx = \frac{x^2}{2}\ln x - \frac{x^2}{4} + c.$$

Így:

$$\mu(T) = -\lim_{\varepsilon\to 0}\int_{\varepsilon}^{1} x\ln x\,dx = -\lim_{\varepsilon\to 0}\left[\frac{x^2}{2}\ln x - \frac{x^2}{4}\right]_{\varepsilon}^{1} = -\left(-\frac{1}{4} - \lim_{\varepsilon\to 0}\left(\frac{\varepsilon^2}{2}\ln\varepsilon - \frac{\varepsilon^2}{4}\right)\right) = \frac{1}{4},$$

mert

$$\frac{1}{2}\lim_{\varepsilon\to 0}\varepsilon^2\ln\varepsilon = (0\cdot\infty) = \frac{1}{2}\lim_{\varepsilon\to 0}\frac{\ln\varepsilon}{\frac{1}{\varepsilon^2}} = \left(\frac{\infty}{\infty}\right) = \frac{1}{2}\lim_{\varepsilon\to 0}\frac{\frac{1}{\varepsilon}}{\frac{-2}{\varepsilon^3}} = -\frac{1}{4}\lim_{\varepsilon\to 0}\varepsilon^2 = 0.$$

4. Számítsuk ki a következő integrálokat!

(a) $\displaystyle\int_0^1 \frac{1}{(4-x)\sqrt{1-x}}\,dx;$

(b) $\displaystyle\int_0^{\sqrt{3}} \frac{1}{\sqrt{3-x^2}}\,dx;$

(c) $\displaystyle\int_1^{\infty} \frac{2x^2+4x+5}{(x^2+4x+5)x^2}\,dx;$

(d) $\displaystyle\int_1^{\infty} x^{-2}\operatorname{arctg} x\,dx;$

(e) $\displaystyle\int_2^{\infty} \frac{x+3}{(x-1)(x^2+1)}\,dx;$

(f) $\displaystyle\int_{-\infty}^{\infty} \frac{x}{(x^2+1)(x^2+2)}\,dx.$

Megoldás:

(a)
$$\int_0^1 \frac{1}{(4-x)\sqrt{1-x}}\,dx = \lim_{\varepsilon\to 0}\int_0^{1-\varepsilon} \frac{1}{(4-x)\sqrt{1-x}}\,dx = \left\{ \begin{array}{c} \sqrt{1-x}=t \\ 1-x=t^2 \\ dx=-2t\,dt \\ 4-x=3+t^2 \\ \begin{array}{|c|c|c|} \hline \text{x} & 0 & 1 \\ \hline \text{t} & 1 & 0 \\ \hline \end{array} \end{array} \right\} =$$

$$= -\lim_{\varepsilon\to 0}\int_{0+\varepsilon}^{1} \frac{(-2t)}{(3+t^2)t}\,dt = 2\lim_{\varepsilon\to 0}\int_{\varepsilon}^{1}\frac{1}{t^2+3}\,dt = \frac{2}{3}\lim_{\varepsilon\to 0}\int_{\varepsilon}^{1}\frac{1}{\left(\frac{t}{\sqrt{3}}\right)^2+1}\,dt =$$

$$= \left\{ \begin{array}{c} \frac{t}{\sqrt{3}}=s \\ t=\sqrt{3}s \\ dt=\sqrt{3}\,ds \\ \begin{array}{|c|c|c|} \hline t & \varepsilon & 1 \\ \hline s & \frac{\varepsilon}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \hline \end{array} \end{array} \right\} = \frac{2}{3}\lim_{\varepsilon\to 0}\int_{\frac{\varepsilon}{\sqrt{3}}}^{\frac{1}{\sqrt{3}}} \frac{\sqrt{3}}{s^2+1}\,ds = \frac{2\sqrt{3}}{3}\lim_{\varepsilon\to 0}\left[\operatorname{arctg} s\right]_{\frac{\varepsilon}{\sqrt{3}}}^{\frac{1}{\sqrt{3}}} =$$

$$= \frac{2\sqrt{3}}{3}\left(\operatorname{arctg}\frac{1}{\sqrt{3}} - \lim_{\varepsilon\to 0}\operatorname{arctg}\frac{\varepsilon}{\sqrt{3}}\right) = \frac{2\sqrt{3}}{3}\left(\frac{\pi}{6}-0\right) = \frac{\sqrt{3}\pi}{9}.$$

