Irredundant Coverings, Tolerances, and Related Algebras

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Abstract This chapter deals with rough approximations defined by tolerance relations that represent similarities between the elements of a given universe of discourse. We consider especially tolerances induced by irredundant coverings of the universe U. This is natural in view of Pawlak's original theory of rough sets defined by equivalence relations: any equivalence E on U is induced by the partition U/E of U into equivalence classes, and U/E is a special irredundant covering of U in which the blocks are disjoint. Here equivalence classes are replaced by tolerance blocks which are maximal sets in which all elements are similar to each other. The blocks of a tolerance R on U always form a covering of U which induces R, but this covering is not necessarily irredundant and its blocks may intersect. In this chapter we consider the semantics of tolerances in rough sets, and in particular the algebraic structures formed by the rough approximations and rough sets defined by different types of tolerances.

1 Tolerances, information systems, and rough approximations

In this section, we show that tolerances can be used for representing information about objects. We also consider incomplete information systems and tolerances determined by them. We define rough approximations and study their properties. In particular, we concentrate on the structures of the ordered sets of lower and upper approximations and show that they form ortholattices.

1.1 Knowledge representation and tolerances

Knowledge about objects may be represented as binary relations . For instance, if we classify all human beings into disjoint sets based on their place of birth, then this classification determines a binary relation R by setting xRy whenever x and y are born in the same place. This relation is *reflexive*, that is, xRx for all human beings x. It is also *symmetric*: if xRy, then x and y were born in the same place and yRx

holds. The relation is *transitive*, because if $x R y$ and $y R z$, then x, y, and z are all born in the same place and also $x R z$ is true.

Originally Z. Pawlak defined rough approximations in terms of *equivalence relations* [16], which are reflexive, symmetric, and transitive relations. Pawlak considered equivalences as *indistinguishability relations*: two objects are equivalent if we cannot distinguish them by using the given information. Reflexivity is a natural property of indistinguishability, because each object is indistinguishable from itself. We may also assume that indistinguishability is symmetric: if x is indistinguishable from y, then y is indistinguishable from x. Transitivity is the most controversial property of indistinguishability: we may have a finite sequence of objects $x_1, x_2, \dots, x_n$ such that each two consecutive objects x_i and x_{i+1} are indistinguishable, but x_1 and x_n are very different from each other. The reason for this is that the difference between x_i and x_{i+1} is so small that it cannot be perceived, but if we go far enough in the chain of indistinguishable objects, we have a clear difference. For instance, if we compare photographs of a person's face, then the photographs taken on consecutive days should not differ much from each other. However, the pictures that are taken with separation of 10 years certainly look different.

In this chapter, we concentrate on cases in which the information about objects is given by a relation which is reflexive and symmetric, but not necessarily transitive. Such a relation is called a *tolerance relation*. The term tolerance relation was introduced in the context of visual perception theory by E. C. Zeeman [22], motivated by the fact that indistinguishability of "points" in the visual world is limited by the discreteness of retinal receptors. We view tolerances as *similarity* relations.

Tolerances correspond to simple graphs. A *simple graph* is an undirected graph that has no loops (edges connected at both ends to the same vertex) and no multiple edges. Any tolerance R on U determines a graph $\mathscr{G} = (U, R)$, where U is interpreted as the set of vertices and R as the set of edges. There is a line connecting x and y if and only if $x R y$. Because each point is R-related to itself, loops connecting a point to itself are not drawn.

Example 1. Assume that $U = \{1,2,3,4,5,6\}$. Let R be a tolerance depicted by the graph $\mathscr{G} = (U, R)$ of Figure 1.

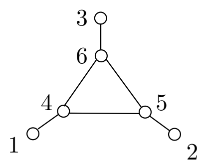


Fig. 1 A graph $\mathscr{G} = (U, R)$

Now, for example, $(1,4), (4,1) \in R$ and $(2,5), (5,2) \in R$, because there is an edge connecting the points 1 and 4, and the points 2 and 5. The elements 1 and 2 are not R-related, because there is no edge connecting them.

Information systems were introduced by Pawlak in [15]. An *information system* is a triple $\mathscr{S} = (U, A, \{V_a\}_{a\in A})$, where U is a nonempty set of *objects*, A is a nonempty set of *attributes*, and $\{V_a\}_{a\in A}$ is an A-indexed family of sets of *attribute values* . Each attribute $a \in A$ is a function $a\colon U \to V_a$. Usually the sets U, A, and V_a are assumed to be finite, which is often a natural assumption. However, in general we do not assume anything about the cardinalities of these sets.

Let $\mathscr{S} = (U, A, \{V_a\}_{a\in A})$ be an information system. An *indistinguishability relation* can be defined for any $B \subseteq A$ by setting

$$IND_B = \{(x,y) \in U \times U \mid a(x) = a(y) \text{ for all } a \in B\}.$$

This means that two objects are B-indistinguishable if and only if their values for all the attributes in B are equal. It is obvious that IND_B is an equivalence for any $B \subseteq A$.

In real-world situations, some attribute values for some objects may be undefined or unknown. Data may be missing for several reasons, but they do not concern us. In [12] these *null values* are marked by $*$. This kind of information systems are called *incomplete information systems*. For each $B \subseteq A$, the following relation is defined:

$$SIM_B = \{(x,y) \in U \times U \mid (\forall a \in B)\, a(x) = a(y) \text{ or } a(x) = * \text{ or } a(y) = *\}.$$

For any $B \subseteq A$, SIM_B is a tolerance on U such that $IND_B \subseteq SIM_B$. For each attribute $a \in A$, let us denote $SIM_{\{a\}}$ simply by SIM_a. It is clear that

$$SIM_B = \bigcap_{a\in B} SIM_a.$$

Example 2. An information system $\mathscr{S}$ in which the sets U and A are finite can be represented by a table. The rows of the table are labelled by the objects and the columns by the attributes of the system $\mathscr{S}$. In the intersection of the row labelled by an object x and the column labelled by an attribute a we find the value $a(x)$.

Let us consider an information system $\mathscr{S} = (U, A, \{V_a\}_{a\in A})$, where the object set $U = \{1,2,3,4,5\}$ consists of five persons called 1, 2, 3, 4 and 5, respectively. The attribute set A has the attributes Age, Eyes, and Height. Let the values of attributes be defined as in Table 1.

Table 1 A simple incomplete information system

	Age	Eyes	Height
1	Young	*	*
2	Middle-aged	Brown	Tall
3	Young	Blue	Short
4	*	Brown	Tall
5	Young	Brown	Tall

The tolerance SIM_A defined in the information system $\mathscr{S}$ is depicted in Figure 2.

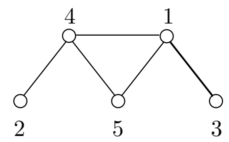


Fig. 2 The tolerance SIM_A on $\{1,2,3,4,5\}$

1.2 Rough approximations defined by tolerances

We begin by defining the rough set approximations based on an arbitrary tolerance R on a universe U. For any $x \in U$, we denote

$$R(x) = \{y \in U \mid x\,R\,y\}.$$

The set $R(x)$ is called the *R-neighbourhood* of x. It consists of the elements that are similar to x in view of the knowledge R. Consider any $X \subseteq U$ and let X^c denote the complement $U \setminus X = \{x \in U \mid x \notin X\}$ in U. For any $x \in U$, we have three possibilities:

(N1) $R(x) \subseteq X$: These elements x are certainly in X in view of the knowledge R, because all elements that are similar to x are in X.
(N2) $R(x) \cap X = \emptyset$, that is, $R(x) \subseteq X^c$: These are the elements x which certainly are not in X, because their R-neighbourhood is totally outside X.
(N3) $R(x) \cap X \neq \emptyset$ and $R(x) \cap X^c \neq \emptyset$: These elements x are such that their belonging to X cannot be decided by the means of the knowledge R; both in X and outside X there are elements which are similar to x.

Example 3. The three kinds of elements with respect to set $X \subseteq U$ are depicted in Figure 3. Element x belongs certainly to X, element y is certainly not in X, and z is such that its belonging to X cannot be decided in view of the knowledge R.

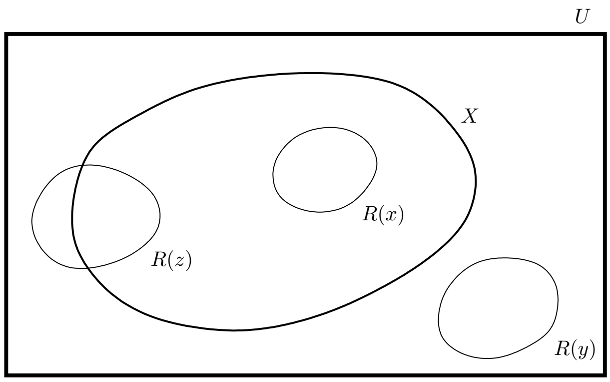


Fig. 3 Three kinds of elements with respect to set $X \subseteq U$

Next we define the rough approximation operators. Let R be a tolerance on a set U. The *upper approximation* of a set $X \subseteq U$ is

$$X^{\blacktriangle} = \{x \in U \mid R(x) \cap X \neq \emptyset\}$$

and the *lower approximation* of X is

$$X^{\blacktriangledown} = \{x \in U \mid R(x) \subseteq X\}.$$

Thus the lower approximation $X^{\blacktriangledown}$ consists of the elements of type (N1) and the upper approximation $X^{\blacktriangle}$ of contains the objects of type (N1) and (N3). The *boundary* $B(X) := X^{\blacktriangle} \setminus X^{\blacktriangledown}$ is the actual area of uncertainty; it consists of the elements of type (N3). Note that if $x \in B(X)$, then $|R(x)| \geq 2$. The set $X^{\blacktriangle c}$ contains the objects of type (N2).

Example 4. Let $U = \{a,b,c\}$ and let R be the tolerance on U defined in Figure 4.

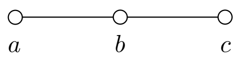


Fig. 4 The graph of a tolerance R is a 3-element chain

The lower and upper approximations of subsets of U are given in Table 2.

Table 2 All lower and upper approximations

X	$X^{\blacktriangledown}$	$X^{\blacktriangle}$
$\emptyset$	$\emptyset$	$\emptyset$
$\{a\}$	$\emptyset$	$\{a,b\}$
$\{b\}$	$\emptyset$	U
$\{c\}$	$\emptyset$	$\{b,c\}$
$\{a,b\}$	$\{a\}$	U
$\{a,c\}$	$\emptyset$	U
$\{b,c\}$	$\{c\}$	U
U	U	U

In the following are listed the basic properties of rough approximations defined by tolerances. Notice that $X^{\blacktriangle\blacktriangledown}$ denotes $(X^{\blacktriangle})^{\blacktriangledown}$ and a similar convention is used in this chapter for combinations of different mappings. The proofs of these claims are easy to verify, and they can also be found in [6, 7].

Proposition 5. *Let R is a tolerance on U and $X,Y \subseteq U$.*

(a) $\emptyset^{\blacktriangledown} = \emptyset^{\blacktriangle} = \emptyset$ *and* $U^{\blacktriangledown} = U^{\blacktriangle} = U$*;*
(b) $X^{\blacktriangledown} \subseteq X \subseteq X^{\blacktriangle}$*;*

(c) $X^{\blacktriangledown\blacktriangle} \subseteq X \subseteq X^{\blacktriangle\blacktriangledown}$;
(d) $X \subseteq Y$ *implies* $X^{\blacktriangledown} \subseteq Y^{\blacktriangledown}$ *and* $X^{\blacktriangle} \subseteq Y^{\blacktriangle}$;
(e) $(X \cup Y)^{\blacktriangle} = X^{\blacktriangle} \cup Y^{\blacktriangle}$ *and* $(X \cap Y)^{\blacktriangledown} = X^{\blacktriangledown} \cap Y^{\blacktriangledown}$;
(f) $X^{\blacktriangledown c} = X^{c\blacktriangle}$ *and* $X^{\blacktriangle c} = X^{c\blacktriangledown}$;
(g) $B(X) = B(X^c)$;
(h) $X^{\blacktriangle\blacktriangledown\blacktriangle} = X^{\blacktriangle}$ *and* $X^{\blacktriangledown\blacktriangle\blacktriangledown} = X^{\blacktriangledown}$.

Remark 6. Let us make some observations concerning Proposition 5. Item (a) says that an element belongs neither certainly nor possibly to the empty set and that every element belongs possibly and certainly to the whole universe.

Statement (b) says that if an element belongs certainly to X in view of the knowledge R, it must be in X. Further, if an element belongs to X, it belongs also possibly to X in the view of knowledge R. This property follows from the relation R being reflexive.

In (c), $X \subseteq X^{\blacktriangle\blacktriangledown}$ says that if $x \in X$, then $R(x) \subseteq X^{\blacktriangle}$. This means that if x belongs to X, then the elements R-related to x are possibly in X. Similarly, if $x \in X^{\blacktriangledown\blacktriangle}$, then x is R-related to some element in $X^{\blacktriangledown}$. But $X^{\blacktriangledown}$ consists of such elements that all element R-related to them are in X. Therefore, also x must be in X. Note that this condition holds since R is symmetric.

Assertion (d) says simply that if all elements of X are in Y, then all elements which are certainly (resp. possibly) in X are also certainly (resp. possibly) in Y.

By (e), the set of elements which are possibly in the union of X and Y equals the set of elements which are possibly in X or possibly in Y. Similarly, the elements which certainly are in the intersection of X and Y are those elements which are certainly in X and certainly in Y. Note that $X^{\blacktriangledown} \cup Y^{\blacktriangledown} \neq (X \cup Y)^{\blacktriangledown}$ may hold. For instance, consider the tolerance of Example 4. If $X = \{a\}$ and $Y = \{b\}$, then $X^{\blacktriangledown} = \emptyset$ and $Y^{\blacktriangledown} = \emptyset$, but $(X \cup Y)^{\blacktriangledown} = \{a,b\}^{\blacktriangledown} = \{a\}$. Additionally, $X^{\blacktriangle} \cap Y^{\blacktriangle} \neq (X \cap Y)^{\blacktriangle}$ for these particular sets X and Y.

Claim (f) says that the operators $^{\blacktriangle}$ and $^{\blacktriangledown}$ are mutually dual. Statement (g) says that if we cannot decide whether an element is in X, we cannot decide whether it is in the set-complement X^c of X either. This natural property follows from (f).

The equalities in (h) are consequences of (c) and (d). Indeed, since $X \subseteq X^{\blacktriangle\blacktriangledown}$, we have $X^{\blacktriangle} \subseteq X^{\blacktriangle\blacktriangledown\blacktriangle}$ by (d). For $X^{\blacktriangle}$, (c) gives $X^{\blacktriangle} \supseteq (X^{\blacktriangle})^{\blacktriangledown\blacktriangle}$. The other part behaves in a similar manner.

Let us denote by $\wp(U)^{\blacktriangledown}$ the set of all lower approximations and by $\wp(U)^{\blacktriangle}$ the set of all upper approximations, that is,

$$\wp(U)^{\blacktriangledown} = \{X^{\blacktriangledown} \mid X \subseteq U\} \qquad \text{and} \qquad \wp(U)^{\blacktriangle} = \{X^{\blacktriangle} \mid X \subseteq U\}.$$

Next we present some lattice-theoretical properties of $\wp(U)^{\blacktriangledown}$ and $\wp(U)^{\blacktriangle}$. For that we need to recall some definitions from the literature [2, 4].

An *order* (or a *partial order*) on a set P is a binary relation $\leq$ such that, for all $a,b,c \in P$, (i) $a \leq a$, (ii) $a \leq b$ and $b \leq a$ imply $a = b$, (iii) $a \leq b$ and $b \leq c$ imply $a \leq c$, that is, the relation $\leq$ is reflexive, antisymmetric, and transitive. A set

P equipped with an order relation $\leq$ is called an *ordered set* (or a *partially ordered set*). We usually denote an ordered set $(P, \leq)$ simply by P.

