Lagrange - Numerical exercise

i	0	1	2	3
$\vec{p}_i$	$\begin{bmatrix}2\\-1\end{bmatrix}$	$\begin{bmatrix}3\\3\end{bmatrix}$	$\begin{bmatrix}-5\\2\end{bmatrix}$	$\begin{bmatrix}6\\4\end{bmatrix}$
u_i	1	3	5	6

Lagrange polynomials

$$L_0^3(u)=\frac{(u-u_1)(u-u_2)(u-u_3)}{(u_0-u_1)(u_0-u_2)(u_0-u_3)}=\frac{(u-3)(u-5)(u-6)}{(1-3)(1-5)(1-6)}=\frac{u^3-14u^2+63u-90}{-40}$$

$$(u^3-8u+15)(u-6)=u^3-8u^2+15u-6u^2+48u-90=u^3-14u^2+63u-90$$

$$L_1^3(u)=\frac{(u-u_0)(u-u_2)(u-u_3)}{(u_1-u_0)(u_1-u_2)(u_1-u_3)}=\frac{(u-1)(u-5)(u-6)}{(3-1)(3-5)(3-6)}=\frac{u^3-12u^2+41u-30}{12}$$

$$(u^2-6u+5)(u-6)=u^3-6u^2+5u-6u^2+36u-30=u^3-12u^2+41u-30$$

$$L_2^3(u)=\frac{(u-u_0)(u-u_1)(u-u_3)}{(u_2-u_0)(u_2-u_1)(u_2-u_3)}=\frac{(u-1)(u-3)(u-6)}{(5-1)(5-3)(5-6)}=\frac{u^3-10u^2+27u-18}{-8}$$

$$(u^2-4u+3)(u-6)=u^3-4u^2+3u-6u^2+24u-18=u^3-10u^2+27u-18$$

$$L_3^3(u)=\frac{(u-u_0)(u-u_1)(u-u_2)}{(u_3-u_0)(u_3-u_1)(u_3-u_2)}=\frac{(u-1)(u-3)(u-5)}{(6-1)(6-3)(6-5)}=\frac{u^3-9u^2+23u-15}{15}$$

$$(u^2-4u+3)(u-5)=u^3-4u^2+3u-5u^2+20u-15=u^3-9u^2+23u-15$$

Lagrange interpolation curve

$$\vec{p}(u)=\sum_{i=0}^{3}\vec{p}_i L_i^3(u)=\begin{bmatrix}2\\-1\end{bmatrix}\frac{u^3-14u^2+63u-90}{-40}+\begin{bmatrix}3\\3\end{bmatrix}\frac{u^3-12u^2+41u-30}{12}+\begin{bmatrix}-5\\2\end{bmatrix}\frac{u^3-10u^2+27u-18}{-8}+\begin{bmatrix}6\\4\end{bmatrix}\frac{u^3-9u^2+23u-15}{15}$$

$$\vec{p}(u)=\begin{bmatrix}\frac{49}{40}u^3-\frac{243}{20}u^2+\frac{1327}{40}u-\frac{81}{4}\\ \frac{7}{24}u^3-\frac{13}{4}u^2+\frac{269}{24}u-\frac{37}{4}\end{bmatrix};\quad \dot{\vec{p}}(u)=\begin{bmatrix}\frac{147}{40}u^2-\frac{243}{10}u+\frac{1327}{40}\\ \frac{7}{8}u^2-\frac{13}{2}u+\frac{269}{24}\end{bmatrix}$$

Tangent vectors at the interpolation points

$$\dot{\vec{p}}(1)=\begin{bmatrix}\frac{251}{20}\\ \frac{67}{12}\end{bmatrix};\quad \dot{\vec{p}}(3)=\begin{bmatrix}-\frac{133}{20}\\ -\frac{5}{12}\end{bmatrix};\quad \dot{\vec{p}}(5)=\begin{bmatrix}\frac{71}{20}\\ \frac{7}{12}\end{bmatrix};\quad \dot{\vec{p}}(6)=\begin{bmatrix}\frac{787}{40}\\ \frac{89}{24}\end{bmatrix}$$

