Sajátérték, sajátvektor

1. Határozzuk meg az alábbi A mátrix valamenyi sajátértékét és a hozzájuk tartozó sajátvektort! Ennek ismeretében döntsön a mátrix szingularitásáról! Állítsa elő a $B = p(A)$ mátrix sajátértékeit, ahol

$$p(z) = -z^3 + z^2 - 10$$

Ha az A mátrix nemszinguláris, akkor határozza meg az inverzének valamenyi sajátértékét is!

$$A = \begin{bmatrix} 3 & 3 & 0 \\ 3 & 3 & 3 \\ 0 & 3 & 3 \end{bmatrix}$$

mo: sajátérték: $\det(A - \lambda \cdot I) = 0$ egyenlet megoldásai.

$$0 = \left| \begin{bmatrix} 3 & 3 & 0 \\ 3 & 3 & 3 \\ 0 & 3 & 3 \end{bmatrix} - \lambda \cdot \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{I} \right| = \left| \begin{matrix} 3-\lambda & 3 & 0 \\ 3 & 3-\lambda & 3 \\ 0 & 3 & 3-\lambda \end{matrix} \right| \begin{matrix} 3-\lambda & 3 \\ 3 & 3-\lambda \\ 0 & 3 \end{matrix} =$$

$$= (3-\lambda)^3 - 9(3-\lambda) - 9(3-\lambda) = (3-\lambda)\left[(3-\lambda)^2 - 9 - 9\right] =$$

$$= (3-\lambda)[9 - 6\lambda + \lambda^2 - 18] = (3-\lambda)[\lambda^2 - 6\lambda - 9] = 0$$

$$\lambda_2 = 3 \qquad \lambda_{1,3} = \frac{6 \pm \overbrace{\sqrt{36+36}}^{8,4853}}{2} \quad \lambda_1 = 7,2427 \quad \lambda_3 = -1,2427$$

sajátértékek: $\lambda_1 = 7,2427$; $\lambda_2 = 3$; $\lambda_3 = -1,2427$.

sajátvektor: $(A - \lambda \cdot I) \cdot x = 0$ LER megoldása

$$\lambda_1 = 7,2427$$

$$\left(\begin{bmatrix} 3 & 3 & 0 \\ 3 & 3 & 3 \\ 0 & 3 & 3 \end{bmatrix} - 7,2427 \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -4,2427 & 3 & 0 \\ 3 & -4,2427 & 3 \\ 0 & 3 & -4,2427 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} -4,2427 & 3 & 0 & 0 \\ 3 & -4,2427 & 3 & 0 \\ 0 & 3 & -4,2427 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} -4,2427 & 3 & 0 & 0 \\ 0 & -2,1214 & 3 & 0 \\ 0 & 3 & -4,2427 & 0 \end{array}\right] \to$$

$$\lambda_{21} = \frac{3}{-4,2427} = -0,7071 \qquad \left(-4,2427 - (-0,7071)\cdot 3\right)$$

$$\lambda_{32} = \frac{3}{-2,1214} = -1,4142 \qquad -4,2427 - (-1,4142)\cdot 3$$

$$\to \left[\begin{array}{ccc|c} -4,2427 & 3 & 0 & 0 \\ 0 & -2,1214 & 3 & 0 \\ 0 & 0 & -0,0001 & 0 \end{array}\right]$$

$$x_3 = \frac{0}{-0,0001} = 0$$

$$x_2 = \frac{1}{-2,1214}\left(0 - 3\cdot 0\right) = 0 \qquad x_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 = \frac{1}{-4,2427}\left(0 - 3\cdot 0\right) = 0$$

$\underline{\lambda_2 = 3}$

$$\begin{bmatrix} 0 & 3 & 0 \\ 3 & 0 & 3 \\ 0 & 3 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 0 & 3 & 0 & 0 \\ 3 & 0 & 3 & 0 \\ 0 & 3 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 3 & 0 & 3 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 3 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 3 & 0 & 3 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\lambda_{32} = \frac{3}{3} = 1$$

$$\left.\begin{array}{l} 3x_1 + 3x_3 = 0 \\ 3x_2 \qquad 0 = 0 \end{array}\right\}$$

$$! \; x_3 = t \qquad x_2 = 0$$

$$x_1 = -x_3 = -t$$

$$x_2 = t\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -t \\ 0 \\ t \end{bmatrix}, \quad t \in \mathbb{Z}$$

$\lambda_3 = -1,2427$

$$\begin{bmatrix} 4,2427 & 3 & 0 \\ 3 & 4,2427 & 3 \\ 0 & 3 & 4,2427 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 4,2427 & 3 & 0 & 0 \\ 3 & 4,2427 & 3 & 0 \\ 0 & 3 & 4,2427 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 4,2427 & 3 & 0 & 0 \\ 0 & 2,1211 & 3 & 0 \\ 0 & 3 & 4,2427 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 4,2427 & 3 & 0 & 0 \\ 0 & 2,1211 & 3 & 0 \\ 0 & 0 & -0,0005 & 0 \end{array}\right]$$

$\lambda_{21} = \frac{3}{4,2427} = 0,7071$ $\qquad 4,2427 - 0,7071 \cdot 3$

$\lambda_{32} = \frac{3}{2,1211} = 1,4144$ $\qquad 4,2427 - 1,4144 \cdot 3$

$x_3 = \frac{0}{-0,0005} = 0$

$x_2 = \frac{1}{2,1211} \left[0 - 3 \cdot 0\right] = 0$ $\qquad x_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$x_1 = 0$

Szingularitás

ha $\det(A) = 0 \Rightarrow$ szinguláris

$\det(A) = \prod \lambda_i = 7,2427 \cdot 3 \cdot (-1,2427) = -27,0015 \neq 0$

nem szinguláris.