Let a and b be elements of an ordered set P. We say that a is *covered by* b (or that b *covers* a), and write $a \prec b$, if $a < b$ and $a \leq c < b$ implies $a = c$. Every finite ordered set can be drawn by using its covering relation $\prec$. The *Hasse diagram* of an ordered set P represents the elements of P with circles, and the circles representing two elements a and b are connected by a line if $a \prec b$ or $b \prec a$. If a is covered by b, the circle representing a is below the circle representing b.

It is clear that for any tolerance R on U, the lower approximations and upper approximations form ordered sets with respect to the set-inclusion order, that is, $(\wp(U)^{\blacktriangledown}, \subseteq)$ and $(\wp(U)^{\blacktriangle}, \subseteq)$ are ordered sets. We usually denote these ordered sets simply by $\wp(U)^{\blacktriangledown}$ and $\wp(U)^{\blacktriangle}$.

Example 7. Let R be the tolerance on Example 4. The Hasse diagrams of $\wp(U)^{\blacktriangledown}$ and $\wp(U)^{\blacktriangle}$ are presented in Figure 5.

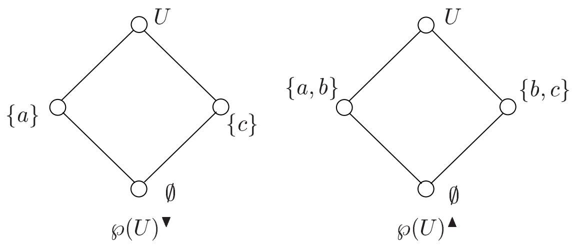


Fig. 5 The ordered sets $\wp(U)^{\blacktriangledown}$ and $\wp(U)^{\blacktriangle}$

If P and Q are ordered sets, then a mapping $\varphi \colon P \to Q$ is an *order-embedding*, if $a \leq b$ in P if and only if $\varphi(a) \leq \varphi(b)$ in Q. Note that an order-embedding is always an injection, because if $\varphi(a) = \varphi(b)$, then $\varphi(a) \leq \varphi(b)$ and $\varphi(a) \geq \varphi(b)$, which imply $a \leq b$ and $a \geq b$, that is, $a = b$. An *order-isomorphism* is a surjective order-embedding. When there exists an order-isomorphism from P to Q, we say that P and Q are *order-isomorphic* and write $P \cong Q$. Since any order-isomorphism $\varphi \colon P \to Q$ is a bijection, φ has an inverse mapping φ^{-1} which also is an order-isomorphism $\varphi^{-1} \colon Q \to P$.

Proposition 8. *If R is a tolerance on U, then $\varphi \colon X^{\blacktriangledown} \mapsto X^{\blacktriangledown\blacktriangle}$ defines an order-isomorphism between $\wp(U)^{\blacktriangledown}$ and $\wp(U)^{\blacktriangle}$.*

Proof. Let $X, Y \subseteq U$. If $X^{\blacktriangledown} \subseteq Y^{\blacktriangledown}$, then $\varphi(X^{\blacktriangledown}) = X^{\blacktriangledown\blacktriangle} \subseteq Y^{\blacktriangledown\blacktriangle} = \varphi(Y^{\blacktriangledown})$. Conversely, if $\varphi(X^{\blacktriangledown}) = X^{\blacktriangledown\blacktriangle} \subseteq Y^{\blacktriangledown\blacktriangle} = \varphi(Y^{\blacktriangledown})$, then $X^{\blacktriangledown} = X^{\blacktriangledown\blacktriangle\blacktriangledown} \subseteq Y^{\blacktriangledown\blacktriangle\blacktriangledown} = Y^{\blacktriangledown}$. Thus, φ is an order-embedding.

If $Z^{\blacktriangle} \in \wp(U)^{\blacktriangle}$, then $Z^{\blacktriangle\blacktriangledown} \in \wp(U)^{\blacktriangledown}$ and $\varphi(Z^{\blacktriangle\blacktriangledown}) = Z^{\blacktriangle\blacktriangledown\blacktriangle} = Z^{\blacktriangle}$. This means that φ is also surjective. □

Let us note that if R is an equivalence on U, then $X^{\blacktriangle\blacktriangledown} = X^{\blacktriangle}$ and $X^{\blacktriangledown\blacktriangle} = X^{\blacktriangledown}$ for all $X \subseteq U$. This implies that $\wp(U)^{\blacktriangledown} = \wp(U)^{\blacktriangle}$.

A family $\mathscr{L}$ of subsets of U is a *closure system* on U if $\mathscr{L}$ is closed under arbitrary intersections. In particular, $\mathscr{L}$ always contains $U = \bigcap \emptyset$. A map $\mathscr{C} \colon \wp(U) \to \wp(U)$ is a *closure operator* on U if for any $X, Y \subseteq U$:

(i) $X \subseteq \mathscr{C}(X)$ (extensive)
(ii) $\mathscr{C}(\mathscr{C}(X)) = \mathscr{C}(X)$ (idempotent)
(iii) $X \subseteq Y$ implies $\mathscr{C}(X) \subseteq \mathscr{C}(Y)$ (order-preserving)

A subset X of U is *closed* (with respect to $\mathscr{C}$) if $\mathscr{C}(X) = X$.

A closure system $\mathscr{L}$ on U defines a closure operator $\mathscr{C}_{\mathscr{L}}$ on U by the rule

$$\mathscr{C}_{\mathscr{L}}(B) = \bigcap \{L \in \mathscr{L} \mid B \subseteq L\}.$$

Conversely, if $\mathscr{C}$ is a closure operator on U, then the family

$$\mathscr{L}_{\mathscr{C}} = \{B \subseteq U \mid \mathscr{C}(B) = B\}$$

of $\mathscr{C}$-closed subsets of U is a closure system. The relationship between closure systems and closure operators is bijective.

Lemma 9. *Let R be a tolerance on U. The family of sets $\wp(U)^{\blacktriangledown}$ is a closure system on U.*

Proof. Consider $\{X^{\blacktriangledown} \mid X \in \mathscr{H}\} \subseteq \wp(U)^{\blacktriangledown}$ for some $\mathscr{H} \subseteq \wp(U)$. We show that

$$\bigcap_{X \in \mathscr{H}} X^{\blacktriangledown} = \big(\bigcap \mathscr{H}\big)^{\blacktriangledown},$$

which means that $\bigcap \{X^{\blacktriangledown} \mid X \in \mathscr{H}\}$ belongs to $\wp(U)^{\blacktriangledown}$. For every $X \in \mathscr{H}$, $\big(\bigcap \mathscr{H}\big)^{\blacktriangledown} \subseteq X^{\blacktriangledown}$ because $\bigcap \mathscr{H} \subseteq X$. Therefore,

$$\big(\bigcap \mathscr{H}\big)^{\blacktriangledown} \subseteq \bigcap_{X \in \mathscr{H}} X^{\blacktriangledown}.$$

On the other hand, if $x \in \bigcap \{X^{\blacktriangledown} \mid X \in \mathscr{H}\}$, then $x \in X^{\blacktriangledown}$ and $R(x) \subseteq X$ for all $X \in \mathscr{H}$. Thus, $R(x) \subseteq \bigcap \mathscr{H}$ and $x \in \big(\bigcap \mathscr{H}\big)^{\blacktriangledown}$. This completes the proof. □

Let R be a tolerance on U. We define a map $\Diamond$ ("diamond") on $\wp(U)$ by setting

$$\Diamond X := X^{\blacktriangle\blacktriangledown} \tag{1}$$

for all $X \subseteq U$.

Lemma 10. *The mapping $\Diamond$ is the closure operator corresponding to the closure system $\wp(U)^{\blacktriangledown}$.*

Proof. This follows from Proposition 5. Indeed, for all $X \subseteq U$, $X \subseteq X^{\blacktriangle\blacktriangledown} = \Diamond X$ and $\Diamond\Diamond X = X^{\blacktriangle\blacktriangledown\blacktriangle\blacktriangledown} = X^{\blacktriangle\blacktriangledown} = \Diamond X$. If $X \subseteq Y$, then $X^{\blacktriangle} \subseteq Y^{\blacktriangle}$ and $\Diamond X = X^{\blacktriangle\blacktriangledown} \subseteq Y^{\blacktriangle\blacktriangledown} = \Diamond Y$. Thus, $\Diamond$ is a closure operator on U. It is clear that $\Diamond X \in \wp(U)^{\blacktriangledown}$, and that if $X^{\blacktriangledown} \in \wp(U)^{\blacktriangledown}$, then $\Diamond X^{\blacktriangledown} = X^{\blacktriangledown\blacktriangle\blacktriangledown} = X^{\blacktriangledown}$. $\square$

Note that since the closure system corresponding $\Diamond$ is $\wp(U)^{\blacktriangledown}$, we have

$$\Diamond X = \bigcap\{A \in \wp(U)^{\blacktriangledown} \mid X \subseteq A\},$$

for any $X \subseteq U$. Therefore, $\Diamond X$ is the least set in $\wp(U)^{\blacktriangledown}$ which contains X. Note also that $X^{\blacktriangle} = (\Diamond X)^{\blacktriangle}$ for all $X \subseteq U$ and $\wp(U)^{\blacktriangledown} = \{\Diamond X \mid X \subseteq U\}$.

A map $\mathscr{I} \colon \wp(U) \to \wp(U)$ is called an *interior operator* on U if for any $X, Y \subseteq U$:

(i) $\mathscr{I}(X) \subseteq X$ (contractive)
(ii) $\mathscr{I}(\mathscr{I}(X)) = \mathscr{I}(X)$ (idempotent)
(iii) $X \subseteq Y$ implies $\mathscr{I}(X) \subseteq \mathscr{I}(Y)$ (order-preserving)

It is known (see e.g. [13]) that each closure operator $\mathscr{C} \colon \wp(U) \to \wp(U)$ defines an interior operator $\mathscr{I}_{\mathscr{C}} \colon \wp(U) \to \wp(U)$ by the rule $\mathscr{I}_{\mathscr{C}}(X) = \mathscr{C}(X^c)^c$. A family $\mathscr{N}$ of subsets of A is said to be an *interior system* if $\mathscr{N}$ is closed under arbitrary unions. Note that interior systems always contain $\emptyset = \bigcup\emptyset$. Also the relationship between interior systems and interior operators is bijective.

Because for any tolerance R on U the mapping $\Diamond$ is a closure operator on U, it defines an interior operator $\Box$ ("box") by the rule

$$\Box X := (\Diamond(X^c))^c = X^{c\blacktriangle\blacktriangledown c} = X^{\blacktriangledown\blacktriangle}. \tag{2}$$

This gives that $\Box(X^c) = (\Diamond X)^c$ and $\Diamond(X^c) = (\Box X)^c$, that is, the operators $\Box$ and $\Diamond$ are mutually dual.

The interior system corresponding to the interior operator $\Box$ is $\wp(U)^{\blacktriangle}$, and

$$\Box X = \bigcup\{A \in \wp(U)^{\blacktriangle} \mid A \subseteq X\}.$$

Now $\Box X$ is the greatest set in $\wp(U)^{\blacktriangle}$ contained in X. Note also that $X^{\blacktriangledown} = (\Box X)^{\blacktriangledown}$ for all $X \subseteq U$ and $\wp(U)^{\blacktriangle} = \{\Box X \mid X \subseteq U\}$. In addition,

$$(\Diamond X)^{\blacktriangle} = \Box(X^{\blacktriangle}) \qquad \text{and} \qquad (\Box X)^{\blacktriangledown} = \Diamond(X^{\blacktriangledown}).$$

An ordered set $(P, \leq)$ is a *complete lattice* if each subset $S \subseteq P$ has a greatest lower bound $\bigwedge S$ and a least upper bound $\bigvee S$.

It is known that if $\mathscr{L}$ is a closure system on U, then the ordered set $(\mathscr{L}, \subseteq)$ is a complete lattice in which for a subset $\mathscr{H}$ of $\mathscr{L}$,

$$\bigwedge \mathscr{H} = \bigcap \mathscr{H} \qquad \text{and} \qquad \bigvee \mathscr{H} = \mathscr{C}_{\mathscr{L}}\big(\bigcup \mathscr{H}\big).$$

Because for a tolerance R on U, $\wp(U)^{\blacktriangledown}$ is a closure system, the ordered set $(\wp(U)^{\blacktriangledown}, \subseteq)$ is a complete lattice in which

$$\bigwedge \mathscr{H} = \bigcap \mathscr{H} \qquad \text{and} \qquad \bigvee \mathscr{H} = \Diamond\big(\bigcup \mathscr{H}\big) \tag{3}$$

for all $\mathscr{H} \subseteq \wp(U)^{\blacktriangledown}$. Similarly, the ordered set $(\wp(U)^{\blacktriangle}, \subseteq)$ is a complete lattice in which

$$\bigwedge \mathscr{H} = \Box\big(\bigcap \mathscr{H}\big) \qquad \text{and} \qquad \bigvee \mathscr{H} = \bigcup \mathscr{H} \tag{4}$$

for all $\mathscr{H} \subseteq \wp(U)^{\blacktriangle}$.

We say that an ordered set $(P, \leq)$ is *bounded* if it has a least element, denoted usually by 0, and a greatest element, denoted by 1. It is obvious and well-known that any complete lattice is bounded. The complete lattices $\wp(U)^{\blacktriangledown}$ and $\wp(U)^{\blacktriangle}$ are bounded in such a way that $\emptyset$ is their smallest element and U is the greatest element.

A mapping $x \mapsto x^{\perp}$ on a bounded lattice L is called an *orthocomplementation*, and $x^{\perp}$ an *orthocomplement* of x, if the following conditions hold for all $x, y \in L$:

(O1) $x \leq y$ implies $y^{\perp} \leq x^{\perp}$ (order-reversing)
(O2) $x^{\perp\perp} = x$ (involution)
(O3) $x \vee x^{\perp} = 1$ and $x \wedge x^{\perp} = 0$ (complement)

An *ortholattice* is a bounded lattice equipped with an orthocomplementation. Note that orthocomplementations are not always unique.

Remark 11. For an ordered set $(P, \leq)$, a mapping $\varphi \colon P \to P$ satisfying (O1) and (O2) is called a *polarity*. . Such a polarity φ is an order-isomorphism from $(P, \leq)$ to its dual $(P, \geq)$. This means that P is *anti-isomorphic* to itself. Hence, the Hasse diagram of P looks the same when it is turned upside-down.

Proposition 12. *Let R be a tolerance on U.*

(a) *The map $X \mapsto X^{c\blacktriangledown}$ is an orthocomplementation in $\wp(U)^{\blacktriangledown}$.*
(b) *The map $X \mapsto X^{c\blacktriangle}$ is an orthocomplementation in $\wp(U)^{\blacktriangle}$.*

Proof. We show that $X^{\perp} = X^{c\blacktriangledown}$ is an orthocomplement of $X \in \wp(U)^{\blacktriangledown}$ which proves (a). Claim (b) may be proved similarly. Suppose $X, Y \in \wp(U)^{\blacktriangledown}$. It is clear that $X^{\perp}$ belongs to $\wp(U)^{\blacktriangledown}$, so the mapping is well defined.

(O1) If $X \subseteq Y$, then $Y^c \subseteq X^c$ and $Y^{\perp} = Y^{c\blacktriangledown} \subseteq X^{c\blacktriangledown} = X^{\perp}$.

(O2) $X^{\perp\perp} = X^{c\blacktriangledown c\blacktriangledown} = X^{cc\blacktriangle\blacktriangledown} = X^{\blacktriangle\blacktriangledown}$. Because $X \in \wp(U)^{\blacktriangledown}$, we have $X = A^{\blacktriangledown}$ for some $A \subseteq U$. Thus, $X^{\perp\perp} = X^{\blacktriangle\blacktriangledown} = A^{\blacktriangledown\blacktriangle\blacktriangledown} = A^{\blacktriangledown} = X$.