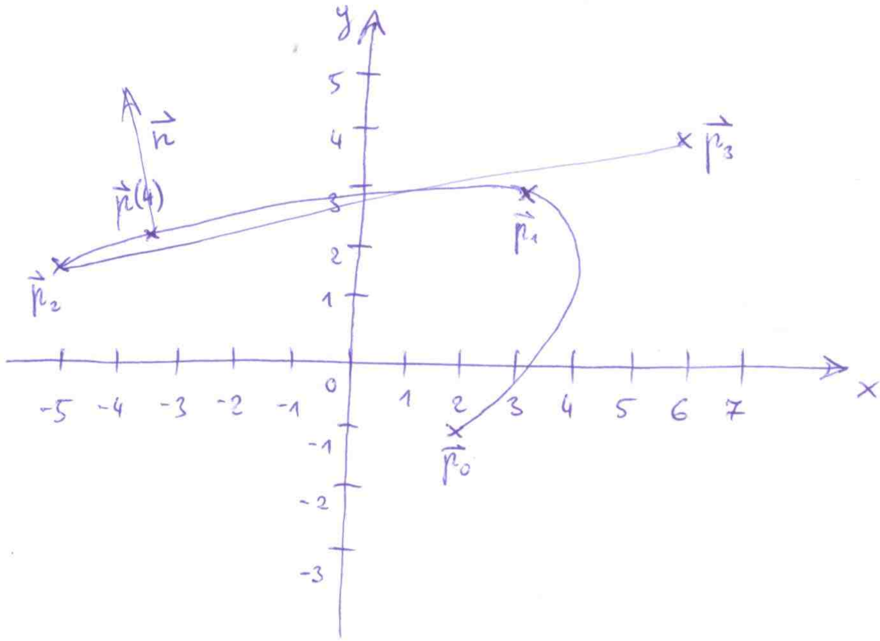
Point of the curve at the parameter u = 4

$$\vec{p}(4)=\begin{bmatrix}-\frac{71}{20}\\ \frac{9}{4}\end{bmatrix}=\begin{bmatrix}-3.55\\ 2.25\end{bmatrix};\quad \dot{\vec{p}}(4)=\begin{bmatrix}-\frac{209}{40}\\ -\frac{19}{24}\end{bmatrix}=\begin{bmatrix}-5.225\\ -0.791\dot{6}\end{bmatrix}$$

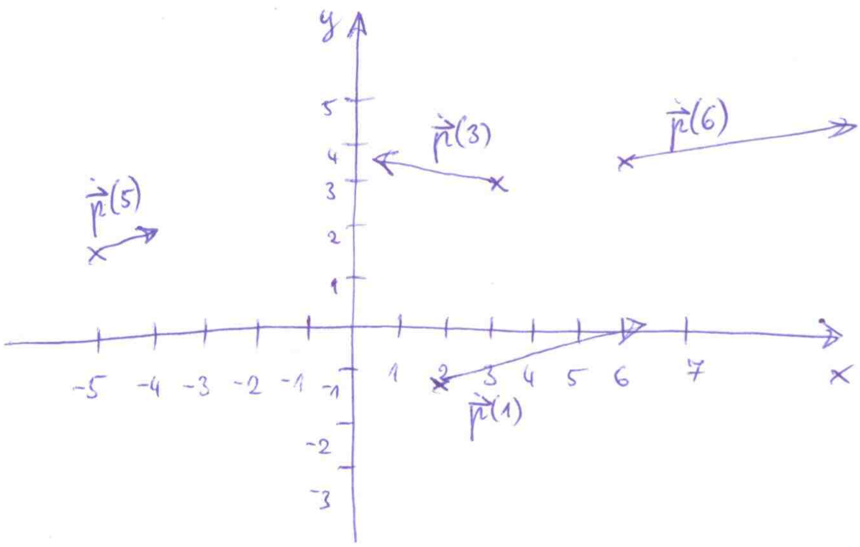
The normal vector at this point

$$\vec{n}=\begin{bmatrix}-0.791\dot{6}\\ +5.225\end{bmatrix}$$

Plot of the interpolation curve



Tangent vectors at the interpolation points



Catmull-Rom – Numerical exercise

i	0	1	2	3	4
$\vec{p}_i$	$\begin{bmatrix}2\\-1\end{bmatrix}$	$\begin{bmatrix}3\\3\end{bmatrix}$	$\begin{bmatrix}-5\\2\end{bmatrix}$	$\begin{bmatrix}6\\4\end{bmatrix}$	$\begin{bmatrix}8\\4\end{bmatrix}$