(b) $$\int_0^{\sqrt{3}} \frac{1}{\sqrt{3-x^2}}\, dx = \frac{\pi}{2}.$$

(c) $$\int_1^{\infty} \frac{2x^2+4x+5}{(x^2+4x+5)x^2}\, dx = \int_1^{\infty} \frac{x^2+(x^2+4x+5)}{(x^2+4x+5)x^2}\, dx =$$

$$= \int_1^{\infty} \frac{1}{x^2+4x+5}\, dx + \int_1^{\infty} \frac{1}{x^2}\, dx = \int_1^{\infty} \frac{1}{(x+2)^2+1}\, dx + \int_1^{\infty} \frac{1}{x^2}\, dx = \frac{\pi}{2} - \text{arctg}3 + 1,$$

mert

$$\int_1^{\infty} \frac{1}{(x+2)^2+1}\, dx = \left\{ \begin{array}{c} x+2=t \\ dx = dt \\ \begin{array}{|c|c|c|} \hline x & 1 & \infty \\ \hline t & 3 & \infty \\ \hline \end{array} \end{array} \right\} = \int_3^{\infty} \frac{1}{t^2+1}\, dt =$$

$$= \lim_{R\to\infty} [\text{arctg}\, x]_3^R = \lim_{R\to\infty} \text{arctg}\, R - \text{arctg}\, 3 = \frac{\pi}{2} - \text{arctg}\, 3,$$

és

$$\int_1^{\infty} \frac{1}{(x)^2}\, dx = \lim_{R\to\infty} \left[\frac{x^{-1}}{-1}\right]_1^R = -\left(\lim_{R\to\infty} \frac{1}{R} - 1\right) = 1.$$

(d) $$\int_1^{\infty} x^{-2}\, \text{arctg}\, x\, dx = \frac{\pi}{4} + \ln\sqrt{2}.$$

(e) $$\int_2^{\infty} \frac{x+3}{(x-1)(x^2+1)}\, dx = \ln 5 - \frac{\pi}{2} + \text{arctg}\, 2.$$

(f) $$\int_{-\infty}^{\infty} \frac{x}{(x^2+1)(x^2+2)}\, dx = 0.$$

Megjegyzés:

$$\int_0^{\infty} \frac{x}{(x^2+1)(x^2+2)}\, dx = \ln\sqrt{2},\ \int_{-\infty}^{0} \frac{x}{(x^2+1)(x^2+2)}\, dx = -\ln\sqrt{2}.$$

5. Legyen $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) = \begin{cases} 0, & \text{ha } x \leq 0; \\ \dfrac{1}{\sqrt{x}}, & \text{ha } 0 < x \leq 1; \\ e^{-x}, & \text{ha } x > 1. \end{cases}$$

Számítsuk ki az

$$\int_{-\infty}^{\infty} f(x)\,dx \qquad \text{és} \qquad \int_{-\infty}^{\infty} xf(x)\,dx$$

improprius integrálokat!