(O3) By straightforward computation,

$$X \wedge X^{\perp} = X \cap X^{c\blacktriangledown} \subseteq X \cap X^c = \emptyset$$

and

$$X \vee X^{\perp} = (X \cup X^{c\blacktriangledown})^{\blacktriangle\blacktriangledown} = (X^{\blacktriangle} \cup X^{\blacktriangle c\blacktriangle})^{\blacktriangledown} \supseteq (X^{\blacktriangle} \cup X^{\blacktriangle c})^{\blacktriangledown} = U^{\blacktriangledown} = U.$$

□

2 Tolerances induced by an irredundant covering

In this section, we first consider blocks of a tolerance. A block of a tolerance can be seen as a counterpart of an equivalence class of an equivalence relation. First we recall the notion of a covering [18]. Each covering induces a tolerance, and in this section tolerances induced by irredundant coverings assume a special role. We characterize the tolerances induced by an irredundant covering and we also give an algorithm which checks whether a tolerance is induced by an irredundant covering. Moreover, in case R is such a tolerance, the algorithm also returns that unique irredundant covering. Subsection 2.3 is devoted to rough approximation operators defined by irredundant coverings. There are many ways to define approximations operators in terms of a covering. We show that if $\mathscr{H}$ is an irredundant covering, many of these operators can be expressed by using rough approximation operators defined by the tolerance induced by $\mathscr{H}$. The last subsection deals with tolerances and formal concept analysis. Many results presented in this section appear already in our previous works [8, 9, 10].

2.1 Blocks of tolerances and set coverings

We begin by considering blocks of tolerances. Blocks of tolerances can be seen as generalizations of equivalence classes, because each block of a tolerance R on U is a maximal set within which all elements are R-related. On the other hand, the R-neighbourhood $R(x)$ of $x \in U$ is not necessarily a block of R. In fact, we shall see that those $R(x)$-neighbourhoods that are blocks play a special role.

Let R be a tolerance on U. A nonempty subset X of U is an R-*preblock* if $X \times X \subseteq R$. Note that if B is an R-preblock, then $B \subseteq R(x)$ for all $x \in B$. An R-*block* is an R-preblock that is maximal with respect to the inclusion relation. Each tolerance R is completely determined by its blocks, that is, $a\,R\,b$ if and only if there exists a block B such that $a, b \in B$. Blocks are "clusters" of similar objects, because each object in a block B is R-related with all other elements in B, and no element outside B is R-related to all elements of B.

Example 13. Let R be a tolerance on $U = \{1,2,3,4\}$ depicted by the graph $\mathscr{G} = (U,R)$ in Figure 6. A nonempty set $X \subseteq U$ is a preblock if and only if all points in X are connected by an edge of $\mathscr{G}$. Blocks are the maximal preblocks, and the blocks of R are $\{1,2\}$, $\{2,3\}, \{3,4\}, \{1,4\}$.

Next we present two lemmas related to R-neighbourhoods of objects.

Lemma 14. *Let R be a tolerance on U and $x \in U$. The following are equivalent:*

(a) $R(x)$ *is a preblock;*
(b) $R(x)$ *is a block.*

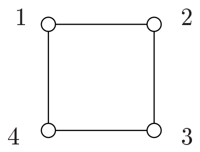


Fig. 6 Tolerance R on $\{1,2,3,4\}$

Proof. It is clear that (b) implies (a). Let $R(x)$ be a preblock. Suppose that $R(x) \subseteq X$ for some preblock X. If $y \in X$, then $x R y$ because $x \in R(x) \subseteq X$ and X is a preblock. Therefore, $y \in R(x)$. Hence, $R(x) = X$ and $R(x)$ is a block. □

By the above lemma, we can also write that for all $x \in U$,

$$R(x) \text{ is a block} \iff R(x) \subseteq R(y) \text{ for all } y \in R(x). \tag{5}$$

As we have noted, if B is a block, then $B \subseteq R(y)$ for all $y \in B$, so "$\Longrightarrow$" follows from this. On the other hand, if $R(x) \subseteq R(y)$ for all $y \in R(x)$, then $a, b \in R(x)$ implies that $a \in R(b)$ and $b \in R(a)$. Therefore, all elements in $R(x)$ are related and by Lemma 14 $R(x)$ is a block.

Lemma 15. *A tolerance R on U is an equivalence if and only if $R(x)$ is a block for each $x \in U$.*

Proof. If R is an equivalence, then each equivalence class $R(x)$ is a block.

On the other hand, suppose that there is $x \in U$ such that $R(x)$ is not a block. By Lemma 14 this means that there are $a, b \in R(x)$ which are not related. Now we have that $a R x$ and $x R b$, but $(a,b) \notin R$. Thus, the relation R is not transitive. □

Example 16. Let us consider the tolerance R on $U = \{a,b,c,d,e\}$ depicted in Figure 7. By Lemma 14, $R(a) = R(b) = \{a,b,e\}$ and $R(c) = R(d) = \{c,d,e\}$ are blocks, because all their elements are R-related. The neighbourhood $R(e) = U$ is not a block, because, for instance, a and c are not related. By Lemma 15, the tolerance R is not an equivalence, because $R(e)$ is not a block.

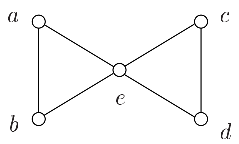


Fig. 7 Tolerance R on $\{a,b,c,d,e\}$

A *partition* π on U is a collection of nonempty subsets of U such that every element x of U belongs to exactly one member of π. For any equivalence E on U, the set of all equivalence classes $U/E = \{[x]_E \mid x \in U\}$ forms a partition of U. On the other hand, each partition π on U defines an equivalence E on U by setting $x E y$ if there is a set $X \in \pi$ such that $x, y \in X$.

Tolerances do not in general determine partitions, but for tolerances the counterparts of partitions are set coverings. A collection $\mathscr{H}$ of nonempty subsets of U is called a *covering* of U if $\bigcup \mathscr{H} = U$. A covering $\mathscr{H}$ is *irredundant* if $\mathscr{H} \setminus \{X\}$ is not a covering for any $X \in \mathscr{H}$.

Example 17. The family $\mathscr{H} = \{\{a\},\{a,b\},\{b,c,d\},\{c,d\}\}$ forms a covering of $U = \{a,b,c,d\}$, but this covering is not irredundant, because an irredundant covering cannot contain sets B and C such that $B \subseteq C$. The subfamilies $\{\{a\},\{b,c,d\}\}$, $\{\{a,b\},\{b,c,d\}\}$, and $\{\{a,b\},\{c,d\}\}$ of $\mathscr{H}$ are irredundant coverings.

The family $\mathscr{K} = \{\{a,b\},\{b,c\},\{c,d\},\{a,d\}\}$ also forms a covering of U. It is not irredundant even $B \nsubseteq C$ for all $B, C \in \mathscr{K}$. The subfamilies $\{\{a,b\},\{c,d\}\}$ and $\{\{a,d\},\{b,c\}\}$ of $\mathscr{K}$ are irredundant coverings.

It is clear that the blocks of a tolerance R on U form a covering of U, because for each $x \in U$, $(x,x) \in R$ means that there must be a block containing x. On the other hand, each covering $\mathscr{H}$ of U defines a tolerance $\bigcup\{X \times X \mid X \in \mathscr{H}\}$, called the *tolerance induced* by $\mathscr{H}$. If R is a tolerance induced by a covering $\mathscr{H}$, then

$$R(x) = \bigcup\{B \in \mathscr{H} \mid x \in B\}.$$

A natural problem is, how to recognize the tolerances that are induced by an irredundant covering. We consider this problem next.

Proposition 18. *Let R be a tolerance induced by an irredundant covering $\mathscr{H}$ of U. Then $\mathscr{H} = \{R(x) \mid R(x)$ is a block$\}$.*

Proof. Let $B \in \mathscr{H}$. Because $\mathscr{H}$ is an irredundant covering, there is an element $x \in B$ such that $x \notin \bigcup(\mathscr{H} \setminus \{B\})$. Since R is induced by $\mathscr{H}$, $x R y$ for all y in B. Therefore, $B \subseteq R(x)$. On the other hand, if $a \in R(x)$, then there is a set $C \in \mathscr{H}$ such that $x, a \in C$. But because x belongs only to B, we have $C = B$ and $a \in B$. Thus, also $R(x) \subseteq B$ and hence $R(x) = B$. Additionally, because $\mathscr{H}$ induces R, we have that $a R b$ for all $a, b \in R(x) = B$. By Lemma 14, this means that $R(x)$ is a block. We have now proved that $\mathscr{H} \subseteq \{R(x) \mid R(x)$ is a block$\}$.

We need to show that also $\{R(x) \mid R(x)$ is a block$\} \subseteq \mathscr{H}$. Assume that $R(x)$ is a block. Because $\mathscr{H}$ is a covering, there is $B \in \mathscr{H}$ such that $x \in B$. If $a \in B$, then because $\mathscr{H}$ induces R, $a \in R(x)$ and thus $B \subseteq R(x)$. Since $\mathscr{H}$ is irredundant covering inducing R, we have by the beginning of the proof that $\mathscr{H} \subseteq \{R(x) \mid R(x)$ is a block$\}$. This means that there is an element $y \in U$ such that $B = R(y)$ and $R(y)$ is a block. Because $R(x)$ and $R(y)$ are blocks, $R(y) = B \subseteq R(x)$ gives $B = R(x)$ and $R(x) \in \mathscr{H}$. □

Proposition 18 says that if R is a tolerance induced by an irredundant covering, then this covering is unique and contains exactly the $R(x)$-neighbourhoods that are blocks. This means that we may simply speak about tolerances induced by an irredundant covering without specifying the covering in question.

Lemma 19. *Let R be a tolerance on U. If $R(x)$ and $R(y)$ are distinct blocks, then $x \notin R(y)$.*

Proof. If $x \in R(y)$, then $R(y) \subseteq R(x)$ by (5) since $R(y)$ is a block. But $x \in R(y)$ means that also $y \in R(x)$. Since $R(x)$ is a block, we have $R(x) \subseteq R(y)$. Hence $R(x) = R(y)$. □

Note that Lemma 19 means that if $\{R(x) \mid R(x) \text{ is a block}\}$ is a covering, then it is irredundant. In fact, we can write the following characterization.

Theorem 20. *Let R be a tolerance on U. The following are equivalent:*

(a) *R is a tolerance induced by an irredundant covering;*
(b) *$\{R(x) \mid R(x)$ is a block$\}$ induces R.*

Proof. That (a) implies (b) is clear by Proposition 18. On the other hand, if $\mathscr{H} = \{R(x) \mid R(x) \text{ is a block}\}$ induces R, then $\mathscr{H}$ is a covering, because for all $y \in U$, $y\,R\,y$ requires that there is an $R(x) \in \mathscr{H}$ such that $y \in R(x)$. By Lemma 19, the covering $\mathscr{H}$ is irredundant, because if $R(x)$ is a block, then x cannot belong to any other $R(y)$ that is a block. Thus, (b) implies (a). □

Example 21. It is possible that the family $\mathscr{H} = \{R(x) \mid R(x) \text{ is a block}\}$ is an irredundant covering, but does not induce R. In this case, R is not a tolerance induced by an irredundant covering. For instance, consider the tolerance of Example 1. Now

$$\mathscr{H} = \{R(x) \mid R(x) \text{ is a block}\} = \{R(1), R(2), R(3)\} = \{\{1,4\},\{2,5\},\{3,6\}\}.$$

The family $\mathscr{H}$ is an irredundant covering of $U = \{1,2,3,4,5,6\}$, but $\mathscr{H}$ does not induce R. For example, $(4,5) \in R$, but there is no block in $\mathscr{H}$ containing 4 and 5.

Example 22. Let us consider the tolerance on $U = \{a,b,c,d,e,f,g\}$ given in Figure 8. This tolerance appears in [18, Figure 3.5]. The neighbourhoods that are blocks are $R(a) = \{a,b,d,e\}$, $R(c) = \{b,c,d,f\}$, and $R(g) = \{d,e,f,g\}$. All edges of the graph are inside these blocks, which means that $\{R(a), R(c), R(g)\}$ induces R. By Theorem 20, R is a tolerance induced by an irredundant covering.

Notice that there may exist blocks which are not R-neighbourhoods of any object. For instance, $\{b,d,e,f\}$ is such a block.

Remark 23. A nonempty set $X \subseteq U$ is an R-preblock if and only if it is a *clique* of the graph $\mathscr{G} = (U,R)$, that is, all pairs of vertices in X are connected by an edge of $\mathscr{G}$. A block of R is thus a maximal clique.

In computer science, the *clique problem* is the decision problem whether a clique of a given size k exists in a graph. A brute-force method for solving this problem is to list all sets of k vertices and to check each one to see whether it forms a clique. For a graph with n vertices, the running time of this algorithm is $\Omega(k^2\binom{n}{k})$. In fact, it is known that the clique problem is NP-complete and therefore an efficient algorithm for the clique problem is unlikely to exist [1].

Fortunately, our problem is simpler. We do not need to find all maximal cliques of a graph. We are only interested in the question whether $R(x)$ is a clique for some

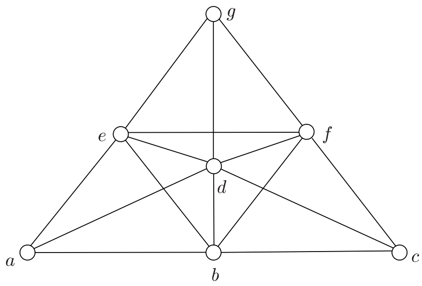


Fig. 8 Tolerance R induced by an irredundant covering

$x \in U$, because such an $R(x)$ is necessarily a maximal clique by Lemma 14. Our Algorithm 31 solves this in $O(|R|)$ steps, where $|R|$ denotes the cardinality of R.

We end this subsection by considering how tolerances induced by an irredundant covering may arise from incomplete information systems.

Let $\mathscr{S} = (U, A, \{V_a\}_{a \in A})$ be an incomplete information system, where the null values are marked by $*$. For each $B \subseteq A$, the tolerance SIM_B is defined as earlier, that is,

$$SIM_B = \{(x,y) \in U \times U \mid (\forall a \in B)\, a(x) = a(y) \text{ or } a(x) = * \text{ or } a(y) = *\}.$$

We define the set $compl_B$ of *B-complete elements* by

$$compl_B = \{x \in U \mid a(x) \neq * \text{ for all } a \in B\}.$$

Our next lemma shows that the neighbourhoods of complete elements are blocks.

Lemma 24. *Let $\mathscr{S} = (U, A, \{V_a\}_{a \in A})$ be an incomplete information system and $B \subseteq A$. For any $c \in compl_B$, the neighbourhood $SIM_B(c)$ is a block.*

Proof. Assume that $x, y \in SIM_B(c)$. Then $a(x) = a(c)$ for all $a \in B$ such that $a(x) \neq *$. Similarly, $a(y) = a(c)$ for all $a \in B$ such that $a(y) \neq *$. This means that $a(x) = a(y)$ and $(x,y) \in SIM_a$ for all $a \in B$ such that $a(x) \neq *$ and $a(y) \neq *$. On the other hand, if $a(x) = *$ or $a(y) = *$, then $(x,y) \in SIM_a$. Thus, $(x,y) \in SIM_a$ for all $a \in B$, which means that $(x,y) \in SIM_B$. Hence $SIM_B(c)$ is a block, according to Lemma 14. □

In what follows, we present a condition under which for each $B \subseteq A$, the tolerance SIM_B is induced by an irredundant covering. Let us begin with an example.

Example 25. Let $\mathscr{S} = (U, A, \{V_a\}_{a \in A})$ be the incomplete information system defined in Table 3.