Calculation of the tangent vectors

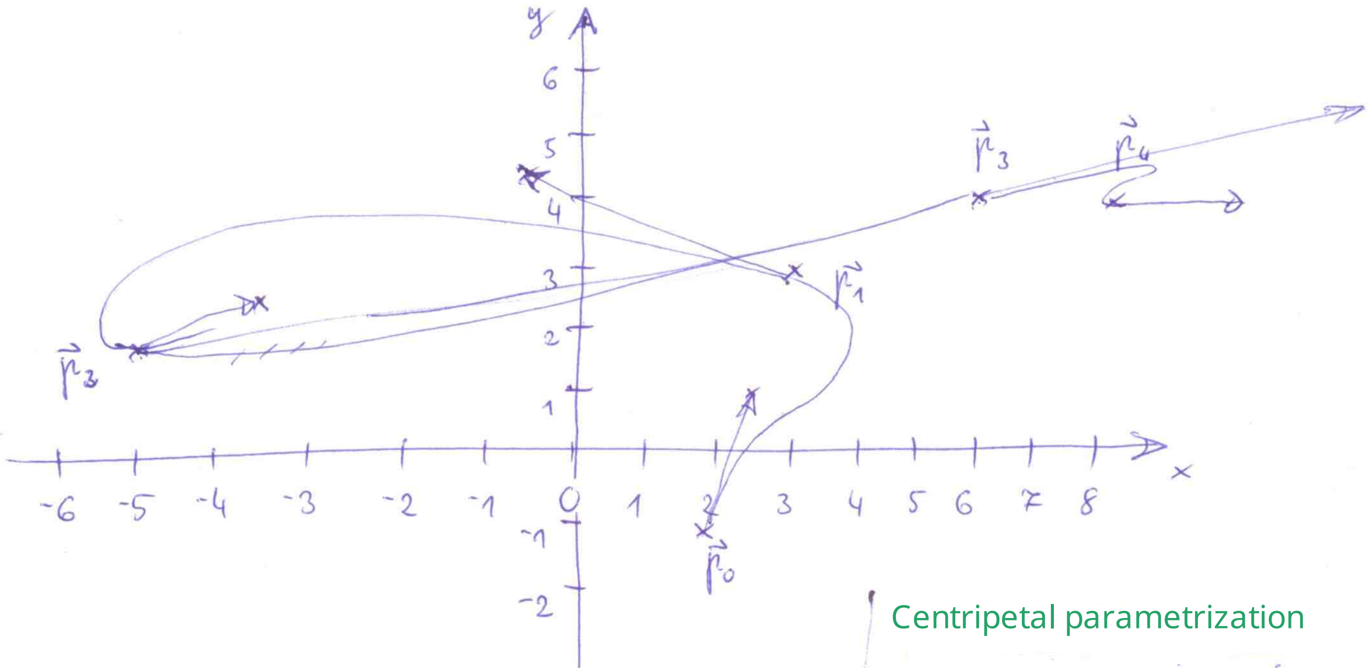
$$\vec{t}_0 = \tau(\vec{p}_1-\vec{p}_0) = \frac{1}{2}\cdot\left(\begin{bmatrix}3\\3\end{bmatrix}-\begin{bmatrix}2\\-1\end{bmatrix}\right) = \begin{bmatrix}0.5\\2\end{bmatrix}$$

$$\vec{t}_1 = \tau(\vec{p}_2-\vec{p}_0) = \frac{1}{2}\cdot\left(\begin{bmatrix}-5\\2\end{bmatrix}-\begin{bmatrix}2\\-1\end{bmatrix}\right) = \begin{bmatrix}-3.5\\1.5\end{bmatrix}$$

$$\vec{t}_2 = \tau(\vec{p}_3-\vec{p}_1) = \frac{1}{2}\cdot\left(\begin{bmatrix}6\\4\end{bmatrix}-\begin{bmatrix}3\\3\end{bmatrix}\right) = \begin{bmatrix}1.5\\0.5\end{bmatrix}$$

$$\vec{t}_3 = \tau(\vec{p}_4-\vec{p}_2) = \frac{1}{2}\cdot\left(\begin{bmatrix}8\\4\end{bmatrix}-\begin{bmatrix}-5\\2\end{bmatrix}\right) = \begin{bmatrix}6.5\\1\end{bmatrix}$$

$$\vec{t}_4 = \tau(\vec{p}_4-\vec{p}_3) = \frac{1}{2}\cdot\left(\begin{bmatrix}8\\4\end{bmatrix}-\begin{bmatrix}6\\4\end{bmatrix}\right) = \begin{bmatrix}2\\0\end{bmatrix}$$



Uniform parametrization

$u_0=0,\ u_1=1,\ u_2=2,\ u_3=3,\ u_4=4$

vagy

$u_0=0,\ u_1=\frac{1}{4},\ u_2=\frac{1}{2},\ u_3=\frac{3}{4},\ u_4=1$

Chord length proportional parametrization

i	$\lVert\vec{p}_i-\vec{p}_{i-1}\rVert$	u_i
1	$\sqrt{17}\approx 4.1231$	0.1625
2	$\sqrt{65}\approx 8.0623$	0.4804
3	$\sqrt{125}\approx 11.18$	0.9212
4	2	1

$\Sigma\approx 25.366$

parameters:

$u_0=0$

$u_1=0.1635$

$u_2=0.4804$

$u_3=0.9212$

$u_4=1$

Centripetal parametrization

i	$\sqrt{\lVert\vec{p}_i-\vec{p}_{i-1}\rVert}$	u_i
1	$\sqrt[4]{17}\approx 2.0305$	0.2109
2	$\sqrt[4]{65}\approx 2.8394$	0.5058
3	$\sqrt[4]{125}\approx 3.3437$	0.8531
4	$\sqrt[4]{4}\approx 1.4142$	1

$\Sigma\approx 9.6279$

parameters:

$u_0=0$

$u_1=0.2109$

$u_2=0.5058$

$u_3=0.8531$

$u_4=1$

Overhauser – Numerical exercise

i	0	1	2	3
$\vec{p}_i$	$\begin{bmatrix}2\\-1\end{bmatrix}$	$\begin{bmatrix}3\\3\end{bmatrix}$	$\begin{bmatrix}5\\2\end{bmatrix}$	$\begin{bmatrix}6\\-4\end{bmatrix}$
u_i	1	3	5	6

$\vec{c}_1(u)$ Bessel parabola calculation

$$L_0^2(u) = \frac{(u-u_1)(u-u_2)}{(u_0-u_1)(u_0-u_2)} = \frac{(u-3)(u-5)}{(1-3)(1-5)}$$

$$L_1^2(u) = \frac{(u-u_0)(u-u_2)}{(u_1-u_0)(u_1-u_2)} = \frac{(u-1)(u-5)}{(3-1)(3-5)}$$

$$L_2^2(u) = \frac{(u-u_0)(u-u_1)}{(u_2-u_0)(u_2-u_1)} = \frac{(u-1)(u-3)}{(5-1)(5-3)}$$

$$\vec{c}_1(u) = \begin{bmatrix}2\\-1\end{bmatrix}\frac{u^2-8u+15}{8} + \begin{bmatrix}3\\3\end{bmatrix}\frac{u^2-6u+5}{-4} + \begin{bmatrix}5\\2\end{bmatrix}\frac{u^2-4u+3}{8}$$

$$\vec{c}_1(u) = \begin{bmatrix}\left(\frac{2}{8}-\frac{3}{4}+\frac{5}{8}\right)u^2 + \left(-\frac{16}{8}+\frac{18}{4}-\frac{20}{8}\right)u + \left(\frac{30}{8}-\frac{15}{4}+\frac{15}{8}\right)\\ \left(-\frac{1}{8}-\frac{3}{4}+\frac{2}{8}\right)u^2 + \left(1+\frac{18}{4}-1\right)u + \left(-\frac{15}{8}-\frac{15}{4}+\frac{6}{8}\right)\end{bmatrix} = \begin{bmatrix}0.125u^2 + 1.875\\ -0.625u^2 + 4.5u - 4.875\end{bmatrix}$$

$\vec{c}_2(u)$ Bessel parabola calculation

$$L_0^2(u) = \frac{(u-u_2)(u-u_3)}{(u_1-u_2)(u_1-u_3)} = \frac{(u-5)(u-6)}{(3-5)(3-6)}$$

$$L_1^2(u) = \frac{(u-u_1)(u-u_3)}{(u_2-u_1)(u_2-u_3)} = \frac{(u-3)(u-6)}{(5-3)(5-6)}$$

$$L_2^2(u) = \frac{(u-u_1)(u-u_2)}{(u_3-u_1)(u_3-u_2)} = \frac{(u-3)(u-5)}{(6-3)(6-5)}$$