$B = p(A)$ mátrix sajátértékei: $p(\lambda_i)$-k.

$p(7,2427) = -7,2427^3 + 7,2427^2 - 10 = -337,4715$

$p(3) = -3^3 + 3^2 - 10 = -28$

$p(-1,2427) = -(-1,2427)^3 + (-1,2427)^2 - 10 = -6,5366$

A^{-1} sajátértékei: $\frac{1}{\lambda_i}$

$\frac{1}{\lambda_1} = \frac{1}{7,2427} = 0,1381$ $\qquad \frac{1}{\lambda_2} = \frac{1}{3} = 0,3333$ $\qquad \frac{1}{-1,2427} = -0,8047$

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2. ugyanez a feladat

$$A = \begin{bmatrix} 2 & 2 & 2 \\ 0 & 2 & 2 \\ 0 & 2 & 2 \end{bmatrix}$$

sajátértékek

$$0 = \begin{vmatrix} 2-\lambda & 2 & 2 \\ 0 & 2-\lambda & 2 \\ 0 & 2 & 2-\lambda \end{vmatrix} \begin{matrix} 2-\lambda & 2 \\ 0 & 2-\lambda \\ 0 & 2 \end{matrix} = (2-\lambda)^3 - 4(2-\lambda) =$$

$$= (2-\lambda)\left[(2-\lambda)^2 - 4\right] = (2-\lambda)(2-\lambda-2)(2-\lambda+2) = (2-\lambda)(-\lambda)(4-\lambda) = 0$$

$\lambda_1 = 4 \quad \lambda_2 = 2 \quad \lambda_3 = 0$

sajátvektorok $\lambda_1 = 4$: $x_1 = \begin{bmatrix} 2t \\ t \\ t \end{bmatrix} = t \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \ t \in \mathbb{Z}$

$$\begin{bmatrix} -2 & 2 & 2 \\ 0 & -2 & 2 \\ 0 & 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} -2 & 2 & 2 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & 2 & -2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} -2 & 2 & 2 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\lambda_{32} = \frac{2}{-2} = -1$

$-2 - (-1) \cdot 2$

szabad tag: $x_3 = t$

$x_2 = \frac{1}{-2}(0 - 2 \cdot t) = t$

$x_1 = \frac{1}{-2}(0 - 2 \cdot t - 2 \cdot t) = 2t$

$\lambda_2 = 2$: $x_2 = \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} = t \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \ t \in \mathbb{Z}$

$$\begin{bmatrix} 0 & 2 & 2 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 0 & 2 & 2 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 \end{array}\right] \rightarrow \begin{array}{c} \begin{matrix} x_2 & x_1 & x_3 & \end{matrix} \\ \left[\begin{array}{ccc|c} 2 & 0 & 2 & 0 \\ 0 & 0 & 2 & 0 \\ 2 & 0 & 0 & 0 \end{array}\right] \end{array} \rightarrow \left[\begin{array}{ccc|c} 2 & 0 & 2 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & -2 & 0 \end{array}\right] \rightarrow \begin{array}{c} \begin{matrix} x_2 & x_3 & x_1 & \end{matrix} \\ \left[\begin{array}{ccc|c} 2 & 2 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & -2 & 0 & 0 \end{array}\right] \end{array}$$

$\lambda_{31} = \frac{2}{2} = 1$

$0 - 1 \cdot 2$

$\lambda_{32} = \frac{-2}{2} = -1$

$$\rightarrow \begin{array}{c} \begin{matrix} x_2 & x_3 & x_1 & \end{matrix} \\ \left[\begin{array}{ccc|c} 2 & 2 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

$x_1 = t \qquad x_3 = \frac{1}{2}(0 - 0 \cdot t) = 0$

$x_2 = \frac{1}{2}(0 - 2 \cdot 0) = 0$

$\lambda_3 = 0$: $\quad x_3 = \begin{bmatrix} 0 \\ -t \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \ t \in \mathbb{Z}$

$$\begin{bmatrix} 2 & 2 & 2 \\ 0 & 2 & 2 \\ 0 & 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 2 & 2 & 2 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 2 & 2 & 2 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\lambda_{32} = \frac{2}{2} = 1$

$x_3 = t$

$x_2 = \frac{1}{2}(0 - 2 \cdot t) = -t$

$x_1 = \frac{1}{2}(0 - 2(-t) - 2 \cdot t) = 0$

szingularitás

$\det A = 4 \cdot 2 \cdot 0 = 0 \Rightarrow$ szinguláris

$B = p(A)$ sajátértékei: $p(\lambda_i)$

$p(4) = -4^3 + 4^2 - 10 = -58$

$p(2) = -2^3 + 2^2 - 10 = -14$

$p(0) = -0^3 + 0^2 - 10 = -10$

nem invertálható.

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