The elements 2 and 3 are A-complete and their SIM_A-neighbourhoods are $\{1,2\}$ and $\{3,4\}$. The tolerance induced by this covering is an equivalence which differs from the tolerance SIM_A, because the objects 1 and 4 are SIM_A-related. This means that the covering $\{\{1,2\},\{3,4\}\}$ does not induce SIM_A.

Table 3 The information system $\mathscr{S}$

	a	b
1	$*$	w_1
2	v_1	w_1
3	v_2	w_2
4	v_2	$*$

Let $\mathscr{S} = (U, A, \{V_a\}_{a \in A})$ be an incomplete information system and $B \subseteq A$. We introduce the following condition:

$$(x,y) \in SIM_B \iff (\exists c \in compl_B)\, x, y \in SIM_B(c) \tag{$\star$}$$

Proposition 26. *Let $\mathscr{S} = (U, A, \{V_a\}_{a \in A})$ be an incomplete information system and let $B \subseteq A$ be such that ($\star$) is satisfied. Then $\mathscr{H}_B = \{SIM_B(c) \mid c \in compl_B\}$ is an irredundant covering and it induces SIM_B.*

Proof. Let $B \subseteq A$. Because $(x,x) \in SIM_B$ for all $x \in U$, by ($\star$) there must be an element $c \in compl_B$ such that $x \in SIM_B(c)$. Therefore, $\mathscr{H}_B$ is a covering.

The covering $\mathscr{H}_B$ is clearly irredundant, because each B-complete element c can belong only to $SIM_B(c)$. By condition ($\star$), $\mathscr{H}_B$ induces SIM_B □

Example 27. Let us consider the incomplete information system of Example 2. The A-complete elements are 2, 3, 5. Now $SIM_A(2) = \{2,4\}$, $SIM_A(3) = \{1,3\}$, and $SIM_A(5) = \{1,4,5\}$. Because every edge of the graph belongs to these blocks, the irredundant covering

$$\{SIM_A(2), SIM_A(3), SIM_A(5)\}$$

induces SIM_A.

Remark 28. For any B-complete element c, $R(c)$ is a cluster in which all elements are similar to each other with respect to the B-attributes. The element c can be seen as a "prototype element" for this cluster because an object $x \in U$ belongs to this cluster if and only if x is B-similar to c. Such prototype elements c are called "medoids" in cluster analysis. If condition ($\star$) holds for some $B \subseteq A$, then this means that two objects x and y are SIM_B-related if and only if there is a B-complete prototype object c such that x and y are SIM_B-related to this c.

2.2 Algorithms

Next we present an algorithm which decides whether a given tolerance is induced by an irredundant covering. If that is the case, then the method also produces the unique irredundant covering.

Since inputs to the algorithms have to be finite, we assume here that U is finite. We suppose that the relation R is given as the collection $\{R(x) \mid x \in U\}$ of all R-neighbourhoods. More precisely, we use an *adjacency list representation* of (U, R)

consisting of an array *Adj* of $|U|$ lists, one for each element in U. For each $x \in U$, the adjacency list $Adj[x]$ contains the members of $R(x)$ (see [1], for example). We also assume that the elements in each adjacency list are ordered according to some given linear order of U. Each adjacency list is implemented as a linked list, where each list element a has a reference $a.next$ to the next element in the list. The end of the list is marked by NULL. The "value" identifying the element stored in the list element a is in $a.key$. For each pair $(x,y) \in R$, the adjacency list $Adj[x]$ contains a list element a such that $a.key = y$. Thus the sum of the lengths of all adjacency lists is $|R|$. Because the relation R is reflexive, $|U| \leq |R|$, and so the amount of memory the adjacency-list representation requires is $O(|R|)$.

Example 29. The adjacency-list representation of the relation R of Example 2 is depicted in Figure 9. The *next*-references are marked by an arrow and the NULL-value is indicated by a diagonal slash.

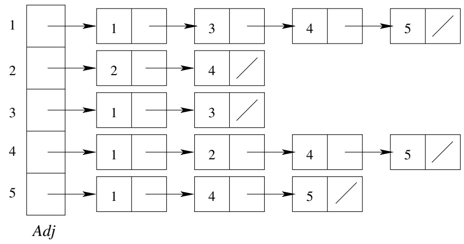


Fig. 9 An adjacency list representation

Our first algorithm checks whether $R(x) \subseteq R(y)$ for some $x,y \in U$.

Algorithm 30 (INCLUSION).
Input: The adjacency list representation *Adj* of (U,R) and two elements $x,y \in U$.
Output: "yes" if $R(x) \subseteq R(y)$; "no" otherwise.

(1) Let a be the first element in the list $Adj[x]$ and let b be the first element in $Adj[y]$.
(2) While $a \neq$ NULL and $b \neq$ NULL, repeat the following:

(a) if $a.key < b.key$, then output "no" and halt.
(b) if $a.key = b.key$, then $a \leftarrow a.next$ and $b \leftarrow b.next$.
(c) if $a.key > b.key$, then $b \leftarrow b.next$.

(3) If $a =$ NULL, then output "yes"; otherwise output "no".

The time-complexity of Algorithm 30 is $O(|Adj[y]|)$. This is because in the worst case $b \leftarrow b.next$ is executed for each element b in $Adj[y]$.

Next we present an algorithm which decides whether $R(x)$ is a block for a given $x \in U$. Using Algorithm 30 it checks whether $R(x) \subseteq R(y)$ holds for each $y \in R(x)$. If this is true, then $R(x)$ is a block by (5).

Algorithm 31 (BLOCK).

Input: The adjacency list representation *Adj* of (U,R) and an element $x \in U$.

Output: "yes" if $R(x)$ is a block; "no" otherwise.

1. For each $y \in R(x) \setminus \{x\}$, test using Algorithm 30 whether $R(x) \subseteq R(y)$. If no, output "no" and halt.
2. If all elements $y \in R(x) \setminus \{x\}$ are checked without halting, output "yes" and halt.

The running time of Algorithm 31 is $O(|R|)$, because when $R(x)$ is a block, we need to check $R(x) \subseteq R(y)$ for all $y \in R(x) \setminus \{x\}$. Each such test takes $O(|Adj[y]|)|$ time. The sum of the lengths of all adjacency lists is $|R|$, from which we get the upper bound.

Our next algorithm decides whether the tolerance R is induced by an irredundant covering. As far as we know, this is the first algorithm solving this problem.

Algorithm 32 (IRREDUNDANT COVERING).

Input: The adjacency list representation *Adj* of (U,R) and the set U.

Output: "yes" if R is induced by an irredundant covering and a set C such that $\{R(x) \mid x \in C\}$ is the irredundant covering inducing R; "no" otherwise.

1. Divide U into $C = \{x \in U \mid R(x) \text{ is a block}\}$ and $D = U \setminus C$.
2. For all $d \in D$ and all $e \in R(d)$, find out whether $\{d,e\} \subseteq R(c)$ for some $c \in C$. If such an element c cannot be found for some (d,e)-pair, output "no" and halt.
3. If all (d,e)-pairs are tested without halting, output "yes" and the set C.

The correctness of the algorithm follows from Theorem 20: if R is induced by an irredundant covering $\mathscr{H}$, then $\mathscr{H}$ must be $\{R(x) \mid R(x) \text{ is a block}\}$. Therefore, the algorithm needs to check whether $\{R(x) \mid R(x) \text{ is a block}\}$ induces R. The algorithm first produces the set C, which can be done in $O(|U| \cdot |R|)$ time (step 1). This is because there are $O(|U|)$ elements in C, and checking whether $R(c)$ is a block for some $c \in C$ takes $O(|R|)$ steps by using Algorithm 31.

Next the algorithm decides whether $\{R(c) \mid c \in C\}$ induces R. This is done by considering the remaining R-neighbourhoods $\{R(d) \mid d \in D\}$. It suffices to check that for any $e \in R(d)$ (meaning that $d\,R\,e$), there is an element $c \in C$ such that $d,e \in R(c)$. Since there are at most $|R|$ this kind of (d,e)-pairs, and the sum of the lengths of the adjacency lists of the elements in C at most $|R|$, the time complexity of step 2 is $O(|R|^2)$. The total running time of Algorithm 32 is therefore $O(|U| \cdot |R|) + O(|R|^2)$ $= O(|R|^2)$.

2.3 *Rough approximations defined by tolerances induced by irredundant coverings*

In Section 1, we showed that $\wp(U)^{\blacktriangledown}$ and $\wp(U)^{\blacktriangle}$ are complete order-isomorphic ortholattices. In this section our aim is to study the properties of these complete lattices defined by tolerances induced by irredundant coverings.

A *distributive lattice* is a lattice L satisfying the *distributive laws*:

(D1) $(\forall x,y,z \in L)\ x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge y)$;
(D2) $(\forall x,y,z \in L)\ x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$.

It is known that a lattice satisfies (D1) if and only if it satisfies (D2). Therefore, checking that a lattice is distributive requires only checking the validity of either (D1) or (D2).

Example 33. In general, $\wp(U)^{\blacktriangledown}$ and $\wp(U)^{\blacktriangle}$ are not distributive. For instance, consider a tolerance R of Example 13. The lattices of lower and upper approximations are given in Figure 10. Note that for the sake of simplicity, we sometimes denote sets by sequences of their elements. For example, $\{1,2,3\}$ is denoted by 123. The lattice $\wp(U)^{\blacktriangledown}$ is not distributive, because

$$(\{1\} \vee \{2\}) \wedge \{3\} = U \wedge \{3\} = \{3\}$$

and

$$(\{1\} \wedge \{3\}) \vee (\{2\} \wedge \{3\}) = \emptyset \vee \emptyset = \emptyset.$$

Because $\wp(U)^{\blacktriangle} \cong \wp(U)^{\blacktriangledown}$, $\wp(U)^{\blacktriangle}$ is not distributive neither.

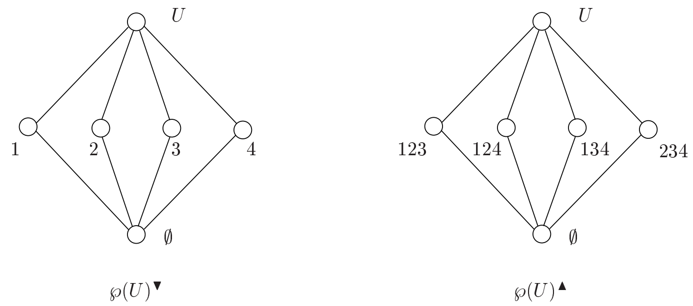


Fig. 10 $\wp(U)^{\blacktriangledown}$ and $\wp(U)^{\blacktriangle}$ are not distributive

Let L be a bounded lattice with least element 0 and greatest element 1. An element $b \in L$ is a *complement* of an element $a \in L$ if

$$a \vee b = 1 \qquad \text{and} \qquad a \wedge b = 0.$$

A *complemented lattice* is a bounded lattice in which every element has a complement. Notice that the orthocomplement of an element is a complement in the sense of the above definition. Complements need not be unique and if a lattice is not distributive, then an element may have several complements. For instance, in $\wp(U)^{\blacktriangledown}$ of Example 33, the element $\{1\}$ has the complements $\{2\}$, $\{3\}$, and $\{4\}$.

A *Boolean lattice* is a complemented distributive lattice. It is known that in a distributive lattice any element can have at most one complement, so in Boolean lattices the complement of any element is unique. The complement of x is denoted by x'. The complement operation in a Boolean lattice has the following properties:

(B1) $0' = 1$ and $1' = 0$;
(B2) $a'' = a$;
(B3) $(a \vee b)' = a' \wedge b'$ and $(a \wedge b)' = a' \vee b'$;
(B4) $a \wedge b = 0$ if and only if $a \leq b'$;
(B5) $a \leq b$ implies $b' \leq a'$.

Let L be a lattice with a least element 0. Then $a \in L$ is called an *atom* of L, if $0 \prec a$. The set of atoms of L is denoted by $\mathscr{A}(L)$. A lattice L with zero is *atomistic* if every element $x \in L$ is a join of atoms.

Let L be an atomistic lattice. For any $x \in L$, let us denote

$$A(x) = \{a \in \mathscr{A}(L) \mid a \leq x\}.$$

Then clearly $x = \bigvee A(x)$, and for any $y \in L$, $x \leq y$ if and only if $A(x) \subseteq A(y)$. Hence $A(x) = A(y)$ if and only if $x = y$. The facts

$$A(x \wedge y) = A(x) \cap A(y) \quad \text{and} \quad A(x \vee y) \supseteq A(x) \cup A(y) \tag{6}$$

follow directly from the definitions of joins and meets.

Next we prove that $\wp(U)^{\blacktriangledown}$ and $\wp(U)^{\blacktriangle}$ are Boolean lattices when R is a tolerance induced by an irredundant covering. For that we will need the following two lemmas.

Lemma 34. *Let R be a tolerance on U.*

(a) *Any atom of $\wp(U)^{\blacktriangle}$ has the form $R(x)$ for some $x \in U$.*
(b) *If $R(x)$ is a block of R, then $R(x)$ is an atom of $\wp(U)^{\blacktriangle}$.*
(c) *If R is induced by an irredundant covering of U, then $\{R(x) \mid R(x)$ is a block$\}$ is the set of the atoms of the lattice $\wp(U)^{\blacktriangle}$.*

Proof. (a) Atoms of $\wp(U)^{\blacktriangle}$ have to be of the form $R(x)$, because the map $^{\blacktriangle}$ is order-preserving and $R(x) = \{x\}^{\blacktriangle}$.

(b) Suppose that $R(x)$ is a block and $R(y) \subseteq R(x)$. Because $y \in R(x)$ and $R(x)$ is a block, we have $R(x) \subseteq R(y)$ by (5). Thus, $R(y) = R(x)$ and $R(x)$ is an atom.

(c) Suppose that $R(y)$ is an atom of $\wp(U)^{\blacktriangle}$. Because $\{R(x) \mid R(x)$ is a block$\}$ is a covering, there exists a block $R(x)$ such that $y \in R(x)$. Since $R(x)$ is a block, $\emptyset \subset R(x) \subseteq R(y)$ by (5). Because $R(y)$ is an atom of $\wp(U)^{\blacktriangle}$, we get $R(y) = R(x)$, and this means that $R(y)$ is a block. □

In a lattice L with a least element 0, an element x^* is a *pseudocomplement* of an element $x \in L$ if, for any $z \in L$, $x \wedge z = 0$ if and only if $z \leq x^*$. Obviously, an element can have at most one pseudocomplement. The lattice L itself is called *pseudocomplemented*, if every element of L has a pseudocomplement. Every pseudocomplemented lattice is necessarily bounded, having 0^* as the greatest element.

Lemma 35. *Any complete atomistic pseudocomplemented lattice L is a Boolean lattice.*

Proof. First, we show that $A(x \vee y) = A(x) \cup A(y)$ for any $x, y \in L$. By (6), it suffices to prove that $A(x \vee y) \subseteq A(x) \cup A(y)$. Take any $a \in \mathscr{A}(L)$ with $a \leq x \vee y$. We show that $a \notin A(x) \cup A(y)$ is not possible. Indeed, if $a \notin A(x)$ and $a \notin A(y)$, then $a \wedge x = 0$ and $a \wedge y = 0$. This gives $x, y \leq a^*$ and $x \vee y \leq a^*$. We get $a = a \wedge (x \vee y) \leq a \wedge a^* = 0$, a contradiction. Hence $a \in A(x) \cup A(y)$, which proves $A(x \vee y) = A(x) \cup A(y)$.

Next, we show that L satisfies identity (D1). Take any $x, y, z \in L$ and observe that

$$\begin{aligned} A(x \wedge (y \vee z)) &= A(x) \cap A(y \vee z) = A(x) \cap (A(y) \cup A(z)) \\ &= (A(x) \cap A(y)) \cup (A(x) \cap A(z)) = A(x \wedge y) \cup A(x \wedge z) \\ &= A((x \wedge y) \vee (x \wedge z)). \end{aligned}$$

Thus $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$, proving that (D1) holds and L is distributive.