Megoldás:

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} 0\,dx + \int_{0}^{1} \frac{1}{\sqrt{x}}\,dx + \int_{1}^{\infty} e^{-x}\,dx = \lim_{\varepsilon\to 0} \int_{\varepsilon}^{1} \frac{1}{\sqrt{x}}\,dx + \lim_{R\to\infty} \int_{1}^{R} e^{-x}\,dx =$$

$$= \lim_{\varepsilon\to 0} \left[\frac{x^{\frac{1}{2}}}{\frac{1}{2}}\right]_{\varepsilon}^{1} + \lim_{R\to\infty} \left[-e^{-x}\right]_1^R = 2\left(1 - \lim_{\varepsilon\to 0}\sqrt{\varepsilon}\right) - \left(\lim_{R\to\infty} \frac{1}{e^R} - \frac{1}{e}\right) = 2 + \frac{1}{e}.$$

$$\int_{-\infty}^{\infty} xf(x)\,dx = \int_{-\infty}^{0} 0\,dx + \int_{0}^{1} \sqrt{x}\,dx + \int_{1}^{\infty} xe^{-x}\,dx = \frac{2}{3} + \frac{2}{e}.$$

6. Milyen $\alpha \in \mathbb{R}$ paraméter esetén teljesül az alábbi egyenlőség?

$$\int_{0}^{\infty} xe^{1-\alpha x} dx = 1$$

Megoldás: $\alpha = \sqrt{e}$.

7. Milyen $\beta \in \mathbb{R}$ paraméter esetén lesznek az alábbi integrálok konvergensek?

(a) $\displaystyle\int_{1}^{2} \frac{1}{x(\ln x)^\beta}\,dx;$ (b) $\displaystyle\int_{2}^{\infty} \frac{1}{x(\ln x)^\beta}\,dx.$

Megoldás:

(a) $\beta < 1$ esetén konvergens az improprius integrál.

(b) $\beta > 1$ esetén konvergens az improprius integrál.

8. Tekintsük az $f : \mathbb{R} \to \mathbb{R}$, $f(x) = e^{-x}$ függvény grafikonja és az x-tengely által határolt síkrészt!

(a) Mekkora az első síknegyedbe eső síkrész területe?

(b) Forgassuk meg az első síknegyedbe eső síkrészt az x-tengely körül! Mekkora a keletkező forgástest térfogata?

(c) Forgassuk meg az első síknegyedbe eső síkrészt az y-tengely körül is! Mekkora a keletkező forgástest térfogata?

Megoldás:

(a) $\mu(T) = \int\limits_0^\infty e^{-x}\,dx = \lim\limits_{R\to\infty} \int\limits_0^R e^{-x}\,dx = \lim\limits_{R\to\infty} \left[-e^{-x}\right]_0^R = -\left(\lim\limits_{R\to\infty} \frac{1}{e^R} - 1\right) = 1.$

(b) $\mu(V_x) = \pi \int\limits_0^\infty e^{-2x}\,dx = \pi \lim\limits_{R\to\infty} \left[\frac{e^{-2x}}{-2}\right]_0^R = \left(-\frac{\pi}{2}\right)\left(\lim\limits_{R\to\infty} \frac{1}{e^{2R}} - 1\right) = \frac{\pi}{2}.$

(c) $\mu(V_y) = \pi \int\limits_0^1 \ln^2 y\,dy = 2\pi.$

9. Vizsgáljuk meg, hogy az alábbi improprius integrálok konvergensek-e! Alkalmazzuk az összehasonlító kritériumot a feladatok megoldása során!

(a) $\int\limits_1^{+\infty} \frac{1}{x^3+1}\,dx;$

(b) $\int\limits_4^{+\infty} \frac{1}{\sqrt{x}-1}\,dx;$

(c) $\int\limits_2^{+\infty} \frac{1}{\sqrt{x-1}}\,dx;$

(d) $\int\limits_1^{+\infty} \frac{1}{\sqrt{x^6+1}}\,dx;$

(e) $\int\limits_1^{+\infty} \frac{\sqrt{x+1}}{x^2}\,dx;$

(f) $\int\limits_\pi^{+\infty} \frac{2+\cos x}{x}\,dx;$

(g) $\int\limits_\pi^{+\infty} \frac{1+\sin x}{x^2}\,dx;$

(h) $\int\limits_2^{+\infty} \frac{1}{\ln x}\,dx;$

(i) $\int\limits_1^{+\infty} \frac{e^x}{x}\,dx;$

(j) $\int\limits_0^{+\infty} \frac{1}{(1+x)\sqrt{x}}\,dx.$

Megoldás:

(a) $\displaystyle\int_1^{+\infty} \frac{1}{x^3+1}\,dx$ konvergens, mert

$$\int_1^{+\infty} \frac{1}{x^3+1}\,dx \leq \int_1^{+\infty} \frac{1}{x^3}\,dx,$$

és

$$\int_1^{+\infty} \frac{1}{x^3}\,dx = \lim_{R\to\infty} \int_1^{+\infty} x^{-3}\,dx = \lim_{R\to\infty} \left[\frac{x^{-2}}{-2}\right]_1^R = \left(-\frac{1}{2}\right)\left(\lim_{R\to\infty} \frac{1}{R^2} - 1\right) = \frac{1}{2}.$$

(b) $\displaystyle\int_4^{+\infty} \frac{1}{\sqrt{x}-1}\,dx$ divergens, mert

$$\int_4^{+\infty} \frac{1}{\sqrt{x}}\,dx \leq \int_4^{+\infty} \frac{1}{\sqrt{x}-1}\,dx$$

és

$$\int_4^{+\infty} \frac{1}{\sqrt{x}}\,dx = +\infty.$$

(c) $\displaystyle\int_2^{+\infty} \frac{1}{\sqrt{x-1}}\,dx$ divergens, mert

$$\int_2^{+\infty} \frac{1}{\sqrt{x}}\,dx \leq \int_2^{+\infty} \frac{1}{\sqrt{x-1}}\,dx$$

és

$$\int_2^{+\infty} \frac{1}{\sqrt{x}}\,dx = +\infty.$$

(d) $\displaystyle\int_1^{+\infty} \frac{1}{\sqrt{x^6+1}}\,dx$ konvergens.

(e) $\displaystyle\int_1^{+\infty} \frac{\sqrt{x+1}}{x^2}\,dx$ konvergens, mert

$$\int_1^{+\infty} \frac{\sqrt{x+1}}{x^2}\,dx \leq \int_1^{+\infty} \frac{\sqrt{x+x}}{x^2}\,dx$$

és

$$\sqrt{2}\int_1^{+\infty}\frac{\sqrt{x}}{x^2}\,dx=\sqrt{2}\int_1^{+\infty}\frac{1}{x\sqrt{x}}\,dx=\sqrt{2}\lim_{R\to\infty}\left[\frac{x^{-\frac{1}{2}}}{-\frac{1}{2}}\right]_1^R=-2\sqrt{2}\left(\lim_{R\to\infty}\frac{1}{\sqrt{R}}-1\right)=2\sqrt{2}.$$

(f) $\displaystyle\int_\pi^{+\infty}\frac{2+\cos x}{x}\,dx$ divergens, mert

$$-1\le\cos x\le 1\qquad\Rightarrow\qquad 1\le 2+\cos x\le 3,$$

miatt

$$\int_\pi^{+\infty}\frac{1}{x}\,dx\le\int_\pi^{+\infty}\frac{2+\cos x}{x}\,dx$$

és

$$\int_\pi^{+\infty}\frac{1}{x}\,dx=+\infty.$$

(g) $\displaystyle\int_\pi^{+\infty}\frac{1+\sin x}{x^2}\,dx$ konvergens, mert

$$\int_\pi^{+\infty}\frac{1+\sin x}{x^2}\,dx\le\int_\pi^{+\infty}\frac{2}{x^2}\,dx$$

és

$$\int_\pi^{+\infty}\frac{2}{x^2}\,dx=\frac{2}{\pi}.$$

(h) $\displaystyle\int_2^{+\infty}\frac{1}{\ln x}\,dx$ divergens.