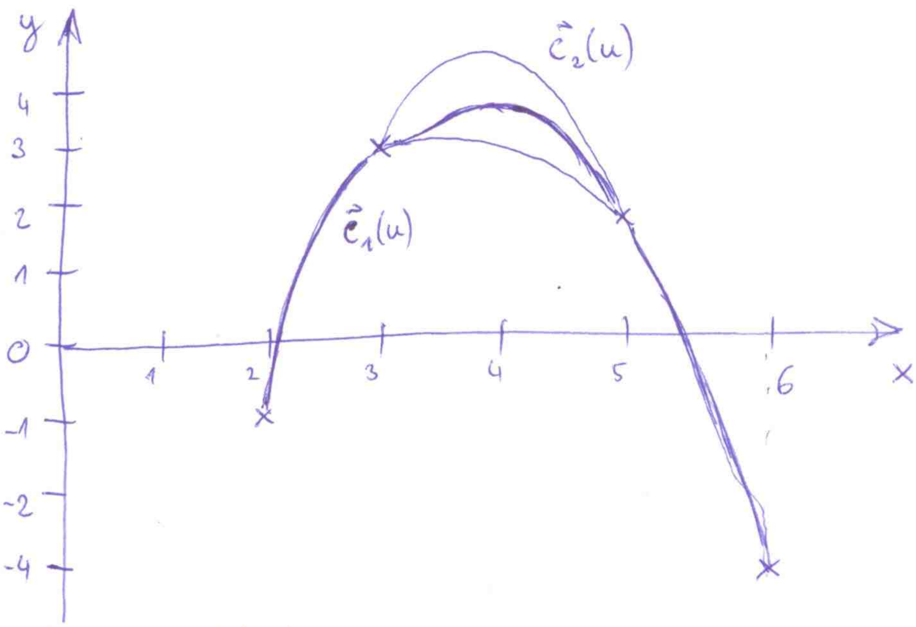
$$\vec{c}_2(u) = \begin{bmatrix}3\\3\end{bmatrix}\frac{u^2-11u+30}{6} + \begin{bmatrix}5\\2\end{bmatrix}\frac{u^2-9u+18}{-2} + \begin{bmatrix}6\\-4\end{bmatrix}\frac{u^2-8u+15}{3}$$

$$\vec{c}_2(u) = \begin{bmatrix}\left(\frac{3}{6}-\frac{5}{2}+\frac{6}{3}\right)u^2 + \left(-\frac{33}{6}+\frac{45}{2}-\frac{48}{3}\right)u + \left(\frac{90}{6}-\frac{90}{2}+\frac{90}{3}\right)\\ \left(\frac{3}{6}-1-\frac{4}{3}\right)u^2 + \left(-\frac{33}{6}+9+\frac{32}{3}\right)u + \left(\frac{90}{6}-18-20\right)\end{bmatrix} = \begin{bmatrix}u\\ -1.8\dot{3}u^2 + 14.1\dot{6}u - 23\end{bmatrix}$$

Parametric form of the spline

$$\vec{c}(u)=\begin{cases}\vec{c}_1(u), & \text{ha } 1\leq u<3,\\ \left(1-\dfrac{5-u}{2}\right)\vec{c}_1(u)+\left(\dfrac{5-u}{2}\right)\vec{c}_2(u), & \text{ha } 3\leq u<5,\\ \vec{c}_2(u), & \text{ha } 5\leq u\leq 6.\end{cases}$$

Plot of the curve



Derivatives of the Bessel parabolas

$$\dot{\vec{c}}_1(u)=\begin{bmatrix}0.25u\\ -1.25u+4.5\end{bmatrix}$$

$$\dot{\vec{c}}_2(u)=\begin{bmatrix}1\\ -\frac{11}{3}u+\frac{85}{6}\end{bmatrix}$$

Tangent vectors at the interpolation points

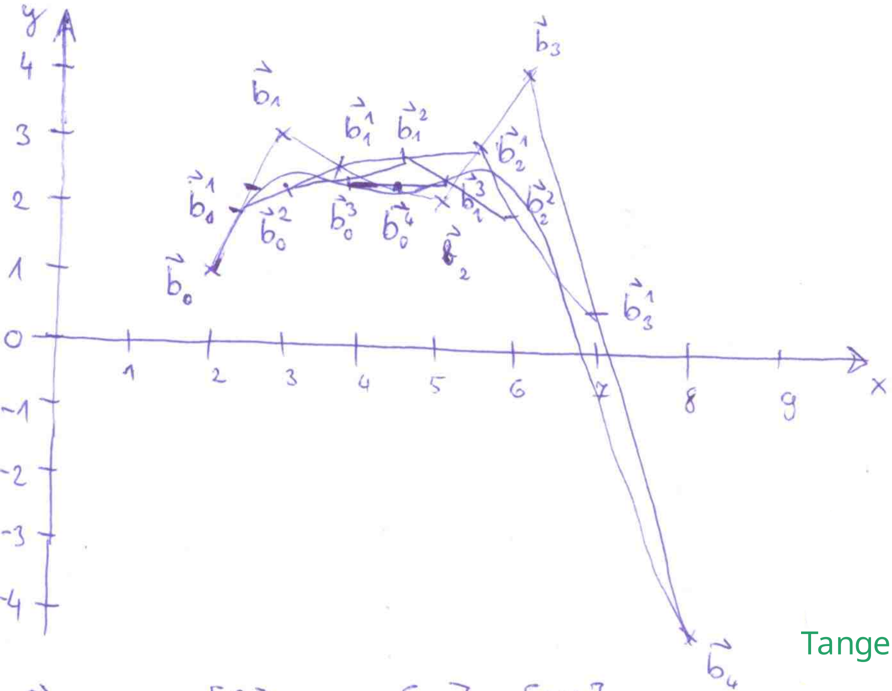
$$\dot{\vec{c}}(1)=\dot{\vec{c}}_1(1)=\begin{bmatrix}\frac{1}{4}\\ \frac{13}{4}\end{bmatrix};\quad \dot{\vec{c}}(3)=\dot{\vec{c}}_1(3)=\begin{bmatrix}\frac{3}{4}\\ \frac{3}{4}\end{bmatrix}$$