To prove that L is complemented, take any $x \in L$. Then, for any $a \in \mathscr{A}(L)$,

$$a \notin A(x) \iff a \nleq x \iff a \wedge x = 0 \iff a \leq x^* \iff a \in A(x^*).$$

This yields $A(x^*) = \mathscr{A}(L) \setminus A(x)$, whence we get

$$A(x \vee x^*) = A(x) \cup A(x^*) = \mathscr{A}(L) = A(1).$$

This implies $x \vee x^* = 1$. Since $x \wedge x^* = 0$, we obtain that x^* is the complement of x in L. Therefore, L is a Boolean lattice. $\square$

Proposition 36. *If R is a tolerance induced by an irredundant covering, then $\wp(U)^\blacktriangle$ is atomistic and pseudocomplemented.*

Proof. Since in the lattice $\wp(U)^\blacktriangle$ the joins coincide with unions, in view of Lemma 34(c), $\wp(U)^\blacktriangle$ is atomistic if for any $A \in \wp(U)^\blacktriangle$,

$$\bigcup\{R(x) \subseteq A \mid R(x) \text{ is a block}\} = A. \tag{7}$$

As the left side of (7) is included in A, we have to show only the converse inclusion. Because $A \in \wp(U)^\blacktriangle$, $A = X^\blacktriangle$ for some $X \subseteq U$. Let $a \in A$. Then $a R b$ for some $b \in X$. Because R is induced by an irredundant covering, by Proposition 18, there is $x \in U$ such that $a, b \in R(x)$ and $R(x)$ is a block. Because $R(x)$ is a block, we have $R(x) \subseteq R(c)$ for all $c \in R(x)$. In particular, $b \in R(c) \cap X$, $R(c) \cap X \neq \emptyset$ and $c \in X^\blacktriangle$ for all $c \in R(x)$. Thus, $R(x) \subseteq X^\blacktriangle = A$ and $a \in \bigcup\{R(x) \subseteq A \mid R(x) \text{ is a block}\}$. This proves (7) and hence $\wp(U)^\blacktriangle$ is atomistic.

Now let $B, C \in \wp(U)^\blacktriangle$ be such that $B \wedge C = \emptyset$. In order to prove that $\wp(U)^\blacktriangle$ is pseudocomplemented, we show that $C \subseteq B^\perp$. Note that we have already proved that $B \wedge B^\perp = \emptyset$ in Proposition 12. Since $\wp(U)^\blacktriangle$ is atomistic, to prove $C \subseteq B^\perp$, it is enough to show that each atom $R(x) \subseteq C$ satisfies $R(x) \subseteq B^\perp = B^{c\blacktriangle}$.

Assume by contradiction that there exists an element $x \in U$ such that $R(x)$ is a block and $R(x) \subseteq C$, but $R(x) \nsubseteq B^{c\blacktriangle} = B^{\blacktriangledown c}$. This means that $R(x) \cap B^\blacktriangledown \neq \emptyset$. Hence there is an element $y \in B^{\blacktriangledown}$ with $y \in R(x)$. Then yRx and $R(y) \subseteq B$. Since $R(x)$ is a block, we get $R(x) \subseteq R(y) \subseteq B$ by (5). Because $R(x) \in \wp(U)^{\blacktriangle}$, $R(x) \subseteq B, C$ yields $B \wedge C \neq \emptyset$, a contradiction. This proves that $B^{\perp}$ is the pseudocomplement of any $B \in \wp(U)^{\blacktriangle}$. □

We can now write the following conclusion of Lemmas 34 and 35, and Proposition 36. Notice that since $^{\blacktriangledown}$ is an isomorphism from $\wp(U)^{\blacktriangle}$ to $\wp(U)^{\blacktriangledown}$, the atoms of $\wp(U)^{\blacktriangledown}$ are the $^{\blacktriangledown}$-images of the atoms of $\wp(U)^{\blacktriangle}$.

Corollary 37. *Let R be a tolerance induced by an irredundant covering of U.*

(a) *$\wp(U)^{\blacktriangle}$ is an atomistic Boolean lattice such that for all $X \in \wp(U)^{\blacktriangle}$, $X' = X^{c\blacktriangle}$. The set of atoms is $\{R(x) \mid R(x)$ is a block$\}$.*
(b) *$\wp(U)^{\blacktriangledown}$ is an atomistic Boolean lattice such that for all $X \in \wp(U)^{\blacktriangledown}$, $X' = X^{c\blacktriangledown}$. The set of atoms is $\{R(x)^{\blacktriangledown} \mid R(x)$ is a block$\}$.*

Example 38. Let us consider the tolerance R of Figure 2 on $U = \{1,2,3,4,5,6\}$. The Boolean lattice $\wp(U)^{\blacktriangle}$ is given in Figure 11. The elements $R(2) = \{2,4\}$, $R(3) = \{1,3\}$, $R(5) = \{1,4,5\}$ are its atoms.

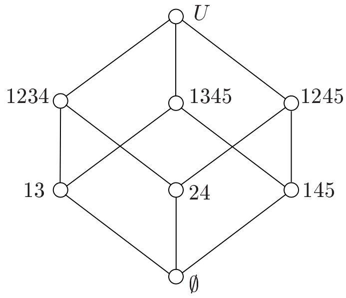


Fig. 11 The Boolean lattice $\wp(U)^{\blacktriangle}$

Note that for tolerances induced by an irredundant covering, the sets in $\wp(U)^{\blacktriangle}$ are in a sense "definable". Let us recall that each $R(x)$ being a block is completely determined by the element x. As we noted, these "prototype elements" are called medoids in cluster analysis. Because $\mathscr{A}(\wp(U)^{\blacktriangle}) = \{R(x) \mid R(x)$ is a block$\}$, each $X^{\blacktriangle}$ is a union of some $R(x)$-neighbourhoods that are blocks. This means that $X^{\blacktriangle}$ can be defined just by listing the appropriate "prototype elements". For instance, in Example 38, the set $\{1,2,4,5\}$ is "defined" by the elements 2 and 5: an element of U belongs to $\{1,2,4,5\}$ if and only if it is R-related to 2 or 5.

Next we show how approximations defined by tolerances and approximations defined by coverings are related when we consider irredundant coverings and tolerances induced by them. For an equivalence relation E, lower and upper approximations for $X \subseteq U$ can be also written in the form

$$X^{\blacktriangledown} = \bigcup\{[x]_E \mid [x]_E \subseteq X\}$$

and

$$X^{\blacktriangle} = \bigcup\{[x]_E \mid [x]_E \cap X \neq \emptyset\},$$

respectively. Because there is a one-to-one correspondence between equivalences and partitions, we could as well define the rough approximations of $X \subseteq U$ in terms of a partition π on U by

$$X^{\blacktriangledown} = \bigcup\{B \in \pi \mid B \subseteq X\}$$

and

$$X^{\blacktriangle} = \bigcup\{B \in \pi \mid B \cap X \neq \emptyset\}.$$

In [21], W. Żakowski presented generalizations of these definitions by replacing the partition π of U by a covering $\mathscr{H}$ of U. His operators do not form a dual pair, but J. A. Pomykała [17] associated with coverings several pairs of mutually dual approximation operators. We recall here a couple of them.

Let $\mathscr{H}$ be a covering of U. For each $x \in U$, the set

$$N(x) = \bigcup\{B \in \mathscr{H} \mid x \in B\}$$

is called the *$\mathscr{H}$-neighbourhood* x. For any $X \subseteq U$, we define

$$\begin{aligned} X^{\blacktriangleleft} &= \{x \in U \mid N(x) \subseteq X\}, \\ X^{\blacktriangleright} &= \bigcup\{B \in \mathscr{H} \mid B \cap X \neq \emptyset\}, \\ X^{\lhd} &= \bigcup\{B \in \mathscr{H} \mid B \subseteq X\}, \text{ and} \\ X^{\rhd} &= \{x \in U \mid B \cap X \neq \emptyset \ \text{ for all } \ B \in \mathscr{H} \text{ with } x \in B\}. \end{aligned}$$

As noted by Pomykała [17], these operators form dual pairs, that is, $X^{\blacktriangleright c} = X^{c \blacktriangleleft}$ and $X^{\rhd c} = X^{c \lhd}$ for all $X \subseteq U$. The operators $^{\blacktriangleright}$ and $^{\lhd}$ are the ones defined by Żakowski.

Let us see how these operators relate to the rough set operators $^{\blacktriangle}$ and $^{\blacktriangledown}$ when $\mathscr{H}$ is a covering of U and R is induced by it. Note that now the $\mathscr{H}$-neighbourhoods and the R-neighbourhoods are equal for all $x \in U$, that is, for any $x \in U$, $N(x) = R(x)$. This means that $X^{\blacktriangledown} = X^{\blacktriangleleft}$ for every $X \subseteq U$. Because also the operators $^{\blacktriangle}$ and $^{\blacktriangledown}$ are dual, we can write the following proposition.

Proposition 39. *If $\mathscr{H}$ is a covering of U and R is induced by $\mathscr{H}$, then*

$$X^{\blacktriangle} = X^{\blacktriangleright} \quad \textit{and} \quad X^{\blacktriangledown} = X^{\blacktriangleleft}$$

for any $X \subseteq U$.

In particular, by Proposition 39 we can write for every $X \subseteq U$,

$$X^{\blacktriangle} = \bigcup\{B \in \mathscr{H} \mid X \cap B \neq \emptyset\}.$$

In (2), the interior operation $\square$ on U is defined by $\square X = X^{\blacktriangledown\blacktriangle}$. The corresponding interior system is $\wp(U)^{\blacktriangle}$ and we noted that

$$\square X = \bigcup\{A \in \wp(U)^{\blacktriangle} \mid A \subseteq X\}.$$

Assume now that the covering $\mathscr{H}$ is irredundant. If R is induced by $\mathscr{H}$, then we have $\mathscr{H} = \{R(x) \mid R(x) \text{ is a block}\}$.

Lemma 40. *If $\mathscr{H}$ is an irredundant covering of U and R is induced by $\mathscr{H}$, then*

$$\bigcup\{A \in \wp(U)^{\blacktriangle} \mid A \subseteq X\} = \bigcup\{B \in \mathscr{H} \mid B \subseteq X\}$$

for every $X \subseteq U$.

Proof. Since for any $B \in \mathscr{H}$, $B = R(x)$ for some $x \in U$, we have $B = \{x\}^{\blacktriangle} \in \wp(U)^{\blacktriangle}$. Hence, $\bigcup\{B \in \mathscr{H} \mid B \subseteq X\}$ is included in $\bigcup\{A \in \wp(U)^{\blacktriangle} \mid A \subseteq X\}$.

Conversely, if $x \in \bigcup\{A \in \wp(U)^{\blacktriangle} \mid A \subseteq X\}$, then $x \in A$ for some $A \in \wp(U)^{\blacktriangle}$, that is, there is a set $Y \subseteq U$ such that $x \in Y^{\blacktriangle} = A$. Since $Y^{\blacktriangle} = \bigcup\{B \in \mathscr{H} \mid Y \cap B \neq \emptyset\}$, this means that $x \in B$ for some $B \in \mathscr{H}$ with $Y \cap B \neq \emptyset$. Then $B \subseteq Y^{\blacktriangle} = A \subseteq X$, which implies $x \in \bigcup\{B \in \mathscr{H} \mid B \subseteq X\}$. Therefore also $\bigcup\{A \in \wp(U)^{\blacktriangle} \mid A \subseteq X\}$ is included in $\bigcup\{B \in \mathscr{H} \mid B \subseteq X\}$. This completes our proof. □

Lemma 40 means that for every $X \subseteq U$,

$$\square X = \bigcup\{B \in \mathscr{H} \mid B \subseteq X\}.$$

Hence, we can write the following proposition (for the definition of $\Diamond$ see (1)).

Proposition 41. *If $\mathscr{H}$ is an irredundant covering of U and R is induced by $\mathscr{H}$, then*

$$X^{\lhd} = \square X \qquad \textit{and} \qquad X^{\rhd} = \Diamond X$$

for any $X \subseteq U$.

2.4 Complement formal contexts based on tolerances

A *formal context* is a triple $\mathscr{K} = (G, M, I)$, where G is a set of *objects*, M is a set of *attributes*, and $I \subseteq G \times M$ is a binary relation called *incidence relation*. The notations $(g, m) \in I$ and $g\, I\, m$ both express that an object g is in relation I with an attribute m, and we read it as "the object g has the attribute m". The basic definitions and results concerning formal concept analysis can be found in [2, 3], for example.

By defining for all subsets $A \subseteq G$ and $B \subseteq M$,

$$A^I = \{m \in M \mid g\,I\,m \text{ for all } g \in A\}$$

and

$$B^I = \{g \in G \mid g\,I\,m \text{ for all } m \in B\},$$

we establish a connection between the powerset lattices $\wp(G)$ and $\wp(M)$. In fact, for any subsets $A, A_1, A_2 \subseteq G$ and $B, B_1, B_2 \subseteq M$ the following hold:

(i) $A_1 \subseteq A_2$ implies $A_2^I \subseteq A_1^I$, and $B_1 \subseteq B_2$ implies $B_2^I \subseteq B_1^I$;
(ii) $A \subseteq A^{II}$ and $B \subseteq B^{II}$;
(iii) $A^I = A^{III}$ and $B^I = B^{III}$.

By these properties, the map $A \mapsto A^{II}$ is a closure operator on G and $B \mapsto B^{II}$ is a closure operator on M.

A small context usually is represented by a table, similar to an information system. The table rows are labelled by objects and the columns are labelled by attributes. A cross ($\times$) in row g and column m means $g\,I\,m$, that is, the object g has the attribute m. In a sense, contexts are like 2-valued information systems, where the values are "cross" and "no cross".

Example 42. A formal context describing some geometrical shapes is given in Table 4.

Table 4 A simple formal context

	Angular	Right angles	Equilateral	Central symmetry
Triangle	×			
Square	×	×	×	×
Circle				×
Rectangle	×	×		×
Rhombus	×		×	×

A formal concept of the context (G,M,I) is a pair $(A,B) \in \wp(G) \times \wp(M)$ with $A^I = B$ and $B^I = A$. The set A is called the *extent* and B the *intent* of the concept (A,B). Hence any concept has the form (A^{II}, A^I) for some $A \subseteq G$, and A is a concept extent if and only if $A^{II} = A$. Similarly, for any $B \subseteq M$, (B^I, B^{II}) is a concept and B is a concept intent if and only if $B^{II} = B$. The set of all concepts of the context (G,M,I) is denoted by $\mathfrak{B}(G,M,I)$.

Let (G,M,I) be a formal context. For any concepts (A_1,B_1) and (A_2,B_2) in $\mathfrak{B}(G,M,I)$, we set $(A_1,B_1) \leq (A_2,B_2)$ if $A_1 \subseteq A_2$. Note that $A_1 \subseteq A_2$ implies that $B_1 = A_1^I \supseteq A_2^I = B_2$ and $A_1^I \supseteq A_2^I$ implies $A_1 = A_1^{II} \subseteq A_2^{II} = A_2$. Therefore,

$$(A_1,B_1) \leq (A_2,B_2) \iff A_1 \subseteq A_2 \iff B_1 \supseteq B_2.$$

It is known [3, Theorem 3] that $\mathfrak{B}(G,M,I)$ forms a complete lattice such that for $\{(A_j,B_j) \mid j \in J\} \subseteq \mathfrak{B}(G,M,I)$,

$$\bigwedge_{j\in J}(A_j,B_j) = \Big(\bigcap_{j\in J} A_j, \big(\bigcup_{j\in J} B_j\big)^{II}\Big), \text{ and } \bigvee_{j\in J}(A_j,B_j) = \Big(\big(\bigcup_{j\in J} A_j\big)^{II}, \bigcap_{j\in J} B_j\Big).$$

This lattice is called *concept lattice*.