(i) $\displaystyle\int_1^{+\infty}\frac{e^x}{x}\,dx$ divergens.

(j) $\displaystyle\int_0^{+\infty}\frac{1}{(1+x)\sqrt{x}}\,dx$ konvergens.

10. Az integrálkritérium segítségével állapítsuk meg, hogy az alábbi sorok közül melyek konvergensek!

(a) $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$;

(b) $\sum_{n=1}^{\infty} \frac{1}{n^2+2}$;

(c) $\sum_{n=1}^{\infty} \frac{n}{n^2+2}$;

(d) $\sum_{n=1}^{\infty} \frac{n}{e^n}$;

(e) $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$;

(f) $\sum_{n=1}^{\infty} \frac{\operatorname{arctg} n}{n^2+1}$.

Megoldás:

Tétel (Integrálkritérium): Legyen $\sum_{n=1}^{\infty} a_n$ pozitív tagú sor és $f : [1,\infty) \to \mathbb{R}$ monoton csökkenő, folytonos függvény, amelyre $f(n) = a_n$. A $\sum_{n=1}^{\infty} a_n$ numerikus sor akkor és csak akkor konvergens, ha az $\int_1^{\infty} f(x)\, dx$ improprius integrál konvergens.

(a) $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ divergens, mert $\int_2^{\infty} \frac{1}{x \ln x}\, dx$ divergens:

$$\int_2^{\infty} \frac{1}{x \ln x}\, dx = \int_2^{\infty} \frac{\frac{1}{x}}{\ln x}\, dx = \lim_{R\to\infty} [\ln(\ln x)]_2^R = \ln\ln R - \ln\ln 2 = +\infty.$$

(b) $\sum_{n=1}^{\infty} \frac{1}{n^2+2}$ konvergens, mert $\int_1^{\infty} \frac{1}{x^2+2}\, dx$ konvergens:

$$\int_1^{\infty} \frac{1}{x^2+2}\, dx \leq \int_1^{\infty} \frac{1}{x^2+1}\, dx$$

és

$$\int_1^{\infty} \frac{1}{x^2+1}\, dx = \lim_{R\to\infty} [\operatorname{arctg} x]_1^R = \lim_{R\to\infty} \operatorname{arctg} R - \operatorname{arctg} 1 = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}.$$

(c) $\sum_{n=1}^{\infty} \frac{n}{n^2+2}$ divergens, mert $\int_1^{\infty} \frac{x}{x^2+2}\, dx$ divergens.

(d) $\sum_{n=1}^{\infty} \frac{n}{e^n}$ konvergens, mert $\int_1^{\infty} \frac{x}{e^x}\,dx$ konvergens:

$$\int_1^{\infty} \frac{x}{e^x}\,dx = -\lim_{R\to\infty} \left[\frac{x+1}{e^x}\right]_1^R = -\left(\lim_{R\to\infty} \frac{R+1}{e^R} - \frac{2}{e}\right) = \frac{2}{e}.$$

(e) $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ konvergens, mert $\int_2^{\infty} \frac{1}{x(\ln x)^2}\,dx$ konvergens:

$$\int_2^{\infty} \frac{1}{x(\ln x)^2}\,dx = \int_2^{\infty} \frac{1}{x}(\ln x)^{-2}\,dx = \lim_{R\to\infty} \left[\frac{(\ln x)^{-1}}{-1}\right]_2^R = -\left(\lim_{R\to\infty} \frac{1}{\ln R} - \frac{1}{\ln 2}\right) = \frac{1}{\ln 2}.$$

(f) $\sum_{n=1}^{\infty} \frac{\operatorname{arctg} n}{n^2+1}$ konvergens, mert $\int_1^{\infty} \frac{\operatorname{arctg} x}{x^2+1}\,dx = \frac{3\pi^2}{32}$.