$$\dot{\vec{c}}(5)=\dot{\vec{c}}_2(5)=\begin{bmatrix}1\\ -\frac{25}{6}\end{bmatrix},\quad \dot{\vec{c}}(6)=\dot{\vec{c}}_2(6)=\begin{bmatrix}1\\ -\frac{47}{6}\end{bmatrix}$$

The point of the curve at the parameter u = 4

$$\vec{c}(4)=\frac{1}{2}\vec{c}_1(4)+\frac{1}{2}\cdot\vec{c}_2(4)=\frac{1}{2}\cdot\begin{bmatrix}3.875\\ 3.125\end{bmatrix}+\frac{1}{2}\cdot\begin{bmatrix}4\\ 4.\dot{3}\end{bmatrix}=\begin{bmatrix}3.9375\\ 3.7291\dot{6}\end{bmatrix}$$

de Casteljau – Numerical exercise



$$\vec b_0 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \vec b_1 = \begin{bmatrix} 3 \\ 3 \end{bmatrix} \quad \vec b_2 = \begin{bmatrix} 5 \\ 2 \end{bmatrix} \quad \vec b_3 = \begin{bmatrix} 6 \\ 4 \end{bmatrix} \quad \vec b_4 = \begin{bmatrix} 8 \\ -4 \end{bmatrix}$$

$$\vec b_0^1 = 0.6 \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 0.4 \cdot \begin{bmatrix} 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 2.4 \\ 1.8 \end{bmatrix}$$

$$\vec b_1^1 = 0.6 \cdot \begin{bmatrix} 3 \\ 3 \end{bmatrix} + 0.4 \cdot \begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 3.8 \\ 2.6 \end{bmatrix}$$

$$\vec b_2^1 = 0.6 \cdot \begin{bmatrix} 5 \\ 2 \end{bmatrix} + 0.4 \cdot \begin{bmatrix} 6 \\ 4 \end{bmatrix} = \begin{bmatrix} 5.4 \\ 2.8 \end{bmatrix}$$

$$\vec b_3^1 = 0.6 \cdot \begin{bmatrix} 6 \\ 4 \end{bmatrix} + 0.4 \cdot \begin{bmatrix} 8 \\ -4 \end{bmatrix} = \begin{bmatrix} 6.8 \\ 0.8 \end{bmatrix}$$

$$\vec b_0^2 = 0.6 \cdot \begin{bmatrix} 2.4 \\ 1.8 \end{bmatrix} + 0.4 \cdot \begin{bmatrix} 3.8 \\ 2.6 \end{bmatrix} = \begin{bmatrix} 2.96 \\ 2.12 \end{bmatrix}$$

$$\vec b_1^2 = 0.6 \cdot \begin{bmatrix} 3.8 \\ 2.6 \end{bmatrix} + 0.4 \cdot \begin{bmatrix} 5.4 \\ 2.8 \end{bmatrix} = \begin{bmatrix} 4.44 \\ 2.68 \end{bmatrix}$$

$$\vec b_2^2 = 0.6 \cdot \begin{bmatrix} 5.4 \\ 2.8 \end{bmatrix} + 0.4 \cdot \begin{bmatrix} 6.8 \\ 0.8 \end{bmatrix} = \begin{bmatrix} 5.96 \\ 2 \end{bmatrix}$$

$$\vec b_0^3 = 0.6 \cdot \begin{bmatrix} 2.96 \\ 2.12 \end{bmatrix} + 0.4 \cdot \begin{bmatrix} 4.44 \\ 2.68 \end{bmatrix} = \begin{bmatrix} 3.552 \\ 2.344 \end{bmatrix}$$

$$\vec b_1^3 = 0.6 \cdot \begin{bmatrix} 4.44 \\ 2.68 \end{bmatrix} + 0.4 \cdot \begin{bmatrix} 5.96 \\ 2 \end{bmatrix} = \begin{bmatrix} 5.048 \\ 2.408 \end{bmatrix}$$

$$\vec b_0^4 = 0.6 \cdot \begin{bmatrix} 3.552 \\ 2.344 \end{bmatrix} + 0.4 \cdot \begin{bmatrix} 5.048 \\ 2.408 \end{bmatrix} = \begin{bmatrix} 4.1504 \\ 2.3696 \end{bmatrix}$$

$$\vec b(0.4) = \begin{bmatrix} 4.1504 \\ 2.3696 \end{bmatrix}$$

The point of the Bézier curve at u = 0.4 is

$$\begin{bmatrix} 4.1504 \\ 2.3696 \end{bmatrix}.$$

Tangent at the given point

$$\vec t = \vec b_1^3 - \vec b_0^3 = \begin{bmatrix} 1.496 \\ 0.064 \end{bmatrix}$$