Example 43. The concept lattice of the context of Example 42 is given in Figure 12. We use shorthand for the geometric shapes: "tr" (triangle), "sq" (square), "cir" (circle), "rec" (rectangle), and "rh" (rhombus). The attributes are denoted simply by capital letters, that is, A denotes "angular", R denotes "right angles", E denotes "equilateral" and C denotes "central symmetry".

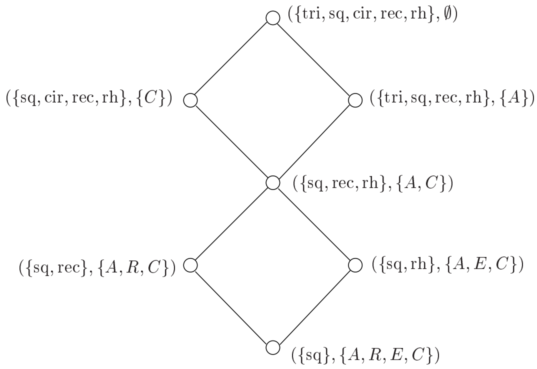


Fig. 12 The concept lattice $\mathfrak{B}(G,M,I)$

If $(P,\leq)$ is a partially ordered set, then it is known that the concept lattice $\mathfrak{B}(P,P,\leq)$ is the smallest complete lattice into which $(P,\leq)$ can be order-embedded [3, Theorem 4]. If R is a symmetric binary relation on U, then (U,U,R) forms a formal context in which the concepts are the maximal pairs $(A,B) \in \wp(U)\times\wp(U)$ such that every element of A is R-related to every element of B. Thus, if $(A,B) \in \mathfrak{B}(U,U,R)$, then also $(B,A) \in \mathfrak{B}(U,U,R)$. The map

$$\sim : \mathfrak{B}(U,U,R) \to \mathfrak{B}(U,U,R), \ (A,B) \mapsto (B,A)$$

is a polarity on $\mathfrak{B}(U,U,R)$. It is easy to see that if the relation R is irreflexive, then the extent and the intent of each concept must be disjoint and we have

$$(A,B)\wedge(B,A) = (\emptyset,M) \ \text{ and } \ (A,B)\vee(B,A) = (G,\emptyset).$$

This means that the map $\sim$ is an orthocomplementation and the concept lattice $\mathfrak{B}(U,U,R)$ is an orthocomplemented lattice.

Now for a tolerance R on U, we consider the context (U,U,R^c), where $R^c = \{(a,b)\in U^2 \mid (a,b)\notin R\}$. Following the terminology by Y. Y. Yao [20], (U,U,R^c) is called a *complement formal context*. Then R^c is an irreflexive and symmetric relation and in view of the previous observations, $\mathfrak{B}(U,U,R^c)$ is an orthocomplemented complete lattice, where the orthocomplement of an element $(A,B) \in \mathfrak{B}(U,U,R^c)$ is just (B,A).

Let us now consider the complement formal context (U,U,R^c) in more detail. For any $X \subseteq U$, we obtain

$$\begin{aligned} X^I &= \{x \in U \mid y R^c x \text{ for all } y \in X\} \\ &= \{x \in U \mid (y,x) \notin R \text{ for all } y \in X\} \\ &= \{x \in U \mid (x,y) \notin R \text{ for all } y \in X\} \\ &= \{x \in U \mid R(x) \cap X = \emptyset\} \\ &= X^{\blacktriangle c} = X^{c \blacktriangledown}. \end{aligned}$$

Thus, $X^{\blacktriangle} = X^{Ic}$ and $X^{\blacktriangledown} = X^{cI}$. From here we get that

$$X^{II} = X^{\blacktriangle c \blacktriangle c} = X^{\blacktriangle \blacktriangledown} = \Diamond X.$$

Since the orthocomplement of $\Diamond X$ in $\wp(U)^{\blacktriangledown}$ equals $(\Diamond X)^{\perp} = X^{\blacktriangle \blacktriangledown c \blacktriangledown} = X^{\blacktriangle c} = X^I$, the concept lattice of the complement context $\mathscr{K} = (U,U,R^c)$ has the form

$$\mathfrak{B}(\mathscr{K}) = \{(X^{\blacktriangle \blacktriangledown}, X^{c \blacktriangledown}) \mid X \subseteq U\} = \{(\Diamond X, (\Diamond X)^{\perp}) \mid X \subseteq U\}.$$

On the other hand, for each $A \in \wp(U)^{\blacktriangledown}$, we have $A = \Diamond A = A^{\blacktriangle \blacktriangledown}$, and so $(A, A^{\perp})$ belongs to $\mathfrak{B}(\mathscr{K})$. Hence,

$$\mathfrak{B}(\mathscr{K}) = \{(A, A^{\perp}) \mid A \in \wp(U)^{\blacktriangledown}\}.$$

We can write the following proposition.

Proposition 44. *Let R be a tolerance on a set U and let $\mathscr{K}$ be the complement formal context (U,U,R^c).*

(a) *$\mathfrak{B}(\mathscr{K})$ is isomorphic to $(\wp(U)^{\blacktriangle}, \subseteq)$ and $(\wp(U)^{\blacktriangledown}, \subseteq)$.*
(b) *$\mathfrak{B}(\mathscr{K})$ is a complete sublattice of the direct product of $(\wp(U)^{\blacktriangledown}, \subseteq)$ and $(\wp(U)^{\blacktriangledown}, \supseteq)$.*

Proof. (a) It is obvious that the map $A \mapsto (A, A^{\perp})$ is an isomorphism between $\wp(U)^{\blacktriangledown}$ and $\mathfrak{B}(\mathscr{K})$. In Proposition 8 we proved that $\wp(U)^{\blacktriangledown}$ and $\wp(U)^{\blacktriangle}$ are isomorphic.

(b) Clearly, $\mathfrak{B}(\mathscr{K}) \subseteq \wp(U)^{\blacktriangledown} \times \wp(U)^{\blacktriangledown}$. Let $\{(A_j, B_j)\}_{j \in J} \subseteq \mathfrak{B}(\mathscr{K})\}$. The join $\bigvee_{j \in J}(A_j, B_j)$ in $\mathfrak{B}(\mathscr{K})\}$ equals $((\bigcup_{j \in J} A_j)^{II}, \bigcap_{j \in J} B_j)$. Because $\bigcap_{j \in J} B_j$ is the meet operation in $(\wp(U)^{\blacktriangledown}, \subseteq)$, it is the join operation in $(\wp(U)^{\blacktriangledown}, \supseteq)$. Moreover, $(\bigcup_{j \in J} A_j)^{II} = \Diamond(\bigcup_{j \in J} A_j)$ is the join in $(\wp(U)^{\blacktriangledown}, \subseteq)$. Therefore, the join of any $\{(A_j, B_j)\}_{j \in J}$ in $\mathfrak{B}(\mathscr{K})$ coincides with join in the direct product of $(\wp(U)^{\blacktriangledown}, \subseteq)$ and $(\wp(U)^{\blacktriangledown}, \supseteq)$. An analogous argument is valid for meets. These facts mean that $\mathfrak{B}(\mathscr{K})$ is a complete sublattice of the direct product of the complete lattices $(\wp(U)^{\blacktriangledown}, \subseteq)$ and $(\wp(U)^{\blacktriangledown}, \supseteq)$. □

Example 45. Let us consider the tolerance R defined in Example 13 on the universe $U = \{1,2,3,4\}$. Then, $R^c = \{(1,3),(3,1),(2,4),(4,2)\}$ and the concept lattice of the complement formal context (U,U,R^c) is depicted in Figure 13. Obviously, the concept lattice $\mathfrak{B}(U,U,R^c)$ is isomorphic to the ortholattices $\wp(U)^\blacktriangle$ and $\wp(U)^\blacktriangledown$ depicted on the Figure 10.

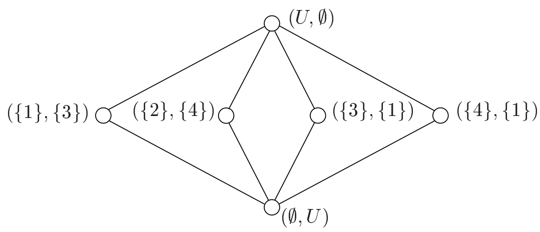


Fig. 13 The concept lattice $\mathfrak{B}(U,U,R^c)$

By Corollary 37 we obtain the following result concerning tolerances induced by irredundant coverings.

Corollary 46. *Let R be a tolerance induced by an irredundant covering of U. Then $\wp(U)^\blacktriangle$, $\wp(U)^\blacktriangledown$ and $\mathfrak{B}(U,U,R^c)$ are isomorphic complete atomistic Boolean lattices.*

It is worth to mention that if L is a complete ortholattice, then there exists a context $\mathscr{K} = (U,U,I)$, where I is an irreflexive and symmetric binary relation on U, such that $L \cong B(\mathscr{K})$ [3, p. 54]. If we set $R = I^c$, then R is a tolerance on U, and by Proposition 44, the rough approximations lattices $(\wp(U)^\blacktriangle, \subseteq)$ and $(\wp(U)^\blacktriangledown, \subseteq)$ are isomorphic to $\mathfrak{B}(\mathscr{K})$. Therefore, we get the following representation theorem for complete ortholattices in terms of rough approximations.

Proposition 47. *A complete lattice L is an ortholattice if and only if there exist a set U and a tolerance R on U such that $L \cong \wp(U)^\blacktriangledown \cong \wp(U)^\blacktriangle$.*

3 Rough set systems determined by tolerances

In this section we consider the rough sets defined by a tolerance R and the order-theoretical properties of the collection RS of them. Section 3.1 is devoted to rough sets defined by tolerances in general. We show that even these structures do not necessarily form lattices, they have a polarity operation, which is an order-isomorphism between $(RS, \leq)$ and its dual $(RS, \geq)$. In Section 3.2 we study rough sets defined by tolerances induced by irredundant coverings. We show that RS forms a Kleene algebra and a double pseudocomplemented lattice. As a double pseudocomplemented lattice, RS is determination trivial. Viewed as a pseudocomplemented Kleene algebra, RS is normal. This chapter ends by Section 3.3, where we show that in case the tolerance is induced by an irredundant covering, the relation-based and covering-based rough sets systems are isomorphic.

3.1 The general case

Originally Pawlak [16, p. 351] defined a rough set as an equivalence class of sets which look the same in view of the knowledge restricted by the given indistinguishability relation, that is, as a class of sets having the same lower approximation and the same upper approximation. This concept generalizes in a natural way to similarity relations.

Let R be a tolerance on U. A relation $\equiv$ is defined on $\wp(U)$ by

$$X \equiv Y \iff X^{\blacktriangledown} = Y^{\blacktriangledown} \text{ and } X^{\blacktriangle} = Y^{\blacktriangle}.$$

The equivalence classes of $\equiv$ are called *rough sets*. Each element in a given rough set looks the same, when observed through the knowledge given by the tolerance R. Namely, if $X \equiv Y$, then exactly the same elements belong certainly or possibly to X and Y.

The order-theoretical study of rough sets was initiated by T. B. Iwiński in [5]. In his approach rough sets on U are the pairs $(X^{\blacktriangledown}, X^{\blacktriangle})$, where $X \subseteq U$. This is justified because if $\mathscr{C} \subseteq \wp(U)$ is a rough set as defined before, that is, $\mathscr{C}$ is an equivalence class of $\equiv$, then $\mathscr{C}$ is uniquely determined by the pair $(X^{\blacktriangledown}, X^{\blacktriangle})$, where X is any member of $\mathscr{C}$: a set $Y \subseteq U$ belongs to $\mathscr{C}$ if and only if $(Y^{\blacktriangledown}, Y^{\blacktriangle}) = (X^{\blacktriangledown}, X^{\blacktriangle})$. Therefore, we call

$$RS = \{(X^{\blacktriangledown}, X^{\blacktriangle}) \mid X \subseteq U\}$$

the *set of rough sets*. The set RS is ordered by the componentwise inclusion:

$$(X^{\blacktriangledown}, X^{\blacktriangle}) \leq (Y^{\blacktriangledown}, Y^{\blacktriangle}) \iff X^{\blacktriangledown} \subseteq Y^{\blacktriangledown} \text{ and } X^{\blacktriangle} \subseteq Y^{\blacktriangle}.$$

Example 48. Let $U = \{1,2,3,4,5\}$ and let R be the tolerance on U depicted in Figure 14.

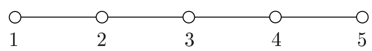


Fig. 14 The graph of a tolerance R

The lower and upper approximations defined by R are presented in Table 5, and the Hasse diagram of RS is given in Figure 15. In the figure, sets are denoted simply by sequences of letters, that is, 124 denotes the set $\{1,2,4\}$. Now RS is not a lattice because, for instance, the elements $(1,123)$ and $(\emptyset,1234)$ do not have a join, because $(12,1234)$ and $(1,U)$ are the minimal upper bounds for $(1,123)$ and $(\emptyset,1234)$, but there is no smallest upper bound. Similarly, we may observe that the elements $(12,1234)$ and $(1,U)$, do not have a meet.

We end this subsection by some basic observations on the structure of RS. Because $(\emptyset^{\blacktriangledown}, \emptyset^{\blacktriangle}) = (\emptyset,\emptyset)$ and $(U^{\blacktriangledown}, U^{\blacktriangle}) = (U,U)$, the ordered set RS is always bounded. The pair $(\emptyset,\emptyset)$ is the least element and (U,U) is the greatest element.

X	$(X^{\blacktriangledown}, X^{\blacktriangle})$	X	$(X^{\blacktriangledown}, X^{\blacktriangle})$
$\emptyset$	$(\emptyset,\emptyset)$	$\{1,2,3\}$	$(12,1234)$
$\{1\}$	$(\emptyset,12)$	$\{1,2,4\}$	$(1,U)$
$\{2\}$	$(\emptyset,123)$	$\{1,2,5\}$	$(1,U)$
$\{3\}$	$(\emptyset,234)$	$\{1,3,4\}$	$(\emptyset,U)$
$\{4\}$	$(\emptyset,345)$	$\{1,3,5\}$	$(\emptyset,U)$
$\{5\}$	$(\emptyset,45)$	$\{1,4,5\}$	$(5,U)$
$\{1,2\}$	$(1,123)$	$\{2,3,4\}$	$(3,U)$
$\{1,3\}$	$(\emptyset,1234)$	$\{2,3,5\}$	$(\emptyset,U)$
$\{1,4\}$	$(\emptyset,U)$	$\{2,4,5\}$	$(5,U)$
$\{1,5\}$	$(\emptyset,1245)$	$\{3,4,5\}$	$(45,2345)$
$\{2,3\}$	$(\emptyset,1234)$	$\{1,2,3,4\}$	$(123,U)$
$\{2,4\}$	$(\emptyset,U)$	$\{1,2,3,5\}$	$(12,U)$
$\{2,5\}$	$(\emptyset,U)$	$\{1,2,4,5\}$	$(15,U)$
$\{3,4\}$	$(\emptyset,2345)$	$\{1,3,4,5\}$	$(45,U)$
$\{3,5\}$	$(\emptyset,2345)$	$\{2,3,4,5\}$	$(345,U)$
$\{4,5\}$	$(5,345)$	U	(U,U)

Table 5 Approximations based on the tolerance R

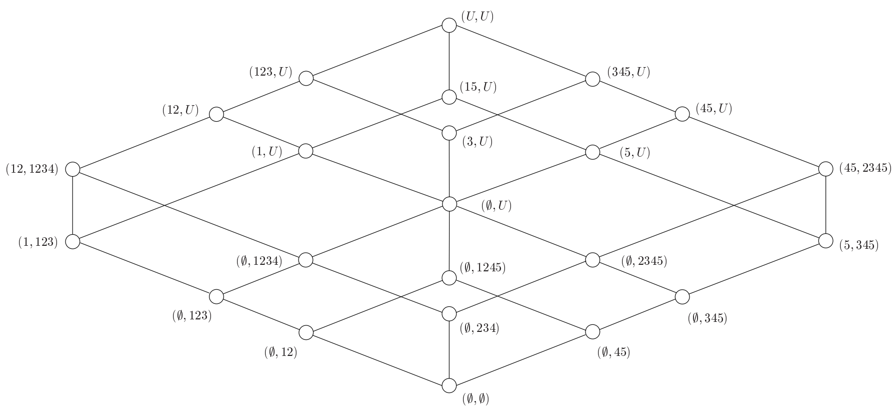


Fig. 15 The ordered set *RS* not forming a lattice

Let us consider the mapping

$$\sim\colon RS \to RS,\ (X^{\blacktriangledown}, X^{\blacktriangle}) \mapsto (X^{\blacktriangle c}, X^{\blacktriangledown c}).$$

It is easy to see that $\sim(X^{\blacktriangledown}, X^{\blacktriangle}) = (X^{c\blacktriangledown}, X^{c\blacktriangle})$, which means that $\sim(X^{\blacktriangledown}, X^{\blacktriangle})$ belongs to *RS*, and that the map $\sim$ is well defined. Clearly, $\sim\sim(X^{\blacktriangledown}, X^{\blacktriangle}) = (X^{\blacktriangledown}, X^{\blacktriangle})$. Furthermore, $(X^{\blacktriangledown}, X^{\blacktriangle}) \leq (Y^{\blacktriangledown}, Y^{\blacktriangle})$ implies $\sim(Y^{\blacktriangledown}, Y^{\blacktriangle}) = (Y^{\blacktriangle c}, Y^{\blacktriangledown c}) \leq (X^{\blacktriangle c}, X^{\blacktriangledown c}) = \sim(X^{\blacktriangledown}, X^{\blacktriangle})$. This means that $\sim$ is a polarity and hence *RS* is anti-isomorphic to itself, that is, *RS* looks the same when turned upside-down (see e.g. Figure 15).

Because $(\wp(U)^{\blacktriangledown}, \subseteq)$ and $(\wp(U)^{\blacktriangle}, \subseteq)$ are complete lattices, their direct product

$$\wp(U)^{\blacktriangledown} \times \wp(U)^{\blacktriangle} = \{(A,B) \mid A \in \wp(U)^{\blacktriangledown} \text{ and } B \in \wp(U)^{\blacktriangle}\}$$

ordered coordinatewise by $\subseteq$ is a complete lattice lattice in which

$$\bigwedge_{i \in I} (A_i, B_i) = \Big(\bigcap_{i \in I} A_i, \Box(\bigcap_{i \in I} B_i) \Big) \tag{8}$$

and

$$\bigvee_{i \in I} (A_i, B_i) = \Big(\Diamond(\bigcup_{i \in I} A_i,), \bigcup_{i \in I} B_i \Big) \tag{9}$$

for all $(A_i, B_i)_{i \in I} \subseteq \wp(U)^{\blacktriangledown} \times \wp(U)^{\blacktriangle}$.

We have proved in [8] that for any tolerance R on U, RS is a complete lattice if and only if it is a complete sublattice of the direct product $\wp(U)^{\blacktriangledown} \times \wp(U)^{\blacktriangle}$. This means that whenever RS is a complete lattice, we know how the joins and meets are defined. Namely, if $\mathcal{H} \subseteq \wp(U)$, then in RS,

$$\bigwedge_{X \in \mathcal{H}} (X^{\blacktriangledown}, X^{\blacktriangle}) = \Big(\bigcap_{X \in \mathcal{H}} X^{\blacktriangledown}, \Box(\bigcap_{X \in \mathcal{H}} X^{\blacktriangle}) \Big) \tag{10}$$

and

$$\bigvee_{X \in \mathcal{H}} (X^{\blacktriangledown}, X^{\blacktriangle}) = \Big(\Diamond(\bigcup_{X \in \mathcal{H}} X^{\blacktriangledown}), \bigcup_{X \in \mathcal{H}} X^{\blacktriangle} \Big). \tag{11}$$

Let us emphasize that showing that RS is a complete lattice is not a simple task, because it needs to show that for any $\mathcal{H} \subseteq \wp(U)$, there are sets $A, B \subseteq U$ such that

$$A^{\blacktriangledown} = \bigcap_{X \in \mathcal{H}} X^{\blacktriangledown} \quad \text{and} \quad A^{\blacktriangle} = \Diamond(\bigcap_{X \in \mathcal{H}} X^{\blacktriangle}) \tag{12}$$

and

$$B^{\blacktriangledown} = \Diamond(\bigcup_{X \in \mathcal{H}} X^{\blacktriangledown}) \quad \text{and} \quad B^{\blacktriangle} = \bigcup_{X \in \mathcal{H}} X^{\blacktriangle}. \tag{13}$$

3.2 *Rough sets defined by tolerances induced by an irredundant covering*

In this section we recall some results which can be found in [8, 10]. We omit the proof of Proposition 49 because it is rather long and technical, and the interested reader may find it in [8].

Proposition 49. *Let R be a tolerance on U induced by an irredundant covering. Then RS is a complete lattice such that for all $\mathcal{H} \subseteq \wp(U)$,*

$$\bigwedge_{X \in \mathcal{H}} (X^{\blacktriangledown}, X^{\blacktriangle}) = \Big(\bigcap_{X \in \mathcal{H}} X^{\blacktriangledown}, \Box(\bigcap_{X \in \mathcal{H}} X^{\blacktriangle}) \Big)$$

and

$$\bigvee_{X \in \mathcal{H}} (X^{\blacktriangledown}, X^{\blacktriangle}) = \Big(\Diamond(\bigcup_{X \in \mathcal{H}} X^{\blacktriangledown}), \bigcup_{X \in \mathcal{H}} X^{\blacktriangle} \Big).$$

□

Notice that if R is a tolerance induced by an irredundant covering, then $\wp(U)^{\blacktriangledown}$ and $\wp(U)^{\blacktriangle}$ are distributive lattices, and hence their direct product

$$\wp(U)^{\blacktriangledown} \times \wp(U)^{\blacktriangle} = \{(A,B) \mid A \in \wp(U)^{\blacktriangledown} \text{ and } B \in \wp(U)^{\blacktriangle}\}$$

is a distributive lattice in which the operations are defined coordinatewise. As a sublattice of the distributive lattice $\wp(U)^{\blacktriangledown} \times \wp(U)^{\blacktriangle}$, also RS is distributive.

Example 50. Let R be the tolerance of Figure 2 on $U = \{1,2,3,4,5\}$. As we have noted, R is a tolerance induced by an irredundant covering, so RS forms a lattice by Proposition 49. The rough approximations of subsets of U are given in Table 6, and the rough set lattice RS is given in Figure 16.

Table 6 Approximations based on the tolerance R of Example 50

X	$(X^{\blacktriangledown}, X^{\blacktriangle})$	X	$(X^{\blacktriangledown}, X^{\blacktriangle})$
$\emptyset$	$(\emptyset,\emptyset)$	$\{1,2,3\}$	$(3,U)$
$\{1\}$	$(\emptyset,1345)$	$\{1,2,4\}$	$(2,U)$
$\{2\}$	$(\emptyset,24)$	$\{1,2,5\}$	$(\emptyset,U)$
$\{3\}$	$(\emptyset,13)$	$\{1,3,4\}$	$(3,U)$
$\{4\}$	$(\emptyset,1245)$	$\{1,3,5\}$	$(3,1345)$
$\{5\}$	$(\emptyset,145)$	$\{1,4,5\}$	$(5,U)$
$\{1,2\}$	$(\emptyset,U)$	$\{2,3,4\}$	$(2,U)$
$\{1,3\}$	$(3,1345)$	$\{2,3,5\}$	$(\emptyset,U)$
$\{1,4\}$	$(\emptyset,U)$	$\{2,4,5\}$	$(2,1245)$
$\{1,5\}$	$(\emptyset,1345)$	$\{3,4,5\}$	$(\emptyset,U)$
$\{2,3\}$	$(\emptyset,1234)$	$\{1,2,3,4\}$	$(23,U)$
$\{2,4\}$	$(2,1245)$	$\{1,2,3,5\}$	$(3,U)$
$\{2,5\}$	$(\emptyset,1245)$	$\{1,2,4,5\}$	$(245,U)$
$\{3,4\}$	$(\emptyset,U)$	$\{1,3,4,5\}$	$(135,U)$
$\{3,5\}$	$(\emptyset,1345)$	$\{2,3,4,5\}$	$(2,U)$
$\{4,5\}$	$(\emptyset,1245)$	U	(U,U)

For a detailed study of the structure of RS, we will need some further notions. A *De Morgan algebra* is a structure $(L,\vee,\wedge,\sim,0,1)$ such that $(L,\vee,\wedge,0,1)$ is a bounded distributive lattice and the operation $\sim$ satisfies the following equations:

(DM1) $\sim(x \wedge y) = \sim x \vee \sim y$;
(DM2) $\sim(x \vee y) = \sim x \wedge \sim y$;
(DM3) $\sim\sim x = x$.

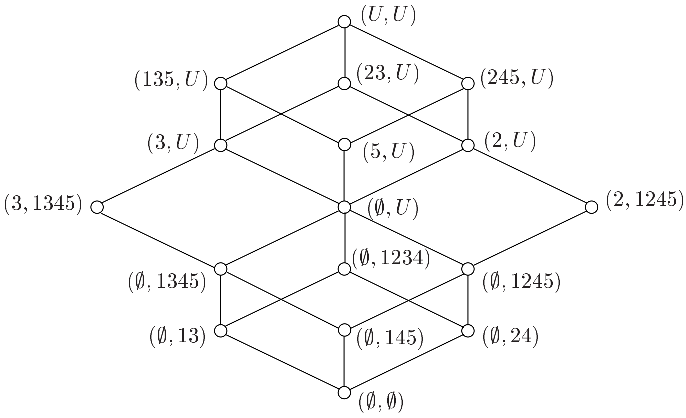


Fig. 16 The lattice *RS*

Remark 51. Notice that under (DM3), (DM1) and (DM2) are equivalent. For example, if (DM1) holds, then

$$\sim(x \vee y) = \sim(\sim\sim x \vee \sim\sim y) = \sim\sim(\sim x \wedge \sim y) = \sim x \wedge \sim y,$$

that is, (DM2) holds. Similarly, (DM2) implies (DM1).

It should be also noted that in a De Morgan algebra $(L, \vee, \wedge, \sim, 0, 1)$, the map $\sim$ is a polarity on L, because (DM3) is part of the definition of an polarity, and $x \leq y$ implies $\sim x = \sim(x \wedge y) = \sim x \vee \sim y$, which is equivalent to $\sim y \leq \sim x$.

On the other hand, if $\sim$ is a polarity on a distributive lattice $(L, \leq)$, then $x, y \leq x \vee y$ implies that $\sim(x \vee y)$ is a lower bound of $\sim x$ and $\sim y$. Assume that z is a lower bound of $\sim x$ and $\sim y$. Because $\sim$ is an involution, $\sim z \geq x$ and $\sim z \geq y$, and therefore $x \vee y \leq \sim z$ and $z \leq \sim(x \vee y)$. Hence, $\sim(x \vee y)$ is the greatest lower bound of $\sim x$ and $\sim y$, that is, $\sim(x \vee y) = \sim x \wedge \sim y$. If L is also bounded by 0 and 1, then $(L, \vee, \wedge, \sim, 0, 1)$ is a De Morgan algebra.

Proposition 52. *If R is a tolerance induced by an irredundant covering, then*

$$(RS, \vee, \wedge, \sim, (\emptyset, \emptyset), (U, U))$$

is a De Morgan algebra.

Proof. If R is a tolerance induced by an irredundant covering, then RS is a distributive lattice by Proposition 49. We have already noted that RS is bounded by $(\emptyset, \emptyset)$ and (U, U). In Section 3.1 we showed that $\sim\colon (X^{\blacktriangledown}, X^{\blacktriangle}) \mapsto (X^{\blacktriangle c}, X^{\blacktriangledown c})$ is a polarity. □

It is quite obvious that even when RS is defined by a tolerance induced by an irredundant covering, it is not in general a Boolean lattice. For instance, in Example 50, the only elements that have complements are $(\emptyset, \emptyset)$ and (U, U). Therefore, the equalities $x \wedge {\sim}x = 0$ or $y \vee {\sim}y = 1$ do not hold, in general. If a De Morgan algebra satisfies the inequality

$$x \wedge {\sim}x \leq y \vee {\sim}y, \tag{K}$$

it is called a *Kleene algebra*.

Proposition 53. *If R is a tolerance induced by an irredundant covering, then*

$$(RS, \vee, \wedge, \sim, (\emptyset, \emptyset), (U, U))$$

is a Kleene algebra.

Proof. Let $X, Y \subseteq U$. Then,

$$(X^{\blacktriangledown}, X^{\blacktriangle}) \wedge {\sim}(X^{\blacktriangledown}, X^{\blacktriangle}) = (X^{\blacktriangledown} \cap X^{c\blacktriangledown}, \Box(X^{\blacktriangle} \cap X^{c\blacktriangle})) = (\emptyset, \Box(X^{\blacktriangle} \cap X^{c\blacktriangle}))$$

and

$$(Y^{\blacktriangledown}, Y^{\blacktriangle}) \vee {\sim}(Y^{\blacktriangledown}, Y^{\blacktriangle}) = (\Diamond(Y^{\blacktriangledown} \cup Y^{c\blacktriangledown}), Y^{\blacktriangle} \cup Y^{c\blacktriangle}) = (\Diamond(Y^{\blacktriangledown} \cup Y^{c\blacktriangledown}), U).$$

From these equations we see directly that

$$(X^{\blacktriangledown}, X^{\blacktriangle}) \wedge {\sim}(X^{\blacktriangledown}, X^{\blacktriangle}) \leq (Y^{\blacktriangledown}, Y^{\blacktriangle}) \vee {\sim}(Y^{\blacktriangledown}, Y^{\blacktriangle}).$$

□

In Section 2.3 we already considered pseudocomplements. Let us recall that in a lattice L with a least element 0, an element denoted by x^* is the *pseudocomplement* of an element $x \in L$, if for any $z \in L$, $x \wedge z = 0$ if and only if $z \leq x^*$, and that L is is said to be *pseudocomplemented* if every element has a pseudocomplement. Analogously, a *dual pseudocomplement* of $x \in L$ is an element x^+ such that $x \vee z = 1$ if and only if $z \geq x^+$. If L is such that each element has a pseudocomplement and a dual pseudocomplement, then L is called a *double pseudocomplemented lattice*.

In the following we list some properties of pseudocomplements. Let L be a double pseudocomplemented lattice and $a, b \in L$. Then

(i) $a \leq b$ implies $b^* \leq a^*$ and $b^+ \leq a^+$,
(ii) $a^{++} \leq a \leq a^{**}$,
(iii) $a^* = a^{***}$ and $a^+ = a^{+++}$.

The next proposition shows that if R is induced by an irredundant covering, then RS is a double pseudocomplemented lattice.

Proposition 54. *Let R be a tolerance induced by an irredundant covering. Then RS is a double pseudocomplemented lattice in which*

$$(A, B)^* = (B^{c\blacktriangledown}, B^{c\blacktriangle}) \qquad \textit{and} \qquad (A, B)^+ = (A^{c\blacktriangledown}, A^{c\blacktriangle})$$

for any $(A, B) \in RS$.