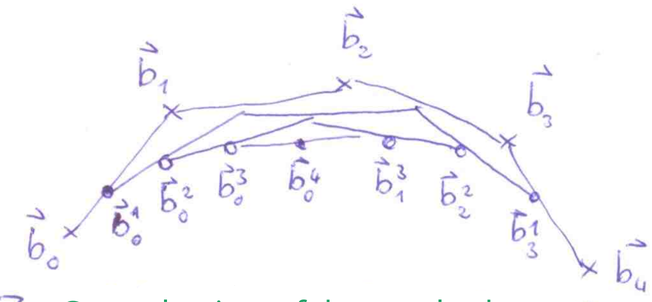
Equation of the tangent line

$$\vec e(\lambda) = \begin{bmatrix} 4.1504 \\ 2.3696 \end{bmatrix} + \lambda \cdot \begin{bmatrix} 1.496 \\ 0.064 \end{bmatrix}, \lambda \in \mathbb{R}$$

Normal vector at the given point

$$\vec n = \begin{bmatrix} 0.064 \\ -1.496 \end{bmatrix}$$

Split of the curve



Control points of the resulted curves

$$\vec b_0 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad \vec b_0^4 = \begin{bmatrix} 4.1504 \\ 2.3696 \end{bmatrix}$$

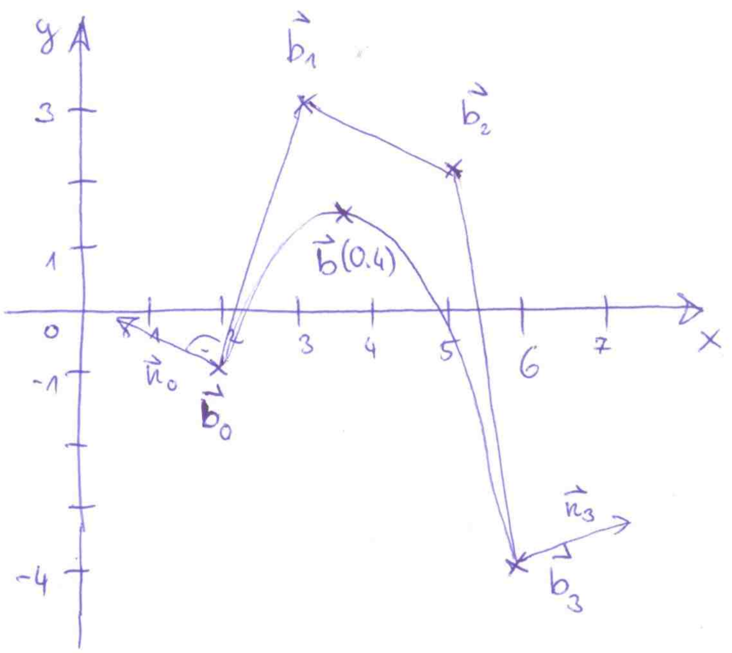
$$\vec b_0^1 = \begin{bmatrix} 2.4 \\ 1.8 \end{bmatrix} \qquad \vec b_1^3 = \begin{bmatrix} 5.048 \\ 2.408 \end{bmatrix}$$

$$\vec b_0^2 = \begin{bmatrix} 2.96 \\ 2.12 \end{bmatrix} \qquad \vec b_2^2 = \begin{bmatrix} 5.96 \\ 2 \end{bmatrix}$$