Proof. This proof uses many of the properties listed in Proposition 5. Let $(A,B) \in RS$. First, we show that $(A,B) \wedge (B^{c\blacktriangledown}, B^{c\blacktriangle}) = (A \cap B^{c\blacktriangledown}, (B \cap B^{c\blacktriangle})^{\blacktriangledown\blacktriangle})$ equals $(\emptyset, \emptyset)$. It suffices to show that the right component $(B \cap B^{c\blacktriangle})^{\blacktriangledown\blacktriangle}$ is $\emptyset$, because then necessarily the left component $A \cap B^{c\blacktriangledown}$ is also empty. Indeed,

$$(B \cap B^{c\blacktriangle})^{\blacktriangledown\blacktriangle} = (B^{\blacktriangledown} \cap B^{c\blacktriangle\blacktriangledown})^{\blacktriangle} = (B^{\blacktriangledown} \cap B^{\blacktriangledown\blacktriangle c})^{\blacktriangle} \subseteq (B^{\blacktriangledown\blacktriangle} \cap B^{\blacktriangledown\blacktriangle c})^{\blacktriangle} = \emptyset^{\blacktriangle} = \emptyset.$$

On the other hand, if $(A,B) \wedge (X,Y) = \emptyset$ for some $(X,Y) \in RS$, then $B \wedge Y = \emptyset$ in the corresponding Boolean lattice $\wp(U)^{\blacktriangle}$. This gives $Y \subseteq B^{c\blacktriangle}$, since $B^{c\blacktriangle}$ is the complement of B in the Boolean lattice $\wp(U)^{\blacktriangle}$ by Corollary 37. To show that $X \subseteq B^{c\blacktriangledown}$ requires more work. Because $(X,Y) \in RS$, $X = Z^{\blacktriangledown}$ and $Y = Z^{\blacktriangle}$ for some $Z \subseteq U$. We have $X^{\blacktriangle} = Z^{\blacktriangledown\blacktriangle} \subseteq Z \subseteq Z^{\blacktriangle\blacktriangledown} = Y^{\blacktriangledown}$. This implies $X^{\blacktriangle\blacktriangle} \subseteq Y^{\blacktriangledown\blacktriangle} \subseteq Y \subseteq B^{c\blacktriangle}$ and further $X^{\blacktriangle} \subseteq (X^{\blacktriangle})^{\blacktriangle\blacktriangledown} \subseteq B^{c\blacktriangle\blacktriangledown}$. Now $B \in \wp(U)^{\blacktriangle}$ means that $B = C^{\blacktriangle}$ for some $C \subseteq U$. We have

$$B^{c\blacktriangle\blacktriangledown} = B^{\blacktriangledown\blacktriangle c} = C^{\blacktriangle\blacktriangledown\blacktriangle c} = C^{\blacktriangle c} = B^{c}.$$

We get by the above that

$$X \subseteq X^{\blacktriangle\blacktriangledown} \subseteq B^{c\blacktriangle\blacktriangledown\blacktriangledown} = B^{c\blacktriangledown}.$$

We have now shown that $(X,Y) \leq (B^{c\blacktriangledown}, B^{c\blacktriangle})$ which completes the proof.

The claim concerning $(A,B)^{+}$ can be proved similarly. □

Let L be a double pseudocomplemented lattice. We say that L is *determination-trivial* if for all $x,y \in L$,

$$x^{*} = y^{*} \text{ and } x^{+} = y^{+} \text{ imply } x = y. \tag{M}$$

Proposition 55. *If R is a tolerance induced by an irredundant covering, then the double pseudocomplemented lattice RS is determination trivial.*

Proof. If $(A,B)^{*} = (C,D)^{*}$, then $B^{\blacktriangledown c} = B^{c\blacktriangle} = D^{c\blacktriangle} = D^{\blacktriangledown c}$. So $B^{\blacktriangledown} = D^{\blacktriangledown}$ and $B^{\blacktriangledown\blacktriangle} = D^{\blacktriangledown\blacktriangle}$. Because $B,D \in \wp(U)^{\blacktriangle}$, $B = B^{\blacktriangledown\blacktriangle} = D^{\blacktriangledown\blacktriangle} = D$. Similarly, $(A,B)^{+} = (C,D)^{+}$ implies $A = C$. We have proved that $(A,B) = (C,D)$. □

A *pseudocomplemented De Morgan algebra* is an algebra $(L, \vee, \wedge, \sim, {}^{*}, 0, 1)$ such that $(L, \vee, \wedge, \sim, 0, 1)$ is a De Morgan algebra and ${}^{*} \colon L \to L$ is a pseudocomplement operation of L. Every pseudocomplemented De Morgan algebra forms a double pseudocomplemented lattice in which the pseudocomplements determine each other by:

$$x^{*} = \sim(\sim x)^{+} \quad \text{and} \quad x^{+} = \sim(\sim x)^{*}. \tag{14}$$

A pseudocomplemented De Morgan algebra $(L, \vee, \wedge, \sim, {}^{*}, 0, 1)$ is *normal* (see [14]), if for all $x \in L$,

$$x^{*} \leq \sim x. \tag{N}$$

Note that if $(L, \vee, \wedge, \sim, {}^{*}, 0, 1)$ is a normal pseudocomplemented De Morgan algebra, then for every $x \in L$ and $y = \sim x$, we have $\sim(\sim y)^{+} = y^{*} \leq \sim y$. Hence $(\sim y)^{+} \geq y$ and so $x^{+} \geq \sim x$. Thus,

$$x^* \leq {\sim}x \leq x^+.$$

It is known [11, 19] that in any distributive double pseudocomplemented lattice, condition (M) is equivalent to condition

$$x \wedge x^+ \leq y \vee y^*. \tag{D}$$

We say that a pseudocomplemented Kleene algebra is normal if the underlying pseudocomplemented De Morgan algebra is normal.

Proposition 56. *If R is a tolerance induced by an irredundant covering, then the pseudocomplemented De Morgan algebra $(RS, \vee, \wedge, \sim, ^*, (\emptyset,\emptyset), (U,U))$ is normal.*

Proof. Let $(A,B) \in RS$. Then by Proposition 54,

$$(A,B)^* = (B^{c\blacktriangledown}, B^{c\blacktriangle}).$$

We also have that

$$\sim (A,B) = (B^c, A^c).$$

Trivially, $B^{c\blacktriangledown} \subseteq B^c$. Since $(A,B) \in RS$, $A = X^\blacktriangledown$ and $B = X^\blacktriangle$ for some $X \subseteq U$. We have

$$A^\blacktriangle = X^{\blacktriangledown\blacktriangle} \subseteq X \subseteq X^{\blacktriangle\blacktriangledown} = B^\blacktriangledown.$$

This, implies

$$B^{c\blacktriangle} = B^{\blacktriangledown c} \subseteq A^{\blacktriangle c} = A^{c\blacktriangledown} \subseteq A^c.$$

We have now proved that $(A,B)^* \leq {\sim}(A,B)$. □

3.3 Covering-based rough set systems

We end this chapter by showing that certain rough set lattices based on irredundant coverings are isomorphic to relation-based rough sets lattices. Therefore, they have all the properties listed in Section 3.2.

In Section 2.3, we defined the operators $^\blacktriangleright$, $^\blacktriangleleft$, $^\triangleright$, and $^\triangleleft$ in terms of a covering $\mathscr{H}$ of U. Let us first consider the operators $^\blacktriangleright$ and $^\triangleleft$ introduced by Żakowski. We denote by RS_0 the set of all rough sets defined by these operators, that is:

$$RS_0 = \{(X^\triangleleft, X^\blacktriangleright) \mid X \subseteq U\}.$$

We order RS_0, similarly as RS, by coordinatewise inclusion, that is to say,

$$(X^\triangleleft, X^\blacktriangleright) \leq (Y^\triangleleft, Y^\blacktriangleright) \iff X^\triangleleft \subseteq Y^\triangleleft \text{ and } X^\blacktriangleright \subseteq Y^\blacktriangleright.$$

Proposition 57. *If R is a tolerance induced by an irredundant covering $\mathscr{H}$, then*

$$RS \cong RS_0.$$

Proof. We prove that the map $\varphi\colon (X^{\blacktriangledown}, X^{\blacktriangle}) \mapsto (X^{\lhd}, X^{\blacktriangleright})$ is the required order-isomorphism. Suppose $(X^{\blacktriangledown}, X^{\blacktriangle}) \leq (Y^{\blacktriangledown}, Y^{\blacktriangle})$. By Proposition 39, $X^{\blacktriangleright} = X^{\blacktriangle} \subseteq Y^{\blacktriangle} = Y^{\blacktriangleright}$. Proposition 41 gives that

$$X^{\lhd} = \Box X = X^{\blacktriangledown\blacktriangle} \subseteq Y^{\blacktriangledown\blacktriangle} = \Box Y = Y^{\lhd}.$$

Thus, $(X^{\lhd}, X^{\blacktriangleright}) \leq (Y^{\lhd}, Y^{\blacktriangleright})$.

On the other hand, assume that $(X^{\lhd}, X^{\blacktriangleright}) \leq (Y^{\lhd}, Y^{\blacktriangleright})$. Then again trivially, $X^{\blacktriangle} = X^{\blacktriangleright} \subseteq Y^{\blacktriangleright} = Y^{\blacktriangle}$. Because

$$X^{\blacktriangledown\blacktriangle} = \Box X = X^{\lhd} \subseteq Y^{\lhd} = \Box Y = Y^{\blacktriangledown\blacktriangle},$$

we have $X^{\blacktriangledown} = X^{\blacktriangledown\blacktriangle\blacktriangledown} \subseteq Y^{\blacktriangledown\blacktriangle\blacktriangledown} = Y^{\blacktriangledown}$. Therefore, also $(X^{\blacktriangledown}, X^{\blacktriangle}) \leq (Y^{\blacktriangledown}, Y^{\blacktriangle})$ and the map φ is an order-embedding.

It is obvious that φ is onto RS_0. So, the map φ is an order-isomorphism. □

By definition, $^{\blacktriangleright}$ and $^{\blacktriangleleft}$ form a pair of dual operators. We denote the rough set system defined by them by RS_1, that is,

$$RS_1 = \{(X^{\blacktriangleleft}, X^{\blacktriangleright}) \mid X \subseteq U\}.$$

In Proposition 39 we showed that if R is a tolerance induced by a covering $\mathscr{H}$ of U, then $X^{\blacktriangle} = X^{\blacktriangleright}$ and $X^{\blacktriangledown} = X^{\blacktriangleleft}$ for every $X \subseteq U$. Therefore, we can write the following proposition.

Proposition 58. *If R is a tolerance induced by a covering $\mathscr{H}$, then*

$$RS = RS_1.$$

We denote the rough set system defined by the dual operators $^{\rhd}$ and $^{\lhd}$ by RS_2, that is,

$$RS_2 = \{(X^{\lhd}, X^{\rhd}) \mid X \subseteq U\}.$$

Again, we may write an isomorphism theorem for RS_2.

Proposition 59. *If R is a tolerance induced by an irredundant covering $\mathscr{H}$, then*

$$RS \cong RS_2.$$

Proof. We noted in Proposition 41 that if $\mathscr{H}$ is an irredundant covering of U and R is induced by $\mathscr{H}$, then for any $X \subseteq U$, $X^{\lhd} = \Box X$ and $X^{\rhd} = \Diamond X$. Because $\Diamond X = X^{\blacktriangle\blacktriangledown}$ and $\Box X = X^{\blacktriangledown\blacktriangle}$, it suffices to show that the map

$$\varphi\colon (X^{\blacktriangledown}, X^{\blacktriangle}) \mapsto (X^{\blacktriangledown\blacktriangle}, X^{\blacktriangle\blacktriangledown})$$

is an order-isomorphism.

If $(X^{\blacktriangledown}, X^{\blacktriangle}) \leq (Y^{\blacktriangledown}, Y^{\blacktriangle})$, then $(X^{\blacktriangledown\blacktriangle}, X^{\blacktriangle\blacktriangledown}) \leq (Y^{\blacktriangledown\blacktriangle}, Y^{\blacktriangle\blacktriangledown})$. Similarly, $(X^{\blacktriangledown\blacktriangle}, X^{\blacktriangle\blacktriangledown}) \leq (Y^{\blacktriangledown\blacktriangle}, Y^{\blacktriangle\blacktriangledown})$ gives $X^{\blacktriangledown} = X^{\blacktriangledown\blacktriangle\blacktriangledown} \subseteq Y^{\blacktriangledown\blacktriangle\blacktriangledown} = Y^{\blacktriangledown}$ and $X^{\blacktriangle} = X^{\blacktriangle\blacktriangledown\blacktriangle} = Y^{\blacktriangle\blacktriangledown\blacktriangle} = X^{\blacktriangle}$. This means that the map φ is an order-embedding. The map φ is clearly onto. □

The next theorem summarizes the results presented in this section.

Theorem 60. *If R is tolerance induced by an irredundant covering $\mathscr{H}$, then*

$$RS = RS_1 \cong RS_0 \cong RS_2.$$

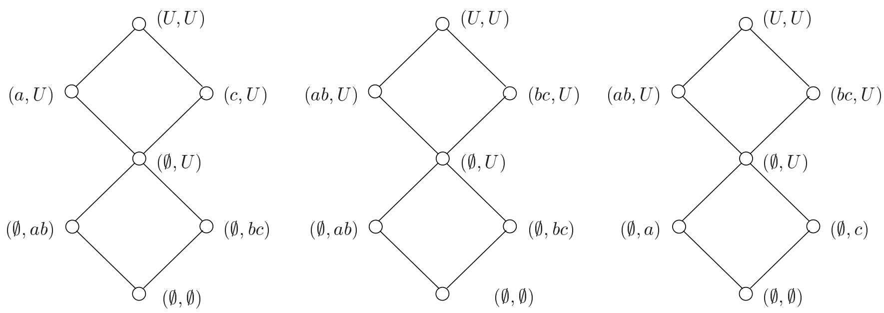
The following example demonstrates that, although the lattices $RS = RS_1$, RS_0, and RS_2 always are isomorphic, their elements may be different, and that different definitions may assign different rough approximations to a given set.

Example 61. Let us consider the tolerance R on $\{a,b,c\}$ of Example 4. The relation R was defined by $R(a) = \{a,b\}$, $R(b) = U$, and $R(c) = \{b,c\}$. The family of sets $\{R(a), R(c)\} = \{\{a,b\},\{b,c\}\}$ is an irredundant covering inducing R. The different approximation operators are presented in Table 7.

Table 7 Different approximation operators

X	$X^{\blacktriangledown}$	$X^{\blacktriangle}$	$X^{\triangleleft}$	$X^{\blacktriangleright}$	$\Box X$	$\Diamond X$
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$\{a\}$	$\emptyset$	$\{a,b\}$	$\emptyset$	$\{a,b\}$	$\emptyset$	$\{a\}$
$\{b\}$	$\emptyset$	U	$\emptyset$	U	$\emptyset$	U
$\{c\}$	$\emptyset$	$\{b,c\}$	$\emptyset$	$\{b,c\}$	$\emptyset$	$\{c\}$
$\{a,b\}$	$\{a\}$	U	$\{a,b\}$	U	$\{a,b\}$	U
$\{a,c\}$	$\emptyset$	U	$\emptyset$	U	$\emptyset$	U
$\{b,c\}$	$\{c\}$	U	$\{b,c\}$	U	$\{b,c\}$	U
U	U	U	U	U	U	U

The isomorphic rough sets lattices $RS = RS_1$, RS_0, and RS_2 are depicted in Figure 17.



$RS = RS_1 = \{(X^{\blacktriangledown}, X^{\blacktriangle}) \mid X \subseteq U\}$ $RS_0 = \{(X^{\triangleleft}, X^{\blacktriangleright}) \mid X \subseteq U\}$ $RS_2 = \{(\Box X, \Diamond X) \mid X \subseteq U\}$

Fig. 17 The lattices $RS = RS_1$, RS_0, and RS_2

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