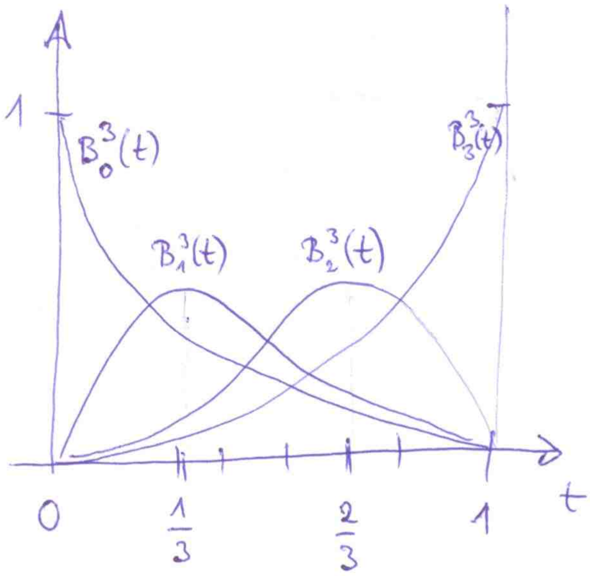
$$\vec b_0^3 = \begin{bmatrix} 3.552 \\ 2.344 \end{bmatrix} \qquad \vec b_3^1 = \begin{bmatrix} 6.8 \\ 0.8 \end{bmatrix}$$

$$\vec b_0^4 = \begin{bmatrix} 4.1504 \\ 2.3696 \end{bmatrix} \qquad \vec b_4 = \begin{bmatrix} 8 \\ -4 \end{bmatrix}$$

Bézier - Numerical exercise



Plot of the Bernstein polynomials



Bernstein polynomials

$$B_0^3(0.4) = \binom{3}{0} 0.4^0 \cdot 0.6^3 = 0.216$$

$$B_1^3(0.4) = \binom{3}{1} 0.4^1 \cdot 0.6^2 = 0.432$$

$$B_2^3(0.4) = \binom{3}{2} 0.4^2 \cdot 0.6^1 = 0.288$$

$$B_3^3(0.4) = \binom{3}{3} 0.4^3 \cdot 0.6^0 = 0.064$$

$$\vec{b}(0.4) = 0.216 \cdot \begin{bmatrix} 2 \\ -1 \end{bmatrix} + 0.432 \cdot \begin{bmatrix} 3 \\ 3 \end{bmatrix} + 0.288 \cdot \begin{bmatrix} 5 \\ 2 \end{bmatrix} + 0.064 \cdot \begin{bmatrix} 6 \\ -4 \end{bmatrix}$$

$$\vec{b}(0.4) = \begin{bmatrix} 3.552 \\ 1.4 \end{bmatrix}$$

Normal vectors at the endpoints

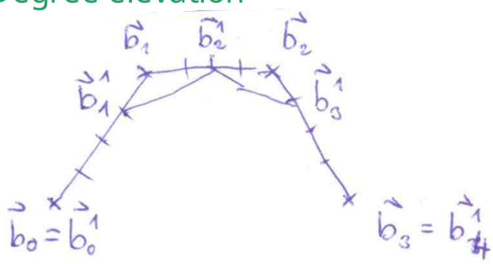
$$\vec{t}_0 = \vec{b}_1 - \vec{b}_0 = \begin{bmatrix} 3 \\ 3 \end{bmatrix} - \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

$$\vec{n}_0 = \frac{1}{\sqrt{17}} \cdot \begin{bmatrix} -4 \\ 1 \end{bmatrix} \approx \begin{bmatrix} -0.9701 \\ 0.2425 \end{bmatrix}$$

$$\vec{t}_3 = \vec{b}_3 - \vec{b}_2 = \begin{bmatrix} 6 \\ -4 \end{bmatrix} - \begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -6 \end{bmatrix}$$

$$\vec{n}_3 = \frac{1}{\sqrt{37}} \cdot \begin{bmatrix} 6 \\ 1 \end{bmatrix} \approx \begin{bmatrix} 0.9864 \\ 0.1644 \end{bmatrix}$$

Degree elevation



$$\vec{b}_0^1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$\vec{b}_1^1 = \frac{1}{4} \cdot \begin{bmatrix} 2 \\ -1 \end{bmatrix} + \frac{3}{4} \cdot \begin{bmatrix} 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 2.75 \\ 2 \end{bmatrix}$$

$$\vec{b}_2^1 = \frac{1}{2} \begin{bmatrix} 3 \\ 3 \end{bmatrix} + \frac{1}{2} \cdot \begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 2.5 \end{bmatrix}$$

$$\vec{b}_3^1 = \frac{3}{4} \cdot \begin{bmatrix} 5 \\ 2 \end{bmatrix} + \frac{1}{4} \cdot \begin{bmatrix} 6 \\ -4 \end{bmatrix} = \begin{bmatrix} 5.25 \\ 0.5 \end{bmatrix}$$

$$\vec{b}_4^1 = \begin{bmatrix} 6 \\ -4 \end{bmatrix}$$

As the result of the degree elevation we obtain the

$$\begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2.75 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 2.5 \end{bmatrix}, \begin{bmatrix} 5.25 \\ 0.5 \end{bmatrix}, \begin{bmatrix} 6 \\ -4 \end{bmatrix}$$

